Polynomial inverse integrating factors of quadratic differential systems

Antoni Ferragut

Departament de Matemàtiques
Universitat Autònoma de Barcelona
08193 Bellaterra, Barcelona

Treball de recerca presentat al
Departament de Matemàtiques
Universitat Autònoma de Barcelona

Directors: Dr. Bartomeu Coll, Dr. Jaume Llibre

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Chapter 1

Introduction

A *(planar) polynomial differential system* defined on $\mathbb{R}^2$ (resp. $\mathbb{C}^2$) is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1.1}$$

where P, Q are real (resp. complex) polynomials in the variables x and y. The *degree* m of this polynomial system is the maximum of the degrees of the polynomials P and Q; that is, $m = \max\{\deg P, \deg Q\}$. Let $X = (P, Q)$ be the vector field associated to system (1.1).

Let U be an open subset of $\mathbb{R}^2$ (resp. $\mathbb{C}^2$). If there exists a non–constant C^1 function $H : U \to \mathbb{R}$ (resp. $\mathbb{C}$), eventually multi–valued, which is constant on all the solutions of X contained in U, then we say that H is a *first integral* of X on U, and that X is *integrable* on U. For our systems it is clear that the existence of a first integral H on U determines their phase portraits on U. Consequently, given a vector field X on $\mathbb{R}^2$ or $\mathbb{C}^2$ it is important to know if this vector field has a first integral.

Suppose now that X is a real polynomial vector field. Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a function whose domain of definition is $\mathbb{R}^2$, and that satisfies the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V. \tag{1.2}$$

Then, V is called an *inverse integrating factor* of system (1.1) in $\mathbb{R}^2$.

We say that the inverse integrating factor V is associated to the first integral H of the vector field X if we have that

$$\dot{x} = \frac{P}{V} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{Q}{V} = \frac{\partial H}{\partial x}.$$

The inverse integrating factors $V : \mathbb{R}^2 \to \mathbb{R}$ defined in the whole $\mathbb{R}^2$ are very important because

(1) $R = 1/V$ defines on $\mathbb{R}^2 \setminus \{V = 0\}$ an integrating factor of system (1.1) (which allows to compute a first integral of the system on $\mathbb{R}^2 \setminus \{V = 0\}$);

(2) $\{V = 0\}$ contains the limit cycles of system (1.1), see [28]. This fact allows to study the limit cycles which bifurcate from periodic orbits of a center (Hamiltonian or not) and compute their shape, see [29] and [30]. For doing that we develop the function V in power series of the small perturbation parameter. A remarkable fact is that the first term of this expansion coincides with the first non–identically zero Melnikov function (see [31]).

(3) The expression of $V(x, y)$ is simpler than the expression of the integrating factors and their associated first integrals, and usually its domain of definition is larger than the domain of definition of the integrating factors or their associated first integrals, for more details see [8].

In this work the *quadratic systems* are the polynomial real differential systems with degree $m = 2$. Our main objective is to classify all the quadratic systems having a polynomial inverse integrating factor $V(x, y)$. For this purpose, we use a classification of the quadratic systems into ten normal forms. This classification can be found in [27]. Due to the extension and, in some cases, to the complexity of the classification of the polynomial inverse integrating factors, in this first part of the work we mainly present all the polynomial inverse integrating factors of six of the ten normal forms.

After this classification, we will study and classify their associated first integrals, and later on we will draw the phase portraits of these systems, grouping those systems which have a topologically equivalent phase portrait. Also, this classification allows to study the trajectories contained in the set $\{V = 0\}$. We believe that, in general, such orbits will be relevant in the global phase portrait of the system, i.e. we hope that almost always they will be separatrices of the system. When this does not occur we shall want to understand the reason.

The set $\{V = 0\}$ contains all the limit cycles of the system, so we will be able to study the possible algebraic limit cycles of the quadratic systems which appear in this classification and their degree. We say that a limit cycle is *algebraic* of *degree* n if it is contained in an irreducible algebraic curve of degree n. Algebraic limit cycles are more important that they seem. For instance, any finite configuration of limit cycles is realizable only by algebraic limit cycles for a convenient polynomial differential system, see [36].

