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On the configuration of Herman rings of meromorphic functions

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ABSTRACT

We prove some results concerning the possible configurations of Herman rings for transcendental meromorphic functions. We show that one pole is enough to obtain cycles of Herman rings of arbitrary period and give a sufficient condition for a configuration to be realizable.

1. Introduction

Given a meromorphic map $f : \mathbb{C} \to \widehat{\mathbb{C}}$, we consider the dynamical system generated by the iterates of f, denoted by $f^n = f \circ \overset{(n)}{\cdots} \circ f$. If f has a limit at ∞, then $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational map. Otherwise, f is a transcendental map, i.e., it has an essential singularity at ∞. If the essential singularity has no preimages, i.e., if f has no *poles*, we speak about entire transcendental functions. Else f is known as a (transcendental) meromorphic function.

There is a dynamically natural partition of the phase space into the *Fatou set* $\mathcal{F}(f)$, where the iterates of f are well defined and form a normal family, and the *Julia set* $J(f)$, which is the complement.

Background on the iteration theory of rational maps can be found for example in [1,2] or [3]. For transcendental maps, the reader can check the survey in [4] or the book [5] or the series of papers on meromorphic functions [6–9].

There are several differences between transcendental and rational maps. One important such difference concerns the *singular values* or singularities of the inverse function. For a rational map f, all branches of the inverse function are locally well defined except on the set of *critical values*, i.e., points $v = f(c)$ where $f'(c) = 0$. If f is transcendental, there is another obstruction: some inverse branches are not well defined either in any neighborhood of the *asymptotic values*. A point $a \in \mathbb{C}$ is called an asymptotic value if there exists a path $\gamma(t) \underset{t\to\infty}{\longrightarrow} \infty$ such that $f(\gamma(t)) \underset{t\to\infty}{\longrightarrow} a$. Although critical values always form a discrete set (even finite in the case of rational maps), this needs not be the case for asymptotic values, unless f is of finite order.

This fact motivated the definition and study of special classes of transcendental maps like, for example, the class $\mathcal{S}$ of functions of *finite type*, which are those with a finite number of singular values. Entire or meromorphic functions in $\mathcal{S}$ share many properties with rational maps, like for example the fact that all components of the Fatou set are eventually periodic [10,11]. There is a classification of the periodic components of the Fatou set for a rational map or a transcendental map in

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