Polynomial inverse integrating factors of quadratic differential systems and other results

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Certifiquem que aquesta memòria ha estat realitzada per Antoni Manel Ferragut i Amengual sota la nostra supervisió i que constitueix la seva Tesi per a aspirar al grau de Doctor en Matemàtiques per la Universitat Autònoma de Barcelona

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A vosaltres tres, que sou i sereu

Els agraïments a una tesi poden esdevenir una part molt difícil d'escriure. Has d'incloure-hi tota la gent que ha passat per la teva vida durant la teva particular Odissea, però sense arribar al punt de donar públicament les gràcies al senyor que te va arreglar la cadena de la bicicleta tres vegades al Brasil. A més, has de mirar de no ser massa extens, que ja tendran temps d'avorrir-se passada la introducció. Fins i tot, has de tenir en compte un cert ordre d'importància. Finalment, i això és primordial, has de fer una gracieta o dues, i sempre, sempre, has d'entendrir els cors que ho necessitin o t'ho demanin.

Aquest topall de porta en potència que teniu a les mans (eventualment, aquests kilobytes a la memòria del vostre computador) és el resultat de $n \in \mathbb{N}$ anys de fruir de la meva passió. Durant aquest temps, ha estat molta la gent que, en petita o gran mesura, m'ha envoltat i acompanyat. Qualcuns els he conegut pel camí, d'altres els he anat perdent a mesura que caminava, els més pacients romanen. A tots vull agrair-los les rialles i les llàgrimes. Prenc llicència per fer-ho així com vull:

Als llaüts de fusta, als olis sobre tela, a les baquetes del 5B i en Manolo, a les orelles socarrimades, al viatge a la París del 76, a l'humit cove de roba bruta, a l'àtic de davant l'escorxador, a la casa amb els moixos, a en Clovis,

a la piscina del *FunCamp* i el mussol que s'hi fixa, a les taronges de la Vall i la somrient marina,

al mirador del Castellet, sa Pastanaga, el metro de *Namesti Republiky*, *as Catedráis*, na Clara, el cel tan i tan i tan obert, els dits sobre l'esquena,

a tots els parcs del món, el mussol de pedra, la *sopaipilla* i el parlar tan intens dels ulls, al Nepal, l'antifaç i els taps, al Samoa, la *crêperie* i l'animat sucre roig, a M. Pons Justo i l'aigua de València, mal anomenada Espanya, a les magdalenes amb xocolata i mugró d'avellana, a les estovalles de quadres, al te,

al *Pacharán* amb metxes vermelles, a les gambes i la confitura de taronja, als Hostalets, al tercers padrins,

als cosmopolites i els punts de la gelera, a la plaça del *guei*, als voltors i les ratetes, a l'edredó individual sobre llit doble, a les *Micheladas*,

a les persianes que queden per pintar, el Risk, el Truc i els espaguetis,

a *Il Groto*, Palace, el Lórien, la Iguana, l'Acros, els tres atxems, la e oberta,

al Bayleys *borasho*, a l'Hole! i el deliciós cargol a la llauna, al *xiringuito* damunt l'arena, als 6×3 i els *To my parents*, al *piti*, en Kitty i en Puces, a la bruixa sense ceba ni formatge, a la paella de Nadal, *aan de mooiste glimlach van de wereld*, a la terrorista de les paraules, *não pode ser um video!*, al carrer de santa Clara, a les girafes que besen el cel, als països tropicals, flors i dones naturals,

a totes les sigles que siguin necessàries...

Al port del Canonge. A la Mar. A la Terra...

La resta és cosmètica.

Qui t'ha parit que t'entengui

Contents

Introduction

Differential equations are mainly used to describe the change of quantities or the behavior of certain systems in the time, such as those governed by Newton's laws in physics. Usually, explicit solutions of the differential equations cannot be found, so we must look for other methods. One approach is to use numerical approximations. However, in most applications, for example in physics, some of the most interesting questions are related to the so-called qualitative properties. If these questions can be answered without solving the differential equations, especially when explicit solutions are unavailable, we can still get a very good understanding of the system.

It is important to learn how to analyze some qualitative properties, such as the existence and uniqueness of solutions, the phase portraits analysis, the dynamics, the stability or the bifurcations of their orbits, the existence of periodic orbits,... of differential equations without solving them explicitly or numerically. Based on these remarks, we conclude that in order to have a better knowledge of the differential equations, without solving them explicitly or numerically, we should use the so called *qualitative theory of differential equations*.

The first part of the work

First integrals. One of the main aims in the qualitative theory of planar differential systems is the existence of a first integral. Given a planar differential system

$$\frac{dx}{dt} = P(x,y), \qquad \frac{dy}{dt} = Q(x,y), \tag{1}$$

where P, Q are real analytic functions in the variables x and y, a non–constant function H defined in an open domain $U \subseteq \mathbb{R}^2$ is a *first integral* of (1) on U if it is constant on all the solutions of the system contained in U. If $H \in \mathcal{C}^1$, this is equivalent to satisfy the equation

$$\dot{H} = P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0 \tag{2}$$

on U.

A system (1) is *Hamiltonian* if there exists a first integral H such that $P = -H_y$ and $Q = H_x$. If $H \in \mathcal{C}^2$, then this is equivalent to the equality

$$\operatorname{div}(P,Q) = P_x + Q_y = -H_{yx} + H_{xy} = 0.$$

The importance of the existence of a first integral is in its level sets. The existence of such a function H on U determines the phase portrait of system (1) on U, because the level sets $\{H(x,y) = h\} \subseteq H(U) \subseteq \mathbb{R}$ contain the orbits of system (1) on U. Consequently, given a system (1), it is important to know if it has a first integral.

In the study of the *polynomial differential systems of degree* $m \in \mathbb{N}$, that is, when P and Q are polynomials and the maximum of the degrees of P and Q is m, one important family of first integrals is the Darboux one. This kind of functions can be defined using invariant algebraic curves and exponential factors.

The curve $f = 0$ is an *invariant algebraic curve* of a polynomial system (1) of degree m if f is a polynomial of $\mathbb{C}[x,y]$ and it is a solution of the equation

$$P\,\frac{\partial f}{\partial x} + Q\,\frac{\partial f}{\partial y} = K\,f, \tag{3}$$

where K is a polynomial of degree lower than m called the *cofactor* of the invariant algebraic curve. An invariant algebraic curve $f = 0$ is *irreducible* if f is irreducible in $\mathbb{C}[x,y]$.

Let g, h be complex polynomials. An *exponential factor* $F = \exp(g/h)$ of a polynomial system (1) of degree m is a solution of the equation

$$P\,\frac{\partial F}{\partial x} + Q\,\frac{\partial F}{\partial y} = L\,F, \tag{4}$$

where L is a polynomial of degree lower than m called the *cofactor* of the exponential factor.

A function H is called *Darboux* if it can be written into the form

$$H = f_1^{\lambda_1} \cdots f_r^{\lambda_r}\, e^{\mu_1 g_1/h_1} \cdots e^{\mu_s g_s/h_s}, \tag{5}$$

where $f_i, g_j, h_j \in \mathbb{C}[x,y]$ and $\lambda_i, \mu_j \in \mathbb{C}$ for $i = 1, \ldots, r$, and $j = 1, \ldots, s$.

A very important result due to Darboux (see [22]) gives a relation between the number of invariant algebraic curves and exponential factors and the existence of a *Darboux first integral*, i.e. a first integral given by a Darboux function. A very short version is given here, for a more complete version see Subsection 1.2.4.

Theorem *If the number of irreducible invariant algebraic curves and exponential factors of a polynomial differential system* (1) *of degree* m *is greater than* $\binom{m+1}{2}$*, then there exists a Darboux first integral for* (1)*. Moreover, if this number is greater than* $\binom{m+1}{2}+1$*, then system* (1) *has a rational first integral, and consequently all orbits are contained into invariant algebraic curves.*

In this theorem an explicit expression for H can be given, as a function of the invariant algebraic curves and the exponential factors. So, in order to obtain a first integral, it suffices to find enough functions of those types.

Inverse integrating factors. Another important tool in the study of planar differential systems is the inverse integrating factor. An *inverse integrating factor* is a solution V, defined in an open set $U \subseteq \mathbb{R}^2$, of the partial differential equation

$$\operatorname{div}\left(\frac{P}{V}, \frac{Q}{V}\right) = 0. \tag{6}$$

If V satisfies this equation, then the system $\dot{x} = P/V$, $\dot{y} = Q/V$, equivalent to (1) after a change of time in the domain $U \setminus V^{-1}(0)$, is Hamiltonian. So when a first integral H and an inverse integrating factor V of system (1) satisfy

$$\frac{P}{V} = -\frac{\partial H}{\partial y}, \qquad \frac{Q}{V} = \frac{\partial H}{\partial x}, \tag{7}$$

we say that H is *associated* to V, and vice versa.

Equation (6) is equivalent to the following one:

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V. \tag{8}$$

From (8) it follows that $V^{-1}(0)$ is invariant under the flow. Then it is formed by orbits of system (1). Given an inverse integrating factor V defined in U, we can compute a first integral H in the set $U \setminus V^{-1}(0)$. The flow associated to a Hamiltonian system is area preserving, so we deduce that the set $V^{-1}(0)$ must contain the orbits of U such that the area of a neighborhood of them is not preserved by the flow of system (1). In particular, a very important property of the inverse integrating factors is stated in the following theorem, proved by Giacomini, Llibre and Viano in [31]:

Theorem *Let* V *be an inverse integrating factor of the polynomial system* (1) *defined in an open subset* U *of* $\mathbb{R}^2$*. If* $\gamma \subset U$ *is a limit cycle of system* (1)*, then* γ *is contained in the set* $V^{-1}(0) = \{(x, y) \in U : V(x, y) = 0\}$*.*

From this theorem it follows that the zero set of V contains all the limit cycles laying in the domain of definition of V. An immediate consequence is that as larger the domain of definition U of V is, more limit cycles of system (1) we can determine, if they exist. In particular, if the domain of V is the whole plane $\mathbb{R}^2$, then the zero set of the inverse integrating factor provides all the limit cycles of the phase portrait. This fact follows immediately when V is a polynomial.

If the factorization of V contains powers of polynomials, then these polynomials satisfy equation (3) for certain cofactors, so they are invariant algebraic curves. Thus, the knowledge of an inverse integrating factor may imply the knowledge of invariant algebraic curves. In particular, if the inverse integrating factor is polynomial, then all its factors define invariant algebraic curves.

In general, the domain of definition of an inverse integrating factor V is larger than the domain of definition of a first integral H. Moreover, usually the expression of V is simpler than the expression of H, see for instance [12]. As a particular case, the domain of definition of a polynomial inverse integrating factor is the whole $\mathbb{R}^2$, but the associated first integrals may have a complicated expression and a restricted domain of definition.

There are many families of planar polynomial differential systems having a polynomial inverse integrating factor, some of them very important. A first example is given by the homogeneous systems, i.e. the polynomial systems (1) with P and Q homogeneous of the same degree. A homogeneous system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$ of degree m has a homogeneous polynomial inverse integrating factor of degree $m + 1$ given by $V = yP - xQ$. And as a second example, polynomial differential systems of degree two having a center have a polynomial inverse integrating factor of degree three or five (see [8] and [40]).

All the facts stated above encourage us to study the inverse integrating factors in addition to the first integrals. Both tools will lead to a complete study of the phase portrait of a planar differential system (1).

Once the study of the inverse integrating factors is motivated, the next step is the computation of such functions V. Equation (8), which defines V, may become very difficult to solve, even if we look for polynomial functions V, considering the functions P and Q as polynomials. Then, the partial differential equation (8) reduces to the computation of a set of conditions on the coefficients of a linear system of equations. For a given system we may fix the degree of V and compute a solution V of (8) by solving a linear system. This fact implies that we must know at least a bound of the degree of V. But the difficulty persists if we do not know such a bound, because we should study an arbitrary number of linear systems.

Polynomial inverse integrating factors of quadratic systems. A *real polynomial differential system of degree two* (or simply a *quadratic system*) is a system of the form (1) with P and Q polynomials such that the maximum of their degrees is two. The main aim of the first part of this work is *to classify all the quadratic systems having a polynomial inverse integrating factor*.

We have taken two restrictions in our discussion about the inverse integrating factors. First we have restricted system (1) to be quadratic, and second we have restricted the inverse integrating factor V to be a polynomial. Then equation (8) becomes a polynomial equation and we may find conditions on the coefficients of P and Q in order that the associated linear system has a solution V. Moreover, we may also find the explicit expression of V if such a solution exists.

Although this may seem an easy method, as we said before it is not a good one because we must compute polynomial inverse integrating factors of an unknown degree. So we must look for other methods.

One way to solve (8) is by grouping the monomials of V. Thus we can write V either as a sum of homogeneous polynomials; or as a polynomial in one of its variables; or even we can do both and consider each homogeneous polynomial as a polynomial in one of the variables. This kind of grouping, combined with some other methods, will make easier the study of the solutions of equation (8).

All the methods we have commented must be applied to a system having twelve parameters, so it would be better to find a way to reduce the difficulty of the problem before starting to solve it. For that purpose, we consider a classification of the quadratic systems into ten normal forms given by Gasull, Sheng Li Ren and Llibre. In [29] they proved that the quadratic systems can be divided into ten (not disjoint) families of quadratic systems. The expression of $P(x, y)$ has no parameters for each normal form of this classification, so the number of parameters is reduced from twelve to six.

In our work using the methods above mentioned and widely explained in Section 2.3, we classify the quadratic systems having a polynomial inverse integrating factor, giving an explicit expression of such polynomial for almost the ten normal forms. In some particular cases of two normal forms it has been not possible for this moment either to find some of the conditions on the coefficients of the system in order that a polynomial inverse integrating factor exists, or to compute an explicit expression of V. In these cases a method for computing the conditions and the explicit expression of V can be followed for a fixed degree $k \in \mathbb{N}$. We call ($\star$) *quadratic systems* the families of quadratic systems for which we have found explicitly a polynomial inverse integrating factor.

