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The Picard–Fuchs equations for complete hyperelliptic integrals of even order curves, and the actions of the generalized Neumann system

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We consider a family of genus 2 hyperelliptic curves of even order and obtain explicitly the systems of 5 linear ordinary differential equations for periods of the corresponding Abelian integrals of first, second, and third kind, as functions of some parameters of the curves. The systems can be regarded as extensions of the well-studied Picard–Fuchs equations for periods of complete integrals of first and second kind on odd hyperelliptic curves. The periods we consider are linear combinations of the action variables of several integrable systems, in particular the generalized Neumann system with polynomial separable potentials. Thus the solutions of the extended Picard–Fuchs equations can be used to study various properties of the actions. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4868965]

I. INTRODUCTION

Given a family of elliptic curves $\mathcal{E} \subset \mathbb{P}^2$ in the Legendre form

$$w^2 = (1 - z^2)(1 - k^2 z^2),$$

it is known that the complete elliptic integrals of first kind

$$K(k) = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}, \quad K'(k) = \int_1^{1/k} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

as functions of the modulus $k \in \mathbb{C}$, give 2 independent solutions of the hypergeometric equation of the Legendre type

$$k(1 - k^2)\frac{d^2 y}{dk^2} - (1 + k^2)\frac{dy}{dk} + ky = 0, \tag{1}$$

that is, $K(k) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1; k^2)$. The equation has singular points $z_{1,2,3} = -1, 0, 1$, which means that the solutions $K(k)$, $K'(k)$ are not single-valued: when k goes around z_i, these functions transform to a linear combination of $K(k)$, $K'(k)$. That is, the solutions $y(k)$ undergo a monodromy.

Equivalently, (1) can be rewritten as a system of first order equations for $K(k)$ and the complete integral of the *second kind*[17]

$$\bar{E}(k) = \int_0^1 \frac{z^2 dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},$$

namely,

$$\frac{dK}{dk} = \frac{1}{k(1 - k^2)}(k^2 K - \bar{E}), \quad \frac{d\bar{E}}{dk} = \frac{k}{1 - k^2}(K - \bar{E}) \tag{2}$$

(see, e.g., Ref. 8).

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