Until now, only seven different families of algebraic limit cycles for quadratic systems have been found: one of degree 2 (see [45]), four of degree 4 (see [48, 26, 10]), one of degree 5 and one of degree 6 (see [18]). Moreover, it is proved that there are no algebraic limit cycles of degree 2 and 4 except these five families, (see [13]). It has been proved that there are no algebraic limit cycles of degree 3 (see [23, 24, 25], and for a short proof see [12]).

It is also known that there are cubic systems having polynomial inverse integrating factors which contain algebraic limit cycles. We want to know if such a situation can occur or not for a quadratic system.

A function of the form $f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp(g/h)$, where f_i, g and h are polynomials in $\mathbb{C}[x, y]$ and $\lambda_i \in \mathbb{C}$, for $i = 1, \ldots, p$, is called *Darbouxian.* We note that eventually such functions can be multi–valued. The quadratic systems with a polynomial inverse integrating factor have always a first integral given by a Darbouxian function, see [9]. Eventually, this Darbouxian function can be a polynomial or a rational function.

Let $H = f/g$ be a rational first integral of a polynomial differential system (1.1). According to Poincaré [44] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable* value of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$. Here, if $c = \infty$ then $f + cg$ denotes g. In [9] it is proved that there are finitely many remarkable values for a given rational first integral H. As far as we know the notion of remarkable values has not been used after Poincaré with the exception of these last years, see for instance [9]

Now suppose that $c \in \mathbb{C}$ is a remarkable value of the rational first integral H and that $u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ is the factorization of the polynomial $f + cg$ into irreducible factors in $\mathbb{C}[x, y]$. If some of the α_i, for $i = 1, \ldots, r$, is larger than 1, then we say that c is a *critical* remarkable value of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable* invariant algebraic curve of system (1.1) with *exponent* α_i. For the definition of an invariant algebraic curve of a polynomial differential system see Section 2.2.1.

In the quadratic systems having a rational first integral we will compute all the critical remarkable values and their critical remarkable invariant algebraic curves, because we want to see the relationships between these values and their corresponding invariant algebraic curves in the phase portrait.

For the quadratic systems having a polynomial inverse integrating factor and a multi–valued first integral, we will like to find some common properties between the polynomial inverse integrating factors.

Finally, we will try to extend some of the obtained results from the quadratic systems to polynomial differential systems of arbitrary degree.

In short, in this work and in the next future we want to give:

(1) A classification of all planar real quadratic systems having a polynomial inverse integrating factor, for this we shall use the ten normal forms.

(2) For such systems we will like to study all the mentioned notions: the first integrals; the orbits contained in $V = 0$, in special the possible algebraic limit cycles; the remarkable values, the critical remarkable values and the invariant algebraic curves associated to these values; the multi–valued first integrals, ...

(3) We will try to extend some of the results obtained for the quadratic systems to polynomial differential systems of higher degree.

In the present moment we provide the classification of all planar real quadratic systems having a polynomial inverse integrating factor for six of the ten normal forms. Moreover, for such systems and using their inverse integrating factors we compute their associated first integrals.

The work is structured as follows. In Chapter 2 we introduce all the definitions and the main results that we will use in the work: first integrals, inverse integrating factors, Darbouxian theory of integrability and remarkable values. Chapter 3, the main chapter of this work, contains the classification of quadratic systems into our ten normal forms and the classification of the polynomial inverse integrating factors of six of these ten normal forms. In Chapter 4 we present the properties which have these quadratic systems with a polynomial inverse integrating factor. We also present the first integral associated to these polynomial inverse integrating factors, classified into polynomial, rational and Darbouxian first integrals which are not of these first two types. Finally, in the rational case, we compute the remarkable values associated to the rational first integrals.