Once this classification is over we want to study the polynomial inverse integrating factors that we have obtained, and deduce from them as many properties as we can. As a first remark, we note that a polynomial inverse integrating factor V may factorize in polynomials of lower degree. Therefore as in our classification we will obtain polynomial inverse integrating factors of arbitrary high degree, it follows that probably we shall obtain invariant algebraic curves of arbitrary high degree. This will be the case.

Darboux first integrals. From the classification of quadratic systems having a polynomial inverse integrating factor we compute, using (7), the first integrals associated to $(\star)$ quadratic systems.

A proof of the next result can be found in [10].

Proposition *If a polynomial differential system has a rational inverse integrating factor V, then it has a Darboux first integral.*

This proposition can be applied to the $(\star)$ quadratic systems, so the first integrals that we obtain for such systems are Darboux functions. Then we classify the $(\star)$ quadratic systems in three types. The systems which have a polynomial first integral, the systems which have a rational first integral and do not have a polynomial first integral, and the systems which have a Darboux first integral and do not have a rational first integral. The first part of this classification is related with the work by Chavarriga, García, Llibre, Pérez del Río and Rodríguez [9], where the quadratic systems having a polynomial first integral are classified. This fact is due to the following result proved in Chapter 4.

Theorem *If a polynomial differential system* (1) *has a polynomial first integral, then it has a polynomial inverse integrating factor.*

Critical remarkable values. The cases in which the system has a rational first integral demand a larger study. Writing a rational first integral H as a quotient of polynomials, $H = f/g$, the orbits of the system must be contained into the algebraic curves $f + cg = 0$ with $c \in \mathbb{R} \cup \{\infty\}$. If $f + cg$ factorizes in $\mathbb{C}[x, y]$, then c is a *remarkable* value. We note that when $c = \infty$, $f + cg$ means g.

The notion of remarkable values is due to Poincaré (see [42]), and it has not been used after Poincaré with the exception of these last years. In an article due to Chavarriga, Giacomini, Giné and Llibre (see [13]) the following result is proved.

Proposition *A rational first integral of a polynomial differential system has finitely many remarkable values.*

We do not compute all the remarkable values of all the rational first integrals, but we compute a particular subset of them, which will give important information on the qualitative behavior of the system. For a given remarkable value $c \in \mathbb{R} \cup \{\infty\}$, let $u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ be the factorization of $f + cg$ into irreducible factors of $\mathbb{C}[x, y]$. If at least one of the values α_i, $i = 1, \ldots, r$, is larger than 1, then c is a *critical* remarkable value. The corresponding curve $u_i = 0$ is a *critical remarkable* invariant algebraic curve of system (1) with *exponent* α_i.

Next proposition, also appearing in [13], shows how many critical remarkable values there are for a polynomial differential system (1) having a rational first integral and a polynomial inverse integrating factor.

Proposition *Suppose that the polynomial differential system* (1) *has a rational first integral H. Then it has a polynomial inverse integrating factor if and only if H has at most two critical remarkable values.*

We compute all critical remarkable values associated to the rational first integrals of the ($\star$) quadratic systems, and also their critical remarkable invariant algebraic curves. From this classification, we obtain the following result, and we see that there are ($\star$) quadratic systems having a rational first integral with 0, 1 or 2 critical remarkable values.

Theorem *Suppose that a* ($\star$) *quadratic system has a polynomial inverse integrating factor V and a rational first integral H. Then, the critical remarkable invariant algebraic curves associated to H are contained in the zero set of V.*

Algebraic limit cycles. As the inverse integrating factors V that we have classified are polynomial, the set $V^{-1}(0)$ contains all the limit cycles of the system, if they exist. Moreover, if there are limit cycles they are *algebraic*, because they are contained into invariant algebraic curves. The following result, due to Llibre and Rodríguez [39], shows the importance of the algebraic limit cycles.

Theorem *Any finite configuration of limit cycles is realizable by algebraic limit cycles for a convenient polynomial differential system.*

As far as we know seven different families of algebraic limit cycles for quadratic systems have been found (see for instance [38]), and from the results of [14] it follows that the corresponding systems do not have a Darboux first integral, and then they do not have a polynomial inverse integrating factor. So these algebraic limit cycles cannot appear in our classification. Moreover, from the expressions of V obtained, we can prove the following result.

Theorem *A* ($\star$) *quadratic system has no algebraic limit cycles.*

Phase portraits. Once the quadratic systems having a polynomial inverse integrating factor are classified, we do the classification of their phase portraits.

We obtain 121 non–topologically equivalent phase portraits from the ($\star$) quadratic systems, and we show them in Section 3.4. In fact in Section 3.4 122 phase portraits appear. The one which for the moment does not appear as a ($\star$) quadratic system is the phase portrait **(92)**. This phase portrait is obtained from systems (I.13) and (I.15), for which we have not proved the existence of a polynomial inverse integrating factor. But some examples of quadratic systems (I.13) and (I.15) having a polynomial inverse integrating factor of degree 6 have been obtained, providing the phase portrait **(92)**. We have numerical evidence that there are no more phase portraits fot the quadratic systems having a polynomial inverse integrating factor than those 122.

The main conclusion from these phase portraits is that the zero set of V provides in the major part of the cases the "skeleton" of the system. That is, it contains all or part of the finite separatrices of the global phase portrait. This is another important property of the inverse integrating factors.

Another conclusion from the study of these 122 phase portrait is that all of them are realizable by a quadratic system having a polynomial inverse integrating factor of degree $k \leq 6$.

Examples. In the classification we find lots of examples of quadratic systems that appeared previously in the literature. We have all the homogeneous quadratic systems (see [51]); the Hamiltonian quadratic systems, see [4]; the quadratic systems having a rational first integral of degree two (see [7]); the quadratic systems having a center (see [48]); the most interesting quadratic foliation (a quadratic system without finite singular points) having three inseparable leaves (the maximum number of inseparable leaves that a quadratic system can exhibit) (see [29]); and the quadratic systems having a polynomial first integral (see [9]).

As a particular example we have obtained a quadratic system having a polynomial inverse integrating factor in which the set $V^{-1}(0)$ contains a closed orbit of a center, see the phase portrait **(16)**. We think that a perturbation of this system might make appear at least one limit cycle from this closed orbit. We will study this in the next future.

Another interesting case comes from the phase portrait **(2)**, for which $V^{-1}(0)$ contains an orbit γ going from infinity to infinity. Since for such systems the infinite line is fulfilled of singular points, the orbit γ plus an arc of infinity forms a very degenerated graphic. We believe that perturbing such graphic inside the class of all quadratic systems we will get a limit cycle bifurcating from this graphic. Again this study will be done in the next future.

The second part of the work

Three articles. In this second part of the work, we present the following three articles, in which the author has collaborated:

A. Ferragut, J. Llibre and A. Mahdi, *Polynomial inverse integrating factors for polynomial vector fields*, to appear in Discrete and Continuous Dynamical Systems.

A. Ferragut, J. Llibre and M.A. Teixeira, *Periodic orbits for a class of $\mathcal{C}^1$ three–dimensional systems*, submitted.

A. Ferragut, J. Llibre and M.A. Teixeira, *Hyperbolic periodic orbits coming from the bifurcation of a 4–dimensional non–linear center*, to appear in International Journal of Bifurcations and Chaos.

Polynomial inverse integrating factors for polynomial vector fields. In this article we give some results about the existence and non–existence of polynomial inverse integrating factors for planar polynomial vector fields. The following result summarizes some relations between the first integrals and the inverse integrating factors for a polynomial vector field in $\mathbb{C}^2$.

Theorem *Let X be a polynomial vector field in $\mathbb{C}^2$.*

(a) If X has a Liouvillian first integral, then it has a Darboux inverse integrating factor.

(b) If X has a Darboux first integral, then it has a rational inverse integrating factor.

(c) If X has a polynomial first integral, then it has a polynomial inverse integrating factor.

We note that in statements (a) and (b) of this theorem the expression of the integrating factor is easier than the expression of the first integral. Looking at the previous theorem a natural question is: if X has a rational first integral, then does X have a polynomial inverse integrating factor? The next proposition is an example of a polynomial vector field which has a rational first integral and has neither a polynomial first integral, nor a polynomial inverse integrating factor.

Proposition *The polynomial vector field*

$$X = 2x(5 + 30x + 40x^2 + 8y^2)\frac{\partial}{\partial x} + y(5 + 44x + 80x^2 + 16y^2)\frac{\partial}{\partial y},$$

has a rational first integral, and has neither a polynomial first integral, nor a polynomial inverse integrating factor.

We prove this proposition finding three critical remarkable values for the rational first integral of the system.

We also give an example of a polynomial vector field having a center and no polynomial inverse integrating factors. This is a result that all the mathematicians working in the area believed, but as far as we know it has not been proved before.

Proposition *The polynomial vector field*

$$X(x,y) = y^3\frac{\partial}{\partial x} - \frac{1}{2}x^2(2x-y^2)\frac{\partial}{\partial y}\ ,$$

has a center and has no polynomial inverse integrating factors.

Finally, we present the following question.

Open Question. *Assume that X is a polynomial vector field having a center. How to characterize if X has a polynomial inverse integrating factor?*

Periodic orbits for a class of $\mathcal{C}^1$ three–dimensional systems. In this second work, we deal with the polynomial differential system of degree 4 in $\mathbb{R}^3$

$$\dot{x} = (y^2+z^2)(1-y^2-z^2), \quad \dot{y} = -z + xy, \quad \dot{z} = y + xz,$$

or equivalently (taking $y = r\cos t$ and $z = r\sin t$)

$$\dot{x} = r^2(1-r^2), \quad \dot{r} = xr, \quad \dot{\theta} = 1,$$

where $x, r \in \mathbb{R}$, $r \geq 0$, and $\theta \in \mathbb{S}^1$. The dot means derivative with respect to the time $t \in \mathbb{R}$. We restrict the system to the set $\overline{D}_3 = H^{-1}([0,1])$, where $H(x,r,\theta) = -2x^2 + 2r^2 - r^4$ is a first integral of the system.

We perturb this system inside a class of $\mathcal{C}^1$ reversible systems. If the perturbation is strongly reversible (that is, the reversible perturbations do not depend on the angle), then the angle θ can be treated as the independent variable to reduce the analysis of the system to a two–dimensional system. Under these assumptions we prove that the dynamics of the perturbed system do not change. If the perturbation is non–strongly reversible, then we show the existence of an arbitrary number of symmetric periodic orbits.

Additionally, we provide a perturbation by a polynomial vector field of degree 4 which has infinitely many limit cycles if a generic assumption is satisfied.

Hyperbolic periodic orbits coming from the bifurcation of a 4–dimensional non–linear center. In this third article, we deal with the polynomial system in $\mathbb{R}^4$

$$\dot{x} = u\,(p + q\,x)^3, \quad \dot{u} = -x, \quad \dot{y} = v\,(p + q\,x)^3, \quad \dot{v} = -y, \tag{9}$$

where $p, q \in \mathbb{R}$. We first prove that the system has a center at the origin if and only if $p > 0$. Next we show the following result.

Theorem *Suppose that $p > 0$. Let g_n and h_n be polynomials of degree n in the variables x, u, y, v, for $n = 2, 4$. Then the system*

$$\begin{aligned}
&\dot{x} = u\,(p + q\,x)^3,\\
&\dot{u} = -x(1 + \varepsilon^2\,g_2(x, u, y, v) + \varepsilon\,g_4(x, u, y, v)),\\
&\dot{y} = v\,(p + q\,x)^3,\\
&\dot{v} = -y(1 + \varepsilon^2\,h_2(x, u, y, v) + \varepsilon\,h_4(x, u, y, v)),
\end{aligned}$$

can have at most 16 hyperbolic periodic orbits bifurcating from the periodic orbits of the center of system (9) for ε sufficiently small using the first order averaging method. Moreover, there are examples of this system having exactly $0, 1, \ldots, 16$ hyperbolic periodic orbits.

The structure of the work. The first part of the work is presented in the Chapters 1 to 3. In Chapter 1 we introduce all the definitions and main results that we will use in the first part, such as first integrals, inverse integrating factors, Darboux theory of integrability and remarkable values. Chapter 2, the main chapter of this first part, contains the classification of quadratic systems into ten normal forms and the classification of quadratic systems having a polynomial inverse integrating factor for each normal form. In Chapter 3 we present the properties which have the quadratic systems having a polynomial inverse integrating factor, giving their phase portraits, and we state some conclusions.

The second part of the work presents the three articles mentioned above. It is formed by Chapters 4, 5 and 6.

Part I

Polynomial inverse integrating factors of quadratic differential systems

Chapter 1

Some preliminary results

In this chapter we present the main definitions and some results on the algebraic theory of planar polynomial differential systems. Although most of them are applied also to non–polynomial systems and/or to complex systems, we restrict this short overview to real polynomial systems.

A *real planar polynomial differential system* of degree m is a system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1.1}$$

where P, Q are real polynomials, $m = \max\{\deg P, \deg Q\}$ and the dot denotes the derivative with respect to the independent variable t. We denote by $X = (P, Q)$ the *vector field* associated to system (1.1) and by

$$\mathbf{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y} \tag{1.2}$$

the linear operator associated to (1.1).

1.1 First integrals. Integrating factors

Let $U \subseteq \mathbb{R}^2$ be an open set. A C^k function $H : U \to \mathbb{R}$, with $k = 0, 1, \ldots, \infty, \omega$, is a *first integral* of system (1.1) in U if H is constant on each solution of this system and H is non–constant on any open subset of U. If $k \geq 1$, then the definition is equivalent to the equality $\mathbf{X}H = 0$ on U.

Example 1.1.1. *The polynomial system*

$$\dot{x} = -y - b(x^2 + y^2), \quad \dot{y} = x, \tag{1.3}$$

$b \in \mathbb{R}$, *has the first integral*

$$H(x, y) = e^{2by}(x^2 + y^2). \tag{1.4}$$

We note that, once we have a first integral, any function of this first integral is also a first integral.

