Polynomial inverse integrating factors of quadratic differential systems and other results

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This thesis is divided into two different parts. In the first one, we study the quadratic systems (polynomial systems of degree two) having a polynomial inverse integrating factor. In the second one, we study three different problems related to polynomial differential systems.

The first part. It is very important, for planar differential systems, the knowledge of a first integral. Its level sets are formed by orbits and they let us draw the phase portrait of the system, which is the main objective of the qualitative theory of planar differential equations.

As it is known, there is a bijection between the study of the first integrals and the study of inverse integrating factors. In fact, it is easier to study the inverse integrating factors than the first integrals.

A widely studied class of planar differential systems is the quadratic one. There are more than a thousand published articles about this subject of differential systems, but we are far away of knowing which quadratic systems are integrable, that is, if they have a first integral.

In this work, we study the quadratic systems having a polynomial inverse integrating factor $V = V(x, y)$, so they also have a first integral, defined where V does not vanish. This class of quadratic systems is important for several reasons:

1. The first integral is always Darboux.
2. It contains the class of homogeneous quadratic system, widely studied (Date, Sibirskii, Vulpe...).
3. It contains the class of quadratic systems having a center, also studied (Dulac, Kapteyn, Bautin...).
4. It contains the class of Hamiltonian quadratic systems (Artés, Llibre, Vulpe).
5. It contains the class of quadratic systems having a polynomial first integral (Chavarriga, García, Llibre, Pérez de Rio, Rodríguez).
6. It contains the class of quadratic systems having a rational first integral of degree two (Cairó, Llibre).

The classification of the quadratic systems having a polynomial inverse integrating factor is not completely finished. There remain near a 5% of the cases to study. We leave their study for an immediate future.

We give some results on the systems we have studied (known as $(\star)$ quadratic systems in the work):

1. If a $(\star)$ quadratic system has a polynomial inverse integrating factor and a rational first integral, then the critical remarkable invariant algebraic curves are contained into the set $\{V = 0\}$.
2. The $(\star)$ quadratic systems have no algebraic limit cycles.
3. There are 122 different phase portraits for these kind of quadratic systems.

4. In the most of the phase portraits, the set $\{V = 0\}$ contains all or most of all the finite separatrices of the system.

5. In some cases where $\{V = 0\}$ does not contain any separatrix, some limit cycles can bifurcate from a perturbation.

The second part. We present the following three articles:

1. A. Ferragut, J. Llibre and A. Mahdi, *Polynomial inverse integrating factors for polynomial vector fields*, to appear in Discrete and Continuous Dynamical Systems.

2. A. Ferragut, J. Llibre and M.A. Teixeira, *Periodic orbits for a class of C^1 three–dimensional systems*, submitted.

3. A. Ferragut, J. Llibre and M.A. Teixeira, *Hyperbolic periodic orbits coming from the bifurcation of a 4-dimensional non-linear center*, to appear in Int. J. Of Bifurcation and Chaos.

In the first article we give three main results. First we prove that a polynomial vector field having a polynomial first integral must have a polynomial inverse integrating factor. The second one is an example of a polynomial vector field having a rational first integral and having neither polynomial first integral nor polynomial inverse integrating factor. It was an open problem to know if there exist polynomial vector fields verifying these conditions. The third one is an example of a polynomial vector field having a center and not having a polynomial inverse integrating factor. An example of this type was expected but unknown in the literature.

In the second article we study reversible polynomial vector fields of degree four in $\mathbb{R}^3$ which have, under certain generic conditions, an arbitrary number of hyperbolic periodic orbits. Without these conditions, they have an arbitrary number of periodic orbits.

Finally, in the third article, we study the perturbation of a center in $\mathbb{R}^4$ which comes from a problem of physics. By the first order averaging theory and perturbing inside the polynomial vector fields of degree four, the perturbed system may have at most sixteen hyperbolic periodic orbits bifurcating from the periodic orbits of the center.