A non–constant, analytic function $R : U \to \mathbb{R}$ is an *integrating factor* of system (1.1) if one of the following three equivalent conditions holds:

1. $\operatorname{div}(RP, RQ) = 0$,
2. $\partial(RP)/\partial x + \partial(RQ)/\partial y = 0$,
3. $\mathbf{X}R + R \operatorname{div}(P, Q) = 0$,

or in an equivalent way

$$P\frac{\partial R}{\partial x} + Q\frac{\partial R}{\partial y} = -\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) R.$$

If R is an integrating factor of system (1.1), then by the change of the independent variable $dt = R(x, y)\, ds$ we obtain the equivalent system

$$x' = R(x, y)P(x, y), \qquad y' = R(x, y)Q(x, y), \tag{1.5}$$

where the prime means derivative with respect to s. The function

$$\begin{aligned} H(x, y) \;=\; & -\int R(x, y)P(x, y)\, dy \\ & + \int \left(R(x, y)Q(x, y) + \frac{\partial}{\partial x}\int R(x, y)P(x, y)\, dy \right) dx, \end{aligned}$$

which is a solution of the system

$$RP = -\frac{\partial H}{\partial y}, \quad RQ = \frac{\partial H}{\partial x}, \tag{1.6}$$

is a first integral of system (1.5) (and, consequently of system (1.1)). Indeed, if Y is the vector field associated to system (1.5) and $\mathbf{Y}$ is its associated linear operator, then

$$\mathbf{Y}H = RP\frac{\partial H}{\partial x} + RQ\frac{\partial H}{\partial y} = R\,\mathbf{X}H = 0.$$

Conversely, and using (1.6), given a first integral H of the vector field X, we can always find an integrating factor R for which (1.6) holds.

Example 1.1.2. *System* (1.3) *has the integrating factor* $R(x, y) = e^{2by}$. *From it we can obtain the first integral* (1.4), *and vice versa.*

Proposition 1.1.3. **(1)** *If system* (1.1) *has two integrating factors* R_1 *and* R_2 *defined in* U*, then the functions* R_1/R_2*, which is defined in* $U \setminus \{R_2 = 0\}$*, and* $R_1R_2/(R_1^2 + R_2^2)$*, which is defined in* $U \setminus (\{R_1 = 0\} \cap \{R_2 = 0\})$*, are first integrals of* (1.1)*.*

(2) *If system* (1.1) *has an integrating factor* R *and a first integral* H*, both defined in* U*, then the function* RH *is another integrating factor defined in* U*.*

Proof: It follows easily from straightforward computations. ∎

1.2 Darboux theory of integrability

In this section we study the existence of first integrals of planar polynomial vector fields through the Darboux theory of integrability. The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert 16^{th} problem, see [36]. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial vector fields and the number of invariant algebraic curves that they have (see [22] and [43]).

Darboux [22] showed how first integrals of planar vector fields having enough invariant algebraic curves can be constructed. In particular, in his work it is proved that if a planar polynomial vector field of degree m has at least $m(m+1)/2$ invariant algebraic curves, then it has a first integral, which can be computed using these invariant algebraic curves. Jouanolou [37] showed that if the number of invariant algebraic curves of a planar polynomial vector field of degree m is at least $m(m+1)/2 + 2$, then the vector field has a rational first integral, and consequently all its solutions are invariant algebraic curves.

1.2.1 Invariant algebraic curves

Let $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$, be an algebraic curve of system (1.1). We say that $f = 0$, or simply f, is *invariant* if $\mathbf{X}f/f = K \in \mathbb{C}[x, y]$. In this case, K is called the *cofactor* of f and it has degree at most $m - 1$. The expression which defines K is often written as

$$\frac{\partial f}{\partial x}P + \frac{\partial f}{\partial y}Q = K\, f.$$

We remark that in the definition of invariant algebraic curve we allow the curve $f = 0$ to be complex. This is due to the fact that sometimes for real vector fields the existence of a real first integral can be forced by the existence of complex invariant algebraic curves.

An *irreducible* invariant algebraic curve $f = 0$ is an invariant algebraic curve such that f is an irreducible polynomial in $\mathbb{C}[x, y]$.

Since the gradient of the polynomial f at the points (x, y) such that $f(x, y) = 0$ is orthogonal to the vector field X, this vector field is tangent to the curve $f = 0$. Hence, the curve $f = 0$ is formed by trajectories of X. A solution of (1.1) either has empty intersection with the zero set of f or is contained in it.

Example 1.2.1. *System* (1.3) *has the complex irreducible invariant algebraic curves $f_1 = x + iy = 0$ and $f_2 = x - iy = 0$; or, equivalently, the real invariant algebraic curve $x^2 + y^2 = 0$.*

We state some properties of invariant algebraic curves.

Proposition 1.2.2. **(1)** *If f is a complex polynomial, then $\bar{f}$ denotes the complex polynomial obtained from f by conjugating all its coefficients. The curve $f = 0$ is an invariant algebraic curve of system* (1.1) *with cofactor K if and only if $\bar{f} = 0$ is an invariant algebraic curve of system* (1.1) *with cofactor $\bar{K}$.*

(2) *Let $n_1, \ldots, n_r \in \mathbb{N}$ and $f_1, \ldots, f_r \in \mathbb{C}[x, y]$. Set $f = f_1^{n_1} \cdots f_r^{n_r}$. Then, $f = 0$ is an invariant algebraic curve with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic curve with cofactor K_{f_i} for all $i \in \{1, \ldots, r\}$. Moreover, the equality $K_f = n_1 K_{f_1} + \cdots + n_r K_{f_r}$ holds.*

(3) *If system* (1.1) *has an integrating factor $R = f_1^{n_1} \cdots f_p^{n_p}$, with $f_i \in \mathbb{C}[x, y]$ and $n_i \in \mathbb{C} \setminus \{0\}$ for all i, $(f_i, f_j) = 1$ if $i \neq j$, then $f_i = 0$ is an invariant algebraic curve of* (1.1) *for all i.*

1.2.2 Exponential factors

In this section we introduce the notion of exponential factors, due to Christopher [18]. An exponential factor appears when an invariant algebraic curve has geometric multiplicity greater than one. The exponential factors play the same role than invariant algebraic curves in order to obtain a first integral for the polynomial system. For more details on exponential factors than the ones given in this section, see [20].

Let $g, h \in \mathbb{C}[x, y]$ be relatively prime polynomials. The function $F = e^{g/h}$ is an *exponential factor* of system (1.1) if $\mathbf{X}F/F = L \in \mathbb{C}[x, y]$. In this case, L is called the *cofactor* of F. It has degree at most $m - 1$. The expression which defines L is often written as

$$P \frac{\partial e^{g/h}}{\partial x} + Q \frac{\partial e^{g/h}}{\partial y} = L\, e^{g/h}. \tag{1.7}$$

Proposition 1.2.3. **(1)** *The function* $e^{g/h}$ *is a complex exponential factor of* (1.1) *with cofactor* L *if and only if* $e^{\bar{g}/\bar{h}} = 0$ *is an exponential factor of system* (1.1) *with complex cofactor* $\bar{L}$.

(2) *Let* $F = e^{g/h}$ *be an exponential factor of system* (1.1) *with cofactor* L. *Then,* $h = 0$ *is an invariant algebraic curve with cofactor* K. *Moreover,*

$$\mathbf{X}g = gK + hL.$$

We remark that the exponential factors of the form $e^{g/h}$ with h constant appear when the straight line at infinity is a solution with multiplicity higher than one for the projectivized version of the vector field.

1.2.3 Independent singular points

We identify the linear vector space $\mathbb{C}_{m-1}[x, y]$ of all complex polynomials in the variables x and y of degree at most $m-1$ with $\mathbb{C}^{m(m+1)/2}$ through the isomorphism

$$\sum_{i+j=0}^{m-1} a_{ij}x^i y^j \leftrightarrow (a_{00}, a_{10}, a_{01}, \ldots, a_{m-1,0}, a_{m-2,1}, \ldots, a_{0,m-1}).$$

We say that r points $(x_k, y_k) \in \mathbb{C}^2$, $k = 1, \ldots, r$, are *independent* with respect to $\mathbb{C}_{m-1}[x, y]$ if the intersection of the r hyperplanes

$$\left\{(a_{ij}) \in \mathbb{C}^{m(m+1)/2} \ : \ \sum_{i+j=0}^{m-1} a_{ij}x_k^i y_k^j = 0, \quad k = 1, \ldots, r\right\},$$

is a linear subspace of $\mathbb{C}^{m(m+1)/2}$ of dimension $m(m+1)/2 - r > 0$.

We note that the maximum number of isolated singular points of the polynomial system (1.1) is m^2, and also that the maximum number of independent isolated singular points of the system can be $m(m+1)/2 - 1$. We remark that $m(m+1)/2 < m^2$ for $m \geq 2$.

A singular point (x_0, y_0) of system (1.1) is *weak* if $\operatorname{div}(P, Q)(x_0, y_0) = 0$.

1.2.4 The Darboux Theorem

The following theorem improves Darboux's theorem (see [22]), essentially because here exponential factors (see [18]) and independent singular points (see [17]) are taken into account, in addition to invariant algebraic curves (see [5], [6]).

Theorem 1.2.4. *Suppose that a polynomial system* (1.1) *of degree m admits*

(a) *p irreducible invariant algebraic curves $f_i = 0$ with respective cofactor K_i, $i = 1, \dots, p$;*

(b) *q exponential factors $F_j = e^{g_j/h_j}$ with respective cofactor L_j, $j = 1, \dots, q$;*

(c) *r independent singular points $(x_k, y_k) \in \mathbb{C}^2$ such that $f_i(x_k, y_k) \neq 0$, for $i = 1, \dots, p$ and $k = 1, \dots, r$.*

Then:

(1) *There exist $\lambda_i, \mu_j \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q)$ if and only if the (multi–valued) function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} \tag{1.8}$$

is an integrating factor of system (1.1)*. If the system is real, then* (1.8) *is real.*

(2) *If $p+q+r = \frac{m(m+1)}{2}$ and the r independent singular points are weak, then the function* (1.8)*, for convenient $\lambda_i, \mu_j \in \mathbb{C}$ not all zero, is a first integral of* (1.1) *if $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ or an integrating factor of* (1.1) *if $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q)$.*

(3) *There exist $\lambda_i, \mu_i \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$ if and only if the function* (1.8) *is a first integral of system* (1.1)*. If the system is real, then* (1.8) *is real.*

(4) *If $p + q + r = \frac{m(m+1)}{2} + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$, not all zero, such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$.*

(5) *If $p + q + r \geq \frac{m(m+1)}{2} + 2$, then system* (1.1) *has a rational first integral, and consequently all orbits are contained into invariant algebraic curves.*

A (multi–valued) function of the form (1.8) is called a *Darboux function.* If a polynomial system (1.1) has a first integral given by a function (1.8), we say that system (1.1) has a *Darboux* first integral, and if a polynomial system (1.1) has an integrating factor given by a function (1.8), we say that system (1.1) has a *Liouvillian* first integral. Roughly speaking, Liouvillian functions are those that can be expressed as composition of elementary functions, for more details see [45].

If among the invariant algebraic curves of system (1.1) a complex conjugate pair $f = 0$ and $\bar{f} = 0$ occurs, then the first integral (1.8) has a factor of the form $f^\lambda \bar{f}^{\bar{\lambda}}$, $\lambda \in \mathbb{C} \setminus \{0\}$, which is the real multi–valued function

$$f^\lambda \bar{f}^{\bar{\lambda}} = \left[(\operatorname{Re} f)^2 + (\operatorname{Im} f)^2\right]^{\operatorname{Re}\lambda} \exp\left\{-2 \operatorname{Im}\lambda \arctan \frac{\operatorname{Im} f}{\operatorname{Re} f}\right\}. \tag{1.9}$$

If among the exponential factors of system (1.1) a complex conjugate pair $e^{g/h}$ and $e^{\bar{g}/\bar{h}}$ occurs, then the first integral (1.8) has a factor of the form $\left(e^{g/h}\right)^\mu \left(e^{\bar{g}/\bar{h}}\right)^{\bar{\mu}}$, $\mu \in \mathbb{C} \setminus \{0\}$, which is the real function $e^{2\operatorname{Re}(\mu g/h)}$.

1.3 Limit cycles

A *closed* or *periodic solution* of system (1.1) is a solution $(x(t), y(t))$ of system (1.1) for which there exists $0 < T < \infty$ such that $x(t) = x(t+T)$ and $y(t) = y(t+T)$, for all $t \in \mathbb{R}$. A closed orbit having a neighborhood in which there are no other closed orbits is called a *limit cycle*. The behavior of the orbits in the neighborhood of a limit cycle is described in the following theorem (see [1] for a proof).

Theorem 1.3.1. *Let L_0 be a limit cycle of a planar differential system. Then, all orbits through points outside L_0 and sufficiently close to L_0 tend to L_0 either as $t \to +\infty$ or as $t \to -\infty$. The same happens to the orbits inside L_0 and sufficiently close to L_0.*

1.4 Inverse integrating factors

The inverse integrating factors are the most important tool in this first part of the work, so we define them for general planar differential systems. Consider the planar differential system

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1.10}$$

where P and Q are $\mathcal{C}^2$–functions in the variables x and y. Let X be its associated vector field and let

$$\mathbf{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be its associated linear operator. Let U be the domain of definition of system (1.10), and let W be an open subset of U. A non–zero $\mathcal{C}^1$ function $V : W \to \mathbb{R}$ is an *inverse integrating factor* of system (1.10) on W if it is a solution of the linear partial differential equation

$$P\,\frac{\partial V}{\partial x} + Q\,\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V, \tag{1.11}$$

also written $\mathbf{X}V = V \operatorname{div} X$. As we deduce from this equation, the gradient $(\partial V/\partial x, \partial V/\partial y)$ of the set of curves $V^{-1}(0)$ is orthogonal to the vector field X. So X is tangent to $\{V = 0\}$, and then this curve is formed by trajectories of X. Moreover, $V^{-1}(0)$ is an invariant algebraic curve of (1.10) with cofactor $\operatorname{div} X$.

Proposition 1.4.1. *Let V be an inverse integrating factor of system* (1.10) *defined in the open subset $W \subseteq \mathbb{R}^2$. Then,*

(1) The function $1/V$, defined in $W \setminus \{V = 0\}$, is an integrating factor of system (1.10)*. Moreover, the function*

$$H(x,y) = -\int \frac{P(x,y)}{V(x,y)}\,dy + \int \left(\frac{Q(x,y)}{V(x,y)} + \frac{\partial}{\partial x} \int \frac{P(x,y)}{V(x,y)}\,dy \right) dx \tag{1.12}$$

is a first integral of (1.10)*.*

(2) If system (1.10) *has a first integral H, then the function*

$$V_H(x,y) = \frac{P}{-\dfrac{\partial H}{\partial y}} = \frac{Q}{\dfrac{\partial H}{\partial x}}$$

is an inverse integrating factor of (1.10)*. Moreover, the system*

$$\dot{x} = \frac{P}{V_H} = -\frac{\partial H}{\partial y}, \qquad \dot{y} = \frac{Q}{V_H} = \frac{\partial H}{\partial x}, \tag{1.13}$$

is Hamiltonian in $W \setminus \{V = 0\}$.

Proof: The first part of the proposition follows from the computation

$$\mathbf{X}\frac{1}{V} = P\left(\frac{1}{V}\right)_x + Q\left(\frac{1}{V}\right)_y = -\frac{\mathbf{X}V}{V^2} = -\frac{1}{V}\operatorname{div} X.$$

The expression of H can be obtained as in Section 1.1.

To prove the second part, we note that $1/V_H$ is an integrating factor of (1.10), so system (1.13) is Hamiltonian in $W \setminus \{V = 0\}$. ■

Remark 1.4.2. Proposition 1.1.3 can be applied also to inverse integrating factors. □

The following lemma (see [11]) gives a linear property of the inverse integrating factors.

Lemma 1.4.3. *Let $V_1, \ldots, V_p$ be inverse integrating factors of system* (1.10) *and $a_1, \ldots, a_p \in \mathbb{R}$. Then, the function $V = \sum_{i=1}^{p} a_i V_i$ is an inverse integrating factor of system* (1.10).

Example 1.4.4.

1. *A linear differential system $x' = ax + by$, $y' = cx + dy$ has always an easy inverse integrating factor $V(x, y) = cx^2 + (d - a)xy - by^2$ (a quadratic homogeneous polynomial), but the first integrals of this system are more complicated functions than this quadratic homogeneous form.*

2. *If P and Q are homogeneous polynomials of the same degree, then the polynomial $V(x, y) = x\,Q - y\,P$ satisfies equation* (1.11). *This follows using the Euler Theorem for homogeneous functions.*

3. *If P and Q are quadratic polynomials and the origin is a center, then there always exists a polynomial $V : \mathbb{R}^2 \to \mathbb{R}$ of degree* 3 *or* 5 *satisfying equation* (1.11), *see [8] and [40].*

4. *If $P(x, y) = -y + P_3(x, y)$ and $Q(x, y) = x + Q_3(x, y)$, with P_3 and Q_3 homogeneous polynomials of degree* 3, *and the origin is a center, then there always exists a function $V : \mathbb{R}^2 \to \mathbb{R}$ of degree at most* 10 *satisfying equation* (1.11), *see [8].*

In all these previous examples, the inverse integrating factor V is a polynomial of small degree, but the first integrals associated are, in general, more complicated functions. Usually the inverse integrating factor have an easy expression than their associated first integral.

Next theorem, proved in [31], gives an important relation between limit cycles and inverse integrating factors.

Theorem 1.4.5. *Consider system* (1.10) *defined in an open set $U \subseteq \mathbb{R}^2$ and let $V(x, y)$ be a $\mathcal{C}^1$ solution of equation* (1.11) *defined in an open subset W of U. If γ is a limit cycle of system* (1.10) *contained in W, then γ is contained in the set $\{(x, y) \in W : V(x, y) = 0\}$.*

The set $V^{-1}(0)$ contains all the limit cycles of system (1.10) which are in W. This fact allows to study the limit cycles which bifurcate from periodic orbits of a center (Hamiltonian or not) and compute their shape. To do this, we develop the function V in power series of the small perturbation parameter. A remarkable fact is that the first term in this expansion coincides with the first non–identically zero Melnikov function, see [32], [33] and [34].

In short, an inverse integrating factor V is a very important function, and perhaps it is the best way to understand the integrability of a two–dimensional differential system, because

1. V allows to compute a first integral.

2. $V^{-1}(0)$ contains all limit cycles lying in the domain of definition of V.

3. In general, the expression of V is simpler than the expression of the integrating factors and the first integrals associated, and its domain of definition is usually larger.

For more information about existence and uniqueness of analytic inverse integrating factors, see [12].

Theorem 1.4.6 (see [13])**.** *Suppose that the polynomial differential system* (1.10) *of degree m, with P and Q relatively prime, has a Darboux first integral H given by* (1.8)*, with the polynomials f_i and h_j irreducible, the polynomials g_j and h_j relatively prime in $\mathbb{C}[x,y]$ and $\lambda_i, \mu_j \in \mathbb{C} \setminus \{0\}$, $i = 1, \dots, p$, $j = 1, \dots, q$. Then, the function $V_{\log H}$, which is an inverse integrating factor of system* (1.10) *associated to the first integral* $\log H$*, is a rational function. It can be written in the form $u_1^{k_1} \cdots u_r^{k_r}$, with $u_i \in \mathbb{C}[x,y]$, $k_i \in \mathbb{Z}$, where each u_i is an irreducible invariant algebraic curve of system* (1.10)*. Moreover, if system* (1.10) *has no rational first integrals, then $V_{\log H}$ is the unique rational inverse integrating factor of system* (1.10)*.*

The following theorem can be found in [27]. Its third part is proved in there.

Theorem 1.4.7. *Let X be a polynomial vector field in $\mathbb{C}^2$.*

(a) If X has a Liouvillian first integral, then it has a Darboux inverse integrating factor.

(b) If X has a Darboux first integral, then it has a rational inverse integrating factor.

(c) If X has a polynomial first integral, then it has a polynomial inverse integrating factor.

Proof: We prove statement (c) (see [27] again). Let H be a polynomial first integral of X. We note that a polynomial function is a particular case of a Darboux function. Therefore, by statement Theorem 1.4.6, X has a rational inverse integrating factor $V = f/g$, where f and g are coprime polynomials. It is known that the curves $f = 0$ and $g = 0$ are invariant algebraic curves of X, see for instance [13].

Let $g_1^{n_1} \cdots g_r^{n_r}$, with $n_1, \dots, n_r \in \mathbb{N}$, be the factorization of g in irreducible factors in $\mathbb{C}[x,y]$. Then, $g_j = 0$ is an invariant algebraic curve of X for $j = 1, \dots, r$. Let h_j be the value of the first integral H on the points of the irreducible invariant algebraic curve $g_j = 0$. Since g_j is irreducible, g_j divides $H - h_j$. Therefore, there exists a polynomial s_j such that $H - h_j = s_j g_j$.

Since H is a polynomial first integral of X, it follows that the function $K = (H-h_1)^{n_1} \cdots (H-h_r)^{n_r}$ is another polynomial first integral of X. Then, $K\,V = f \prod_{i=1}^{r} s_i^{n_i}$ is a polynomial inverse integrating factor of X. Hence, the statement is proved. ∎

1.5 Rational first integrals. Remarkable values

We introduce in this section some properties of the polynomial differential systems having a rational first integral. The results of this section can be found in [13].

Let H be a polynomial first integral of degree n. We say that the degree of H is *minimal* in the set of the degrees of all the polynomial first integrals of system (1.1) if any other polynomial first integral of system (1.1) has degree at least n.

Let $H = f/g$ be a rational first integral. We say that H has *degree* $n = \max\{\deg f, \deg g\}$. The degree of H is *minimal* in the set of the degrees of all the rational first integrals of system (1.1) if any other rational first integral of (1.1) has degree at least n.

Lemma 1.5.1. *If a polynomial system* (1.1) *has a minimal rational first integral $H = f/g$ which is not a polynomial, then it is not restrictive to assume that f and g are polynomial functions of the same degree and that they are irreducibles.*

Proof: Suppose that $\deg(f) \neq \deg(g)$. Without losing generality, we can assume that $\deg(f) < \deg(g)$. Then, the rational function $\tilde{f}/g = (f + c_1 g)/g$, for a convenient $c_1 \in \mathbb{C}$, is another rational first integral of system (1.1) and $\deg(\tilde{f}) = \deg(g)$.

Now suppose that $\deg(f) = \deg(g)$ and $(f, g) = 1$. If f is not irreducible, then we take the first integral $(f + cg)/g$, for a certain c such that $f + cg$ is irreducible. Now if g is not irreducible, then we take the first integral $(g + d(f + cg))/(f + cg)$, for a certain d such that $g + d(f + cg)$ is irreducible. ∎

Let $H = f/g$ be a minimal rational first integral of a polynomial system (1.1). According to Poincaré [43] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable value* of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$ for $c \in \mathbb{C}$ or if g is reducible in $\mathbb{C}[x, y]$ for $c = \infty$.

Proposition 1.5.2. *Assume that a polynomial differential system* (1.1) *has a first integral H given by expression* (1.8) *which is rational and minimal. Then, H has finitely many remarkable values.*

Suppose that $c \in \mathbb{C}$ is a remarkable value of a rational first integral H and that $u_1^{\alpha_1} \dots u_r^{\alpha_r}$ is the factorization of the polynomial $f + cg$ into irreducible factors in $\mathbb{C}[x, y]$. If some of the α_i, for $i = 1, \dots, r$, is larger than one, then c is a *critical* remarkable value of H, and $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable* invariant algebraic curve of system (1.1) with *exponent* α_i.

Proposition 1.5.3. *Let $H = f/g$ be a minimal rational first integral of system* (1.1). *Assume that $f = f_1^{\alpha_1} \cdots f_r^{\alpha_r}$ and $g = g_1^{\beta_1} \cdots g_s^{\beta_s}$, for some irreducible polynomials f_i, g_j, $i = 1, \dots, r$, $j = 1, \dots, s$, and $r, s \in \mathbb{N}$. Let $\delta_i = \deg f_i$, $\gamma_j = \deg g_j$. Assume that $\deg f = \deg g$ and $(f, g) = 1$ and define the rational first integral*

$$\tilde{H} = \frac{\tilde{f}}{\tilde{g}} = \frac{c_2 f + (c_1 c_2 + 1) g}{f + c_1 g}.$$

We take $c_1, c_2 \in \mathbb{C}$ such that $\tilde{f}$ and $\tilde{g}$ are irreducible. Then,

(1) If f (resp. g) is reducible, then $c = -c_2 - c_1^{-1}$ (resp. $c = -c_2$) is a remarkable value of $\tilde{H}$.

(2) If $\alpha_i > 1$ for some $i \in \{1, \dots, r\}$ (resp. $\beta_j > 1$ for some $j \in \{1, \dots, s\}$), then $c = -c_2 - c_1^{-1}$ (resp. $c = -c_2$) is critical, and $f_i = 0$ (resp. g_j) is a critical remarkable invariant algebraic curve with exponent α_i (resp. β_j).

Proof: The equation $\tilde{f} + c\tilde{g} = 0$ can be written as $(c+c_2)f + (c_1(c+c_2)+1)g = 0$. Then, the proposition follows. ■

If f is a polynomial, let $\tilde{f}$ be the homogeneous part of f of highest degree. If H is the function given by the expression (1.8), we define

$$\tilde{H} = \tilde{f}_1^{\lambda_1} \cdots \tilde{f}_p^{\lambda_p} \left(e^{\tilde{g}_1/\tilde{h}_1^{n_1}} \right)^{\mu_1} \cdots \left(e^{\tilde{g}_q/\tilde{h}_q^{n_q}} \right)^{\mu_q}.$$

Theorem 1.5.4. *Suppose that the polynomial differential system* (1.1) *of degree m, with P and Q relatively prime, has a Darboux first integral H given by* (1.8), *where the polynomials f_i and h_j are irreducible, the polynomials g_j and h_j are relatively prime in $\mathbb{C}[x, y]$ and $\lambda_i, \mu_j \in \mathbb{C} \setminus \{0\}$, $i = 1, \dots, p$, $j = 1, \dots, q$. Then, the following statements hold.*

(1) Suppose that H is a minimal rational first integral, $H = f/g$, and that system (1.1) *has no polynomial first integrals. It is not restrictive to assume that f and g are irreducible. Then:*

(a) The rational function

$$V_H = \frac{g^2}{\prod\limits_i u_i^{\alpha_i - 1}},$$

where the product runs over all the critical remarkable invariant algebraic curves $u_i = 0$ having exponent $\alpha_i > 1$, is an inverse integrating factor.

(b) System (1.1) *has a polynomial inverse integrating factor if and only if H has, at most, two critical remarkable values.*

(2) Furthermore, if we assume that f_i and h_j are different for $i = 1, \dots, p$ and $j = 1, \dots, q$, then the following statements hold.

If system (1.1) *has no rational first integrals, then*

$$V_{\log H} = f_1 \cdots f_p \, h_1^{n_1+1} \cdots h_q^{n_q+1} \in \mathbb{R}[x, y].$$

Moreover, if $\tilde{H}$ is a multi–valued function and e^{g_j/h_j} are exponential factors for $j = 1, \dots, q$, then $V_{\log H} = f_1 \cdots f_p \, h_1^{n_1+1} \cdots h_q^{n_q+1}$ is a polynomial of degree $m + 1$.

Polynomial inverse integrating factors of quadratic systems

In this first part of the work the *quadratic systems* are the polynomial real differential systems of type (1.1) of degree $m = 2$. Our main objective is to classify all the quadratic systems having a polynomial inverse integrating factor $V(x, y)$.

In Section 2.1 we classify the quadratic systems into ten normal forms (see [29]). Once this classification is done, in Section 2.2 we present the methods we use to find quadratic systems having a polynomial inverse integrating factor. The rest of the chapter is dedicated to find the quadratic systems having a polynomial inverse integrating factor, using the classification of Section 2.1 and the methods described in Section 2.2.

2.1 Normal forms of the quadratic systems

We classify the quadratic systems in ten normal forms, passing from a quadratic system with the usual 12 parameters to a quadratic system with 6 parameters. We use these normal forms to find the quadratic systems having an inverse integrating factor. To do this classification we must transform the quadratic systems by using affine changes and scaling time, but we must be sure that with these changes a polynomial inverse integrating factor becomes a polynomial inverse integrating factor.

If $v(x, y)$ is an inverse integrating factor of a polynomial system and we apply a change of time $T = \gamma t$, then clearly $v(x, y)$ is an inverse integrating factor of the new system. The next proposition solves the question of the affine change.

Proposition 2.1.1. *Any inverse integrating factor associated to a polynomial system is transformed into an inverse integrating factor if the system is changed by an affine transformation.*

Proof: Consider the polynomial system

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y). \tag{2.1}$$

Let $x_1 = ax + by + \alpha, y_1 = cx + dy + \beta$, with $ad - bc \neq 0$. Then,

$$x = \frac{d(x_1 - \alpha) - b(y_1 - \beta)}{ad - bc}, y = \frac{a(y_1 - \beta) - c(x_1 - \alpha)}{ad - bc}$$

and

$$\dot{x_1} = p_1(x_1, y_1) = ap(x, y) + bq(x, y), \quad \dot{y_1} = q_1(x_1, y_1) = cp(x, y) + dq(x, y). \tag{2.2}$$

Let $V(x_1, y_1)$ be an inverse integrating factor of (2.2) and let $v(x, y) = V(x_1, y_1)$. It is clear that $v(x, y)$ is a polynomial if $V(x_1, y_1)$ is a polynomial. We want to prove that $v(x, y)$ is an inverse integrating factor of (2.1).

As V is an inverse integrating factor of system (2.2), it satisfies the equation

$$p_1 \frac{\partial V}{\partial x_1} + q_1 \frac{\partial V}{\partial y_1} = \left(\frac{\partial p_1}{\partial x_1} + \frac{\partial q_1}{\partial y_1} \right) V. \tag{2.3}$$

We next write this equation in terms of p, q and x, y. First of all, the functions $\partial p_1 / \partial x_1$ and $\partial q_1 / \partial y_1$ are written as

$$\begin{aligned}
\frac{\partial p_1}{\partial x_1} &= a \left(\frac{\partial p}{\partial x} \frac{d}{a\,d - b\,c} - \frac{\partial p}{\partial y} \frac{c}{a\,d - b\,c} \right) + b \left(\frac{\partial q}{\partial x} \frac{d}{a\,d - b\,c} - \frac{\partial q}{\partial y} \frac{c}{a\,d - b\,c} \right) \\
&= \frac{1}{a\,d - b\,c} \left(a\,d \frac{\partial p}{\partial x} - a\,c \frac{\partial p}{\partial y} + b\,d \frac{\partial q}{\partial x} - b\,c \frac{\partial q}{\partial y} \right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial q_1}{\partial y_1} &= c \left(-\frac{\partial p}{\partial x} \frac{b}{a\,d - b\,c} + \frac{\partial p}{\partial y} \frac{a}{a\,d - b\,c} \right) + d \left(-\frac{\partial q}{\partial x} \frac{b}{a\,d - b\,c} + \frac{\partial q}{\partial y} \frac{a}{a\,d - b\,c} \right) \\
&= \frac{1}{a\,d - b\,c} \left(-b\,c \frac{\partial p}{\partial x} + a\,c \frac{\partial p}{\partial y} - b\,d \frac{\partial q}{\partial x} + a\,d \frac{\partial q}{\partial y} \right).
\end{aligned}$$

Adding these two expressions, we obtain

$$\frac{\partial p_1}{\partial x_1} + \frac{\partial q_1}{\partial y_1} = \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}. \tag{2.4}$$

On the other hand, the expressions of $\partial V / \partial x_1$ and $\partial V / \partial y_1$ are, respectively,

$$\frac{\partial V}{\partial x_1} = \frac{1}{a\,d - b\,c} \left(d \frac{\partial v}{\partial x} - c \frac{\partial v}{\partial y} \right)$$

and

$$\frac{\partial V}{\partial y_1} = \frac{1}{a\,d - b\,c} \left(-b \frac{\partial v}{\partial x} + a \frac{\partial v}{\partial y} \right).$$

Then, the expression $p_1\, \partial V / \partial x_1 + q_1\, \partial V / \partial y_1$ becomes

$$\frac{a\,p + b\,q}{a\,d - b\,c} \left(d \frac{\partial v}{\partial x} - c \frac{\partial v}{\partial y} \right) + \frac{c\,p + d\,q}{a\,d - b\,c} \left(-b \frac{\partial v}{\partial x} + a \frac{\partial v}{\partial y} \right) = p \frac{\partial v}{\partial x} + q \frac{\partial v}{\partial y}. \tag{2.5}$$

So by equations (2.4) and (2.5), v is an inverse integrating factor of (2.1). ∎

Next proposition can be found in [29]. Since it plays a main role in our study, we give its proof.

Proposition 2.1.2. *Any real quadratic system is affine–equivalent, rescaling the time variable if necessary, to system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, where $Q(x,y) = d + ax + by + lx^2 + mxy + ny^2$ and $\dot{x} = P(x,y)$ is one of the following ten normal forms:*

$$
\begin{array}{llll}
(I) & \dot{x} = 1 + xy, & (VI) & \dot{x} = 1 + x^2, \\
(II) & \dot{x} = xy, & (VII) & \dot{x} = x^2, \\
(III) & \dot{x} = y + x^2, & (VIII) & \dot{x} = x, \\
(IV) & \dot{x} = y, & (IX) & \dot{x} = 1, \\
(V) & \dot{x} = -1 + x^2, & (X) & \dot{x} = 0.
\end{array}
$$

Proof: We can write a real polynomial quadratic differential system as

$$
\begin{aligned}
\dot{x} &= d_1 + a_1 x + b_1 y + l_1 x^2 + m_1 xy + n_1 y^2, \\
\dot{y} &= d_2 + a_2 x + b_2 y + l_2 x^2 + m_2 xy + n_2 y^2,
\end{aligned}
\tag{2.6}
$$

where all the parameters are assumed to be real. We claim that we can take $n_1 = 0$. Indeed, if $n_1 l_2 \neq 0$ then system (2.6) becomes a quadratic system without term y^2 in $\dot{x}$ by the change of variables $x_1 = y - rx$, $y_1 = y$, where $r \neq 0$ satisfies

$$
l_2 + (m_2 - l_1)r + (n_2 - m_1)r^2 - n_1 r^3 = 0. \tag{2.7}
$$

If $l_2 = 0$, that is, if the x^2 term does not appear in $\dot{y}$, then it is sufficient to interchange x and y. In short, we can assume that

$$
\dot{x} = d_1 + a_1 x + b_1 y + l_1 x^2 + m_1 xy \tag{2.8}
$$

and that $\dot{y} = Q(x,y)$ is an arbitrary quadratic polynomial. If $m_1 \neq 0$, then we introduce the translation $x + b_1 m_1^{-1} \to x$, and then (2.8) becomes $\dot{x} = d' + a'x + l'x^2 + m_1 xy$. Now, the change $a' + l'x + m_1 y \to y$ converts this new system into $\dot{x} = d' + xy$, $\dot{y} = Q(x,y)$. If $d' \neq 0$, then we make the change $(d')^{-1}x \to x$ to get (I). If $d' = 0$, then we have (II).

If $m_1 = 0$ and $b_1 \neq 0$, then the change $d_1 + a_1 x + b_1 y \to y$ converts (2.8) into $\dot{x} = y + l_1 x^2$.

Now if $l_1 \neq 0$, then we make the change $l_1^{-1} y \to y$, $l_1 t \to t$ to get (III). If $l_1 = 0$, then we have (IV).

If $m_1 = b_1 = 0$ and $l_1 \neq 0$, then we define $k = a_1^2 - 4d_1 l_1$. If $k \neq 0$, then by the change $2l_1|k|^{-1/2}(x + a_1(2l_1)^{-1}) \to x$, $2^{-1}|k|^{1/2}t \to t$ we get (V) or (VI) according to whether k is positive or negative. If $k = 0$, then by the change $x + a_1(2l_1)^{-1}x \to x$, $l_1 t \to t$ we get (VII).

If $m_1 = b_1 = l_1 = 0$ and $a_1 \neq 0$, then by the change $x + d_1(a_1)^{-1} \to x$, $a_1 t \to t$ we get (VIII).

Lastly, suppose that $m_1 = b_1 = l_1 = a_1 = 0$. If $d_1 \neq 0$, then we get (IX) by change $d_1 t \to t$; if $d_1 = 0$, then we have (X). ∎

Remark 2.1.3.
1. The ten cases obtained in Proposition 2.1.2 do not need to have empty intersection.
2. By Proposition 2.1.1, if a quadratic system has a polynomial inverse integrating factor, then its normal form given in Proposition 2.1.2 has a polynomial inverse integrating factor. So in order to find all the families of quadratic systems having a polynomial inverse integrating factor we need to compute the polynomial inverse integrating factors for the normal forms obtained.

□

2.2 Methods for computing polynomial inverse integrating factors

We consider the real planar quadratic system

$$\begin{aligned} \dot{x} &= P(x,y) = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} &= Q(x,y) = d + ax + by + lx^2 + mxy + ny^2. \end{aligned} \tag{2.9}$$

We assume that P and Q have no common factors, otherwise the system can be transformed into a linear one. In order to find a polynomial inverse integrating factor of this system, we use the classification of quadratic systems given in Proposition 2.1.2. We will find *polynomial inverse integrating factors* of degree $k \in \mathbb{N}$ by using some different methods of solving the equation

$$P\,\frac{\partial V}{\partial x} + Q\,\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V.$$

We denote this equation by (★). As the term in y^2 of $P(x,y)$ is zero in all the normal forms, we take $a_{02} = 0$ in (2.9).

Remark 2.2.1.
1. In case (X), equation (★) has always the polynomial solution $V = Q$. The cases where $P \equiv 0$ or $Q \equiv 0$ are not interesting for us, so from now and on we exclude them from our classification.

2. The equation $\operatorname{div}(P,Q) = \partial P/\partial x + \partial Q/\partial y = 0$ is equivalent to $b = -a_{10}$, $m = -2a_{20}$, $n = -a_{11}/2$. In this case, the system is Hamiltonian, so $V = 1$ is always a polynomial solution of ($\bigstar$). Moreover,

$$H(x,y) = dx - a_{00}y + \frac{a}{2}x^2 - a_{10}xy - \frac{a_{01}}{2}y^2 + \frac{l}{3}x^3 - a_{20}x^2y - \frac{a_{11}}{2}xy^2$$

is a polynomial first integral of system (2.9), and also a polynomial inverse integrating factor of the system.

This case appears for every normal form in our classification.

$\square$

Next we describe the methods we use to find polynomial inverse integrating factors of degree $k > 0$.

Method 1 Since we are looking for real polynomial inverse integrating factors of degree $k \in \mathbb{N}$, we write V as

$$V(x,y) = \sum_{i+j=0}^{k} v_{i,j}x^iy^j,$$

where $v_{i,j} \in \mathbb{R}$. Equation ($\bigstar$) is a polynomial equation since P, Q and V are polynomial functions, and it can be written as a linear system with unknowns $v_{i,j}$, $i+j = 0,\ldots,k$.

If $k > 1$, we define in a recursive way the matrix A_k,

$$A_k = \left(\begin{array}{c|c} & 0 \\ A_{k-1} & C_{1,k} \\ & C_{2,k} \\ \hline 0 & C_{3,k} \end{array}\right), \tag{2.10}$$

where

$$C_{1,k} = \begin{pmatrix} ka_{00} & d & & & \\ & (k-1)a_{00} & 2d & & \\ & & \ddots & \ddots & \\ & & & a_{00} & kd \end{pmatrix}, \tag{2.11}$$

$$C_{2,k} = \begin{pmatrix} (k-1)a_{10} - b & a & & & \\ ka_{01} & (k-2)a_{10} & 2a & & \\ & (k-1)a_{01} & \ddots & \ddots & \\ & & \ddots & \ddots & ka \\ & & & a_{01} & -a_{10} + (k-1)b \end{pmatrix}, \tag{2.12}$$

$$C_{3,k} = \begin{pmatrix} (k-2)a_{20}-m & l & & & \\ (k-1)a_{11}-2n & (k-3)a_{20} & 2l & & \\ & (k-2)a_{11}-n & \ddots & \ddots & \\ & & \ddots & \ddots & kl \\ & & & \ddots & -2a_{20}+(k-1)m \\ & & & & -a_{11}+(k-2)n \end{pmatrix} \tag{2.13}$$

and

$$A_1 = \begin{pmatrix} -a_{10}-b & a_{00} & d \\ -2a_{20}-m & -b & a \\ -a_{11}-2n & a_{01} & -a_{10} \\ 0 & -a_{20}-m & l \\ 0 & -2n & -2a_{20} \\ 0 & 0 & -a_{11}-n \end{pmatrix}. \tag{2.14}$$

Let $\mathbf{0}$ be the zero vector, of convenient dimension wherever it may appear. We state the following result.

Proposition 2.2.2. *Let $V(x,y)$ be a polynomial inverse integrating factor of degree k of system* (2.9)*. Then, equation (★) is equivalent to the homogeneous linear system*

$$A_k V^k = \mathbf{0}, \tag{2.15}$$

where A_k is the matrix defined in (2.10) *and*

$$V^k = (v_{0,0},\ v_{1,0},\ v_{0,1},\ v_{2,0},\ v_{1,1},\ v_{0,2},\dots,v_{k,0},\ v_{k-1,1},\dots,v_{1,k-1},\ v_{0,k})^T$$

is the vector of the coefficients of $V(x,y)$.

Proof: Equation (★) is a polynomial equation of degree $k+1$ in two variables. Then it can be written as

$$\begin{aligned} \sum_{i+j=0}^{k+1} \Big[&\big(a_{11}(i-1)+n(j-3)\big)v_{i,j-1} + \big(a_{20}(i-3)+m(j-1)\big)v_{i-1,j} \\ &+l(j+1)v_{i-2,j+1} + a_{00}(i+1)v_{i+1,j} + \big(a_{10}(i-1)+b(j-1)\big)v_{i,j} \\ &+a_{01}(i+1)v_{i+1,j-1} + d(j+1)v_{i,j+1} + a(j+1)v_{i-1,j+1}\Big] x^i y^j = 0, \end{aligned} \tag{2.16}$$

where $v_{r,s} = 0$ if $r, s, r+s \notin \{0,\dots,k\}$. As all the coefficients of this polynomial must be zero, we must take

$$\begin{aligned} &\big(a_{11}(i-1)+n(j-3)\big)v_{i,j-1} + \big(a_{20}(i-3)+m(j-1)\big)v_{i-1,j} \\ &\quad +l(j+1)v_{i-2,j+1} + a_{00}(i+1)v_{i+1,j} + \big(a_{10}(i-1)+b(j-1)\big)v_{i,j} \\ &\quad +a_{01}(i+1)v_{i+1,j-1} + d(j+1)v_{i,j+1} + a(j+1)v_{i-1,j+1} = 0, \end{aligned}$$

for $0 \leq i + j \leq k + 1$. In order to obtain equation (2.15), we just have to write these equalities as a linear system with unknowns $v_{i,j}$. ■

Remark 2.2.3. 1. We note that, for a given $k \in \mathbb{N}$, the matrix A_k has $(k+2)(k+3)/2$ rows and $(k+1)(k+2)/2$ columns. Then, the (homogeneous) linear system is over–determined and for each case we have to eliminate some equations in order to get a non–trivial solution. Then, we must take some conditions on the coefficients d, a, b, l, m, n (the coefficients of the polynomial Q, see Proposition 2.1.2), in order to obtain a polynomial inverse integrating factor of the corresponding system.

2. We look for conditions on the coefficients of $Q(x, y)$ so that the null–space of the matrix A_k has dimension one.
3. This first Method is very tedious for k large, so we use it to compute solutions of ($\bigstar$) of small degree, usually degree $k \leq 6$.
4. This Method, adapted and improved, is used in [46] for finding invariant algebraic curves of a given degree for planar polynomial systems.

□

Method 2 We compute a first integral $H(x, y)$ of system (2.9) and then we obtain an inverse integrating factor $V(x, y)$ from the equations $P/V = -H_y$, $Q/V = H_x$ (see Section 1.4). This Method is used in some cases when we cannot bound the degree of V. The difficulty in this Method is to find the convenient first integral such that the inverse integrating factor V is polynomial.

Method 3 We write V as a polynomial in the variable y (resp. x): $V(x, y) = \sum_{i=0}^{s} W_i(x)\, y^i$ (resp. $V(x, y) = \sum_{i=0}^{r} \tilde{W}_i(y)\, x^i$), $0 \leq r, s \leq k$. Then, ($\bigstar$) can be written as a polynomial equation in y (resp. in x), and we can solve it starting by the terms of highest or lowest degree in y (resp. in x). This Method is useful if we need to compute the degree of V in y (resp. x).

Method 4 We write V as

$$V(x, y) = V_0 + V_1(x, y) + \cdots + V_{k-1}(x, y) + V_k(x, y), \tag{2.17}$$

where $V_i(x, y)$ is a homogeneous polynomial of degree i, for $i = 1, \ldots, k$, and $V_0 \in \mathbb{R}$. Then, ($\bigstar$) becomes a system of homogeneous differential equations of degree from 0 to $k + 1$, which are solved recursively. Moreover, from the Euler Theorem for homogeneous functions, we have

$$xV_x + yV_y = \sum_{j=0}^{k} jV_j.$$

Then, equation (★) is transformed, after multiplying by x (resp. y), into a system of $k+2$ ordinary differential equations, where there only appear the polynomials V_j and their derivative with respect to y (resp. x). This Method is the one we more use to solve (★). In Section 2.3.8 we combine it with the decomposition of each V_j in powers of linear polynomials, as explained in Proposition 2.3.2, Corollary 2.3.3 and Lemma 2.3.4.

Method 5 We write V as in (2.17) and then we write each homogeneous polynomial $V_i(x,y)$ as a polynomial in y (resp. x): $V_i(x,y) = \sum_{j=0}^{i} v_{i-j,j} x^{i-j} y^j$ (resp. $V_i(x,y) = \sum_{j=0}^{i} v_{j,i-j} y^{i-j} x^j$).

Remark 2.2.4. There are solutions of (★) of arbitrary degree k. In some of these cases, it is very difficult for us to find an explicit expression for V. But in these cases, this expression can be computed for fixed k using Method 4. □

2.3 Finding polynomial inverse integrating factors

In this section we find the quadratic systems which have a polynomial inverse integrating factor for each one of the nine normal forms (I)–(IX). For the normal forms where P contains the monomial x^2, we will use the following lemma.

Lemma 2.3.1. *Consider the quadratic system*

$$\dot{x} = a_{00} + a_{01}\, y + x^2, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2. \tag{2.18}$$

The following statements hold.

(1) Assume that $a_{00} \in \{0, 1, -1\}$, $a_{01} \in \{0, 1\}$, and $n \neq 0$. Let $V(x,y)$ be a polynomial inverse integrating factor of (2.18)*. Then, the degree of $V(x,y)$ with respect to y is two.*

(2) Assume $a_{00} = 0$, $a_{01} \in \{0, 1\}$, and $n = 0$. Let $V(x,y)$ be a polynomial inverse integrating factor of degree $k > 4$ of (2.18)*. Then,*

$$m = 1 - \frac{k-3}{p} \neq 1,$$

where $p \in \{-1, 1, 2, 3, \dots, k-1\}$.

Proof: First we prove statement 1. We write $V(x,y) = \sum_{i=0}^{s} W_i(x) y^i$. Then equation (★) is a polynomial equation in y. The equation corresponding to y^{s+1} is

$$n(s-2)W_s(x) + a_{01} W_s'(x) = 0.$$

If $a_{01} = 1$ then $W_s(x) = e^{-n(s-2)}$. As $W_s(x)$ is a polynomial and $n \neq 0$, we take $s = 2$, and then $W_s(x) = W_2(x) \equiv 1$. If $a_{01} = 0$, as $n \neq 0$ and $W_s(x) \not\equiv 0$, again we must take $s = 2$.

Next we prove statement 2. We write $V(x, y)$ as in (2.17). The homogeneous equation of degree $k + 1$ of equation ($\bigstar$) is

$$-(m+2)xV_k + x(lx + my)\frac{\partial V_k}{\partial y} + x^2\frac{\partial V_k}{\partial x} = 0.$$

If $m = 1$, then $V_k(x, y) = x^3 F(y/x - l \log x)$, so $\deg(V_k) = 3$, in contradiction with the assumption $k > 4$. If $m \neq 1$, then

$$V_k(x, y) = x^{m+2} F\left(\frac{l}{m-1}x^{1-m} + x^{-m}y\right),$$

where F is an arbitrary function. As $V_k(x, y)$ is a homogeneous polynomial of degree k, the function F must be of the form $F(z) = z^{p+1}$, with $p \in \mathbb{N} \cup \{-1\}$. We discard $p = 0$ because, in this case, we would get $k = 3$. So

$$V_k(x, y) = x^{2-pm}\left(\frac{l}{m-1}x + y\right)^{p+1}.$$

Then $k = 3 - p(m - 1)$, and from this equality we get $m = 1 - (k - 3)/p$, $p \in \mathbb{N} \cup \{-1\}$. We must also take $p < k$, because $V_k(x, y)$ contains the monomial $x^{k-p-1}y^{p+1}$. ■

Next results can be found in [9]. The computations in the cases where P contains the monomial xy are based in those results, so we also give their proofs.

Proposition 2.3.2. *We consider the ordinary differential equation*

$$NH + UH_y = M^s L, \tag{2.19}$$

where N, U, M and L are polynomials. Assume that M is a polynomial of degree τ such that M divides U, and that M has neither common factors with $N + jM_yU/M$ for $j = 0, 1, \ldots, s-1$, nor with L. If equation (2.19) has a solution H given by a polynomial of degree m, then $H = M^s W$ where W is a polynomial of degree $m - s\tau$ such that

$$\left[N + sM_y\frac{U}{M}\right]W + UW_y = L.$$

Furthermore, if M and H are homogeneous then W is also homogeneous.

Proof: Since M divides U but it does not divide N, from (2.19) it follows that there exists $j \in \mathbb{N}$ such that $H = M^j A$, where A is a polynomial of degree $m - \tau j$. Going back to (2.19) with this expression of H, we obtain that $M^j\big([N + jM_yU/M]A + UA_y\big) = M^s L$. We have that $j \leq s$, otherwise M divides L in contradiction with the assumptions. On the other hand, if $j < s$, then M must divide $[N + jM_yU/M]A + UA_y$. Then, since M divides U and does not divide A we get that M divides $N + jM_yU/M$, again in contradiction with the hypotheses. Hence $j = s$, and furthermore A satisfies the equation $[N + sM_yU/M]A + UA_y = L$. If M and H are homogeneous, then from $H = M^s W$ we deduce that W is homogeneous and the proof of the proposition is completed. ■

Corollary 2.3.3. *We consider the differential equation*

$$KH + TH_y = F^q G^s E, \tag{2.20}$$

where K and E are homogeneous polynomials of degree 1, $F = m(p-1)x + (k - 3 + 2(p-1)(n-1))y$ and $G = \alpha x + \beta y$, $T = FG$.

(a) Suppose that $q > 0$ and G does not divide $K + jG_yF$ for $j = 0, 1, \ldots, s-1$, and that F does not divide $K + sG_yF + iF_yG$ for $i = 0, 1, \ldots, q-1$. If there exists a solution of (2.20) given by a homogeneous polynomial H of degree m, then $H = F^q G^s V_{m-s-q}$, where V_{m-s-q} satisfies

$$DV_{m-s-q} + TV_{m-s-q,y} = E, \tag{2.21}$$

with

$$D = K + sG_yF + qF_yG. \tag{2.22}$$

(b) Suppose that $q = 0$ and $G \nmid (K + jG_yF)$ for $j = 0, 1, \ldots, s-1$. If there exists a solution H of (2.20) given by one homogeneous polynomial of degree m, then $H = G^s V_{m-s}$, where V_{m-s} satisfies $DV_{m-s} + TV_{m-s,y} = E$, with $D = K + sG_yF$.

Proof: (a) Applying Proposition 2.3.2 with $N = K$, $U = T$, $L = F^q E$ and $M = G$ (and hence $t = 1$), we have that $H = G^s W_{m-s}$, where W_{m-s} is the solution of $[K + sG_yF]\, W_{m-s} + TW_{m-s,y} = F^q E$. Applying again Proposition 2.3.2 to this equation with $N = K + sG_yF$, $U = T$, $L = E$ and $M = F$ we get $W_{m-s} = F^q V_{m-s-q}$, with V_{m-s-q} a solution of (2.21).

(b) This statement is the consequence of applying Proposition 2.3.2 with $N = K$, $U = T$, $L = E$ and $M = G$. ■

Lemma 2.3.4. *Let* $V = \sum_{i=0}^{m} \alpha_i x^{m-i} y^i$ *be a homogeneous polynomial of degree* m *solution of* $DV + TV_y = E$*, where* $T = dx^2 + exy + (f-1)y^2$*,* $D = \widetilde{e}x + \widetilde{f}y$ *and* $E = \sum_{r=0}^{m+1} q_r x^{m+1-r} y^r$*. Let* $M(l+2, l+1)$ *be the* $(l+2) \times (l+1)$ *matrix*

$$\begin{pmatrix} \widetilde{f} + l(f-1) & & & & & \\ \widetilde{e} + le & \widetilde{f} + (l-1)(f-1) & & & & \\ ld & \widetilde{e} + (l-1)e & \widetilde{f} + (l-2)(f-1) & & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & & 2d & \widetilde{e} + e & \widetilde{f} \\ & & & & d & \widetilde{e} \end{pmatrix}.$$

Then the coefficients α_i *of* V *are the solution* $Z = (\alpha_m, \alpha_{m-1}, \ldots, \alpha_0)^t$ *of the system* $M(m+2, m+1)Z = b$ *with* $m+2$ *equations and* $m+1$ *unknowns, where* $b = (q_{m+1}, q_m, \ldots, q_0)^t$*. Furthermore, if there exists* $s \in \{0, 1, \ldots, m\}$ *such that* $\widetilde{f} + j(f-1) \neq 0$ *and* $q_{j+1} = 0$ *for all* $j \geq s$*, then the existence of* V *is equivalent to the compatibility of system* $M(s+1, s)\widetilde{Z} = \widetilde{b}$*, where* $\widetilde{Z} = (\alpha_{s-1}, \alpha_{s-2}, \ldots, \alpha_0)^t$ *and* $\widetilde{b} = (q_s, q_{s-1}, \ldots, q_0)^t$*. Moreover, the coefficients of* V *are* $(0, \ldots, 0, \alpha_{s-1}, \ldots, \alpha_0)$*.*

Proof: Taking into account the expressions of D, T and V, we can write

$$\begin{aligned}
& DV + TV_y \\
&= \sum_{t=0}^{m} \widetilde{e}\alpha_t x^{m-t+1} y^t + \sum_{t=0}^{m} \widetilde{f}\alpha_t x^{m-t} y^{t+1} + \sum_{t=1}^{m} td\alpha_t x^{m-t+2} y^{t-1} + \\
&\quad \sum_{t=1}^{m} te\alpha_t x^{m-t+1} y^t + \sum_{t=0}^{m} t(f-1)\alpha_t x^{m-t} y^{t+1} \\
&= \sum_{t=0}^{m} (\widetilde{f} + t(f-1))\alpha_t x^{m-t} y^{t+1} + \sum_{t=0}^{m} (\widetilde{e} + te)\alpha_t x^{m-t+1} y^t + \\
&\quad \sum_{t=1}^{m} td\alpha_t x^{m-t+2} y^{t-1} \\
&= (d\alpha_1 + \widetilde{e}\alpha_0)x^{m+1} + \sum_{j=0}^{m-2} ((j+2)d\alpha_{j+2} + (\widetilde{e} + (j+1)e)\alpha_{j+1} + \\
&\quad (\widetilde{f} + j(f-1))\alpha_j) x^{m-j} y^{j+1} + ((\widetilde{e} + m)e)\alpha_m + \\
&\quad (\widetilde{f} + (m-1)(f-1))\alpha_{m-1}) xy^m + (\widetilde{f} + m(f-1))\alpha_{m-1} y^{m+1}.
\end{aligned}$$

Then equation $DV + TV_y = E$ is equivalent to the linear system $M(m+2, m+1)Z = b$.

Now, we suppose that there exists $s \ \epsilon \ \{0, 1, .., m\}$ such that $\widetilde{f} + j(f-1) \neq 0$

and $q_{j+1}=0$ for all $j\geq s$. If we define X as the matrix

$$\left(\begin{array}{cccccc}
\widetilde{f}\text{+m(f-1)} & & & & & \\
\widetilde{e}\text{+me} & \widetilde{f}\text{+(m-1)(f-1)} & & & & \\
\text{md} & \widetilde{e}\text{+(m-1)e} & \widetilde{f}\text{+(m-2)(f-1)} & & & \\
 & \cdots & \cdots & \cdots & \cdots & \\
 & & & \widetilde{e}\text{+(s+2)e} & \widetilde{f}\text{+(s+1)(f-1)} & \\
 & & & \text{(s+2)d} & \widetilde{e}\text{+(s+1)e} & \widetilde{f}\text{+s(f-1)}
\end{array}\right),$$

we can write $M(m+2,m+1)$ as

$$M(m+2,m+1)=\left(\begin{array}{cc} X & 0 \\ Y & M(s+1,s) \end{array}\right).$$

On the other hand, we have $Z=(\alpha_m,\alpha_{m-1},.,\alpha_s,|\alpha_{s-1},..,\alpha_0)^t=(\Omega^t|\widetilde{Z}^t)^t$ and $b=(0|q_s,q_{s-1},..q_0)^t=(0|\widetilde{b}^t)^t$. Now, the system that corresponds to the first $m-s+1$ unknowns is $X\Omega=0$ and since $\det(X)=\prod_{j=s}^{m}(\widetilde{f}+j(f-1))\neq 0$ we deduce that $\Omega=0$, that is, $\alpha_j=0$ for all $j\geq s$. Finally, the remainder equations becomes $Y\Omega+M(s+1,s)\widetilde{Z}=\widetilde{b}$. As $\Omega=0$, the proof is finished. ∎

From now on in this chapter, we find the quadratic systems having a polynomial inverse integrating factor, taking into account the normal forms given in Proposition 2.1.2.

2.3.1 The case $P(x,y)\equiv 1$

We consider the quadratic system

$$\dot{x}=1,\quad \dot{y}=d+ax+by+lx^2+mxy+ny^2, \tag{2.23}$$

where $d,a,b,l,m,n\in\mathbb{R}$. If $n\neq 0$ then this system is transformed into

$$\dot{x}=1,\quad \dot{y}=\frac{b_{00}}{4}-\frac{b_{10}}{2}x+\frac{b_{20}}{4}x^2+y^2, \tag{2.24}$$

by the affine change $mx/2+ny+b/2\to y$, where $b_{00}=4dn-b^2+2m$, $b_{10}=bm-2an$ and $b_{20}=4ln-m^2$.

The set of conditions on the coefficients of system (2.23) with $n=0$ and system (2.24) in order to have a polynomial inverse integrating factor are stated in the following two propositions.

Proposition 2.3.5. *A system of type* (2.23) *with* $n=0$ *and having a polynomial inverse integrating factor* $V(x,y)$ *can be written, after an affine change of variables and a rescaling of the time if it is necessary, as* $\dot{x}=1$, $\dot{y}=Q(x,y)$, *where* Q *is one of the polynomials below.*

(IX.1) *$Q(x,y) = x(\delta + x)$, where $\delta = 0, 1$. The system is Hamiltonian, so we have $V(x,y) = 1$.*

(IX.2) *$Q(x,y) = y + x^2$, and we get $V(x,y) = 1 + y + (1+x)^2$.*

(IX.3) *$Q(x,y) = (b_{01} + \delta x)y$, where $\delta = \pm 1$ and $b_{01} \in \mathbb{R}$, and we get $V(x,y) = y$.*

Proof: If $b = m = 0$, then the system is Hamiltonian. We also have $l \neq 0$, otherwise the system is linear. If $a \neq 0$, then by the change $lx/a \to x$, $-dl^2x/a^3 + l^2y/a^3 \to y$, $lt/a \to t$ we have $\dot{y} = x(1+x)$. If $a = 0$, then by the change $-dx/l + y/l \to y$ we get $\dot{y} = x^2$. We obtain statement (IX.1).

Assume $b^2 + m^2 \neq 0$. Next we find the maximum degree in y of a solution V of degree k. For that purpose, we write V as a polynomial of degree $s \geq 0$ in y:

$$V(x,y) = \sum_{i=0}^{s} W_i(x) y^i.$$

We can write equation ($\bigstar$) as a polynomial equation in y. Then, all the coefficients of the new equation (which depend on x) must vanish. The coefficient of y^{s+1} is zero and the coefficient of y^s is given by

$$(b + mx)(s-1)W_s(x) + W_s'(x) = 0.$$

Solving this equation, we obtain $W_s(x) = e^{-(s-1)(b+mx/2)x}$. As $W_s(x)$ is a polynomial and we are assuming $b^2 + m^2 \neq 0$, we take $s = 1$. So we get $V(x,y) = W_0(x) + y$. Now the whole equation ($\bigstar$) becomes

$$d + ax + lx^2 - (b + mx)W_0(x) + W_0'(x) = 0. \tag{2.25}$$

We distinguish two cases. If $m = 0$ (and so $bl \neq 0$), then equation (2.25) becomes

$$d + ax + lx^2 - bW_0(x) + W_0'(x) = 0,$$

from which we obtain

$$W_0(x) = \frac{1}{b^3}(2l + ab + b^2d + (ab + 2l)bx + b^2lx^2) + C_0 e^{bx},$$

where $C_0 \in \mathbb{R}$. As $b \neq 0$, we take $C_0 = 0$. Then, we get the polynomial

$$V(x,y) = 2l + ab + b^2d + (ab + 2l)bx + b^2lx^2 + b^3y.$$

After the change $(bx, abx/l + b^3y/l + (a + db^2)/l, bt) \to (x, y, t)$, we get the system and the solution stated in (IX.2).

If $m \neq 0$, then from (2.25) we get

$$W_0(x) = \frac{am - bl + lmx}{m^2} + C_0 e^{bx+mx^2/2}$$
$$-\sqrt{\frac{\pi}{2}} \frac{(b^2+m)l - m(ab-dm)}{m^{5/2}} e^{bx+mx^2/2} \Phi\left(\frac{b+mx}{\sqrt{2m}}\right),$$

where $\Phi(x)$ is the error function $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$ and $C_0 \in \mathbb{R}$. As $W_0(x)$ is a polynomial, we take $C_0 = 0$ and $(b^2+m)l - m(ab-dm) = 0$. We obtain the polynomial

$$V(x,y) = am - bl + lmx + m^2 y.$$

After the change $(\sqrt{|m|}x, lx + my + (am-bl)/m, \sqrt{|m|}t) \to (x,y,t)$, we obtain the system and the solution stated in (IX.3), where δ is the sign of m and $b_{01} = b/\sqrt{|m|}$. ∎

Proposition 2.3.6. *A system of type* (2.24) *having a polynomial inverse integrating factor* $V(x,y)$ *can be written, after an affine change of variables and a rescaling of the time if it is necessary, as* $\dot{x} = 1$, $\dot{y} = Q(x,y)$, *where*

(IX.4) $Q(x,y) = \delta + y^2$, *with* $\delta = -1, 0, 1$. *The expression of a polynomial inverse integrating factor is* $V(x,y) = \delta + y^2$.

Proof: First we write V as a polynomial of degree s in y:

$$V(x,y) = \sum_{i=0}^{s} W_i(x) y^i,$$

with $W_s(x) \not\equiv 0$. The equation corresponding to the coefficient of y^{s+1} in (★) is $(s-2)W_s(x) = 0$. Then, $s = 2$, which means $k \geq 2$. Next we write $V(x,y)$ as in (2.17). So equation (★) can be transformed into a system of $k+2$ homogeneous equations. The homogeneous equation of degree $k+1$ is

$$-8yV_k + (b_{20}x^2 + 4y^2)\frac{\partial V_k}{\partial y} = 0.$$

Solving this equation we obtain $V_k(x,y) = (b_{20}x^2 + 4y^2) f_k(x)$, where $f_k(x)$ is an arbitrary non–zero function of x. As V_k is an homogeneous polynomial of degree k, we take $f_k(x) = x^{k-2}$, and then

$$V_k(x,y) = (b_{20}x^2 + 4y^2)\, x^{k-2}.$$

The homogeneous equation of degree k is

$$-16b_{10}x^{k-1}y - 8yV_{k-1} + (b_{20}x^2 + 4y^2)\frac{\partial V_{k-1}}{\partial y} = 0,$$

from which we get

$$V_{k-1}(x,y) = -2b_{10}x^{k-1} + C_{k-1}(b_{20}x^2 + 4y^2)\,x^{k-3},$$

with $C_{k-1} \in \mathbb{R}$. From the homogeneous equation of degree $k-1$,

$$4x^{k-3}(b_{20}kx^2 - 2(2b_{10}C_{k-1} - b_{00})xy - 4(k-2)y^2) - 8yV_{k-2} + (b_{20}x^2 + 4y^2)\frac{\partial V_{k-2}}{\partial y} = 0,$$

we get

$$V_{k-2}(x,y) = C_{k-2}x^{k-4}(b_{20}x^2 + 4y^2) + x^{k-3}((b_{00} - 2b_{10}C_{k-1})x - 4y) + \frac{2(k-1)x^{k-4}(b_{20}x^2 + 4y^2)}{\sqrt{-b_{20}}}\ \text{arctanh}\left(\frac{2y}{\sqrt{-b_{20}}x}\right)$$

if $b_{20} < 0$,

$$V_{k-2}(x,y) = C_{k-2}x^{k-4}(b_{20}x^2 + 4y^2) + x^{k-3}((b_{00} - 2b_{10}C_{k-1})x - 4y) - \frac{2(k-1)x^{k-4}(b_{20}x^2 + 4y^2)}{\sqrt{b_{20}}}\ \arctan\left(\frac{2y}{\sqrt{b_{20}}x}\right)$$

if $b_{20} > 0$, and

$$V_{k-2}(x,y) = C_{k-2}x^{k-4}y^2 - (2b_{10}C_{k-1} - b_{00})x^{k-2} + 4(k-2)x^{k-3}y$$

if $b_{20} = 0$. In all cases, $C_{k-2} \in \mathbb{R}$. As $k \geq 2$, we must take $b_{20} = 0$. From the homogeneous equation of degree $k-2$, we obtain

$$V_{k-3}(x,y) = 4(k-3)C_{k-1}x^{k-4}y - \frac{2b_{00}C_{k-1} - b_{10}C_{k-2}}{2}x^{k-3} + C_{k-3}x^{k-5}y^2 - \frac{2b_{10}(2k-3)}{3y}\,x^{k-2},$$

with $C_{k-3} \in \mathbb{R}$. In order to obtain a polynomial, we must take $b_{10} = 0$. Then, $\dot{y} = b_{00}/4 + y^2$. In this case we have a solution of degree 2, which is, after either the change $(\sqrt{|b_{00}|}x/2, 2y/\sqrt{|b_{00}|}, \sqrt{|b_{00}|}t/2) \to (x,y,t)$ if $b_{00} \neq 0$, or the change $(x/2, 2y, t/2) \to (x,y,t)$ if $b_{00} = 0$, the solution stated in (IX.4). The parameter δ corresponds to the sign of b_{00}. ∎

2.3.2 The case $P(x, y) = x$

We consider the quadratic system

$$\dot{x} = x, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.26}$$

where $d, a, b, l, m, n \in \mathbb{R}$. If $n = 0$ and $m \neq 0$, then this system is transformed into

$$\dot{x} = x, \quad \dot{y} = b_{00} + by + xy, \tag{2.27}$$

where $b_{00} = (bl(b-1) - m(ab - dm))/m^2$, by the affine change $mx \to x$, $lx/m + y + a/m - (b-1)l/m^2 \to y$. If $n \neq 0$ then system (2.26) becomes

$$\dot{x} = x, \quad \dot{y} = \frac{b_{00}}{4} - \frac{b_{10}}{2}x + \frac{b_{20}}{4}x^2 + y^2, \tag{2.28}$$

where $b_{00} = 4dn - b^2$, $b_{10} = (b-1)m - 2an$ and $b_{20} = 4ln - m^2$, by the affine change $mx/2 + ny + b/2 \to y$.

The set of conditions on the coefficients of system (2.26) with $n = m = 0$, system (2.27) and system (2.28) in order to have a polynomial inverse integrating factor are stated in the following three propositions.

Proposition 2.3.7. *A system of type (2.26) with $n = m = 0$ and $l \neq 0$ has always a polynomial inverse integrating factor $V(x, y)$ (the case $b = d = 0$ is excluded because the system would be equivalent to a linear one). In order to get its expression, we distinguish four cases, depending on the value of the parameter b. The system can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = x$, $\dot{y} = Q(x, y)$, where Q is one of the polynomials below.*

(VIII.1) $Q(x, y) = -y + x^2$. *The system is Hamiltonian, so we have* $V(x, y) = 1$.

(VIII.2) $Q(x, y) = \delta + x^2$ *where* $\delta = \pm 1$, *and we get* $V(x, y) = x$.

(VIII.3) $Q(x, y) = \delta x + y + x^2$ *where* $\delta = 0, 1$, *and we get* $V(x, y) = x^2$.

(VIII.4) $Q(x, y) = by + x^2$ *with* $b \neq -1, 0, 1$, *and* $V(x, y) = x((b-2)y + x^2)$.

Proof: If $b = -1$, then the system is Hamiltonian. By the change $-ax/(2l) + y/l - d/l \to y$, we get (VIII.1). From system $A_1 V^1 = \mathbf{0}$,

$$\begin{pmatrix} -1-b & 0 & d \\ & -b & a \\ & & -1 \\ & & l \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \end{pmatrix} = \mathbf{0},$$

we get $V(x,y) = x$ and $b = 0$. So statement (VIII.2) follows after the change $(lx, -alx + ly) \to (x,y)$, where δ is the sign of $dl \neq 0$. From system $A_2 V^2 = \mathbf{0}$,

$$\begin{pmatrix} -1-b & 0 & d & & & \\ & -b & a & 0 & d & \\ & & -1 & 0 & 0 & 2d \\ & & l & 1-b & a & 0 \\ & & & & 0 & 2a \\ & & & & 0 & b-1 \\ & & & & l & 0 \\ & & & & & 2l \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2} \end{pmatrix} = \mathbf{0},$$

we get $V(x,y) = x^2$ and the condition $b = 1$. So statement (VIII.3) follows after applying either the change $(lx/a, ly/a^2 + dl/a^2) \to (x,y)$ if $a \neq 0$, or the change $(lx, ly + dl) \to (x,y)$ if $a = 0$. Finally, assuming $b \neq -1, 0, 1$, we consider system $A_3 V^3 = \mathbf{0}$,

$$\begin{pmatrix} -1-b & 0 & d & & & & & & & \\ & -b & a & 0 & d & & & & & \\ & & -1 & 0 & 0 & 2d & & & & \\ & & l & 1-b & a & 0 & 0 & d & & \\ & & & & 0 & 2a & 0 & 0 & 2d & \\ & & & & 0 & b-1 & 0 & 0 & 0 & 3d \\ & & & & l & 0 & 2-b & a & 0 & 0 \\ & & & & & 2l & 0 & 1 & 2a & 0 \\ & & & & & & & 0 & b & 3a \\ & & & & & & & 0 & 0 & 2b-1 \\ & & & & & & & l & 0 & 0 \\ & & & & & & & & 2l & 0 \\ & & & & & & & & & 3l \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2} \\ v_{3,0} \\ v_{2,1} \\ v_{1,2} \\ v_{0,3} \end{pmatrix} = \mathbf{0}.$$

We get the solution shown in statement (VIII.4) after the change of variables $(lx, alx/(b-1) + ly + dl/b) \to (x,y)$. ■

Proposition 2.3.8. *A system of type* (2.27) *having a polynomial inverse integrating factor* $V(x,y)$ *can be written, after an affine change of variables and a rescaling of the time if it is necessary, as* $\dot{x} = x$, $\dot{y} = Q(x,y)$, *where* Q *is one of the polynomials below.*

(VIII.5) $Q(x,y) = \delta x + (x-1)y$ *where* $\delta = 0, 1$, *and we get* $V(x,y) = \delta(1 + x) + xy$.

(VIII.6) $Q(x,y) = (b+x)y$, *and we have* $V(x,y) = xy$.

Proof: We write $V(x,y)$ as in (2.17). The homogeneous equation of degree $k+1$ of equation (★) is

$$-xV_k + xy\frac{\partial V_k}{\partial y} = 0.$$

We get $V_k(x,y) = x^{k-1}y$. The homogeneous equation of degree k is

$$(k-2)x^{k-1}y - xV_{k-1} + xy\frac{\partial V_{k-1}}{\partial y} = 0.$$

From this equation,

$$V_{k-1}(x,y) = (C_{k-1} - (k-2)\log y)x^{k-2}y,$$

$C_{k-1} \in \mathbb{R}$. Then, a polynomial solution V must have degree $k=2$. Now system $A_2V^2 = \mathbf{0}$ is

$$\begin{pmatrix} -(b+1) & 0 & b_{00} & & & \\ -1 & -b & 0 & 0 & b_{00} & \\ & 0 & -1 & 0 & 0 & 2b_{00} \\ & -1 & 0 & 1-b & 0 & 0 \\ & & & 0 & 0 & b-1 \\ & & & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2} \end{pmatrix} = \mathbf{0}.$$

From this system we get the polynomial $V(x,y) = b_{00} + xy$ and the condition $(b+1)b_{00}v_{1,1} = 0$. If $b=-1$ then we obtain statement (VIII.5) using the change $y/b_{00} - 1 \to y$ if $b_{00} \neq 0$. If $b = b_{00} = 0$ then the system has a common factor. If $b \neq -1, 0$ and $b_{00} = 0$, then statement (VIII.6) follows. ■

Proposition 2.3.9. *A system of type* (2.28) *having a polynomial inverse integrating factor $V(x,y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = x$, $\dot{y} = Q(x,y)$, where Q is one of the polynomials below. The expression of a polynomial inverse integrating factor is given in all the cases.*

(VIII.7) $Q(x,y) = -1/4 + \delta x^2 + y^2$, *where* $\delta = -1, 0, 1$. *We get* $V(x,y) = 4\delta x^2 + (2y-1)^2$.

(VIII.8) $Q(x,y) = b_{00}/4 + y^2$, *where* $b_{00} \in \mathbb{R}$. *We have* $V(x,y) = x(b_{00} + 4y^2)$.

Proof: We write $V(x,y)$ as in (2.17). Equation (★) can be written as a system of $k+2$ homogeneous equations. From the homogeneous equation of degree $k+1$,

$$-8yV_k + (b_{20}x^2 + 4y^2)\frac{\partial V_k}{\partial y} = 0,$$

we get $V_k(x,y) = (b_{20}x^2 + 4y^2)\, x^{k-2}$. The homogeneous equation of degree k is

$$4x^{k-2}\big((k-1)b_{20}x^2 + 4b_{10}xy - 4(k-3) - 8yV_{k-1} + (b_{20}x^2+4y^2)\frac{\partial V_{k-1}}{\partial y} = 0,$$

from which we obtain

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}(b_{20}x^2+4y^2) - 2x^{k-2}(b_{10}x+2y) - \frac{2(k-2)x^{k-3}(b_{20}x^2+4y^2)}{\sqrt{b_{20}}} \arctan\left(\frac{2y}{\sqrt{b_{20}}x}\right)$$

if $b_{20} > 0$,

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}(b_{20}x^2+4y^2) - 2x^{k-2}(b_{10}x+2y) + \frac{2(k-2)x^{k-3}(b_{20}x^2+4y^2)}{\sqrt{-b_{20}}} \operatorname{arctanh}\left(\frac{2y}{\sqrt{-b_{20}}x}\right)$$

if $b_{20} < 0$, and

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}y^2 - 2b_{10}x^{k-1} + 4(k-3)x^{k-2}y$$

if $b_{20} = 0$. In all cases, $C_{k-1} \in \mathbb{R}$. If $k = 2$ then system $A_2V^2 = \mathbf{0}$,

$$\begin{pmatrix} -1 & 0 & b_{00}/4 & & & \\ 0 & 0 & -b_{10}/2 & 0 & b_{00}/4 & \\ -2 & 0 & -1 & 0 & 0 & b_{00}/2 \\ & 0 & b_{20}/4 & 1 & -b_{10}/2 & 0 \\ & -2 & 0 & 0 & 0 & -b_{10} \\ & & -1 & 0 & 0 & -1 \\ & & & 0 & b_{20}/4 & 0 \\ & & & -2 & 0 & b_{20}/2 \\ & & & & -1 & 0 \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2} \end{pmatrix} = \mathbf{0},$$

gives us the conditions and the solution of statement (VIII.7), using the change $\sqrt{|b_{20}|}x/2 \to x$ if $b_{20} \neq 0$. The parameter δ is the sign of b_{20}.

If $k \neq 2$, then we must take $b_{20} = 0$. From the homogeneous equation of degree $k-1$ we obtain the expression of $V_{k-2}(x,y)$,

$$V_{k-2}(x,y) = C_{k-2}x^{k-4}y^2 + C_{k-1}(k-4)x^{k-3}y + \frac{2b_{00} - C_{k-1}b_{10} + 4(k-3)^2}{2}x^{k-2} - \frac{2b_{10}(2k-5)}{3y}x^{k-1},$$

$C_{k-2} \in \mathbb{R}$. In order to obtain a polynomial, we must take $b_{10} = 0$, so $\dot{y} = b_{00}/4 + y^2$. In this case, the polynomial of degree 3 stated in (VIII.8) is a solution of ($\bigstar$). We note that this case includes (VIII.7) with $\delta = 0$. ∎

2.3.3 The case $P(x,y) = y$

We consider the quadratic system

$$\dot{x} = y, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.29}$$

where $d, a, b, l, m, n \in \mathbb{R}$. If $n = 0$ and $m \neq 0$, then this system is transformed into

$$\dot{x} = a_{00} + a_{10}x + y, \quad \dot{y} = b_{00} + b_{01}y + xy, \tag{2.30}$$

by the affine change $m^2x \to x$, $(Ax+y+B)m^3 \to y$, $t/m \to t$, where $a_{00} = -Bm^3$, $a_{10} = -l$, $b_{00} = (d-(A+b)B)m^4$, $b_{01} = l + bm$ and $A = l/m$, $B = (am^2 - l^2 - blm)/m^3$. If $n \neq 0$ then system (2.29) becomes

$$\dot{x} = a_{00} + a_{10}x + y, \quad \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + y^2, \tag{2.31}$$

where

$$\begin{array}{lll} a_{00} = -(m+2bn)/2, & a_{10} = -m, & b_{00} = 4dn^3 - (m+2bn)^2/4, \\ b_{10} = -m^2 - 2n(bm - 2an), & b_{20} = 4ln - m^2, & \end{array}$$

by the affine change $nx \to x$, $mnx + 2n^2y + (m+2bn)/2 \to y$, $2nt \to t$.

The set of conditions on the coefficients of system (2.29) with $n = m = 0$, system (2.30) and system (2.31) in order to have a polynomial inverse integrating factor are stated in the following three propositions.

Proposition 2.3.10. *A system of type* (2.29) *with $n = m = 0$ and $l \neq 0$ having a polynomial inverse integrating factor $V(x,y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = y$, $\dot{y} = Q(x,y)$, where*

(IV.1) $Q(x,y) = -b_{00} + x^2$ *where $b_{00} \in \mathbb{R}$. The system is Hamiltonian, so we have $V(x,y) = 1$.*

Proof: Under the hypotheses of the proposition, if $b = 0$ then we get (IV.1) after the change $(2lx + a, 2\sqrt{2}ly, t/\sqrt{2}) \to (x,y,t)$, where $b_{00} = a^2 - 4dl$.

We shall prove that (★) has no polynomial solution under the hypotheses of the proposition and assuming $b \neq 0$. If $k \leq 3$, straightforward computations show that there is no solution, so we assume $k > 3$. Now we transform our system into

$$\dot{x} = y, \quad \dot{y} = D + Ax + y + x^2,$$

by the change $lx/b^2 \to x$, $ly/b^3 \to y$, $bt \to t$, where $D = dl/b^4$ and $A = a/b^2$. Next we write a solution $V(x,y)$ of degree k as in (2.17). From the homogeneous equation of degree $k+1$ of equation ($\bigstar$), that is $x^2\partial V_k/\partial y = 0$, we get $V_k(x,y) = x^k$. From the homogeneous equation of degree k, which is

$$kx^{k-1}y - x^k + x^2\frac{\partial V_{k-1}}{\partial y} = 0,$$

we obtain $V_{k-1}(x,y) = x^{k-3}\big(C_{k-1}x^2 + xy - ky^2/2\big)$, $C_{k-1} \in \mathbb{R}$. If $j \in \{1, \ldots, k-1\}$, then the homogeneous equation of degree $k-j$ is

$$D\frac{\partial V_{k-j+1}}{\partial y} - V_{k-j} + y\frac{\partial V_{k-j}}{\partial x} + (Ax+y)\frac{\partial V_{k-j}}{\partial y} + x^2\frac{\partial V_{k-j-1}}{\partial y} = 0. \qquad (2.32)$$

An easy induction argument shows that the degree in y of V_{k-j} is $2j$, for all j. Let $v_{k-3j,2j}$ and $v_{k-3j+1,2j-1}$ be the respective coefficients of the monomials $x^{k-3j}y^{2j}$ and $x^{k-3j+1}y^{2j-1}$ of V_{k-j}. Now we take, from equation (2.32), the two equations associated to the coefficients of the monomials of maximum degree in y which are, respectively, of degree $2j+1$ and $2j$:

$$2(j+1)v_{k-3(j+1),2(j+1)} + (k-3j)v_{k-3j,2j} = 0, \qquad (2.33)$$

$$(2j+1)v_{k-3j-2,2j+1} + (k-3j+1)v_{k-3j+1,2j-1} + (2j-1)v_{k-3j,2j} = 0. \qquad (2.34)$$

We remark that $v_{k,0} = 1$. From (2.33), we get for $j = 1, \ldots, [k/3]$

$$v_{k-3j,2j} = \frac{1}{j!(-2)^j}\prod_{i=0}^{j-1}(k-3i) \neq 0. \qquad (2.35)$$

If $k = 3p + l$, $p \in \mathbb{N}$, $l \in \{1,2\}$, then we consider equation (2.33) with $j = (k-l)/3 = p \in \mathbb{N}$:

$$2(p+1)v_{l-3,2(p+1)} + l\, v_{k-3p,2p} = 0.$$

Since $0 < l < 3$, we have $v_{l-3,2(p+1)} = 0$, and then $v_{k-3p,2p} = 0$ in contradiction with (2.35). So we must take $k = 3p$, $p \in \mathbb{N}\setminus\{1\}$. If $1 \le j \le p$ then we can isolate $v_{3(p-j),2j}$ and $v_{3(p-j)+1,2j-1}$ from equations (2.33) and (2.34), respectively:

$$v_{3(p-j),2j} = \frac{3^j}{j!(-2)^j}\prod_{i=0}^{j-1}(p-i) \neq 0,$$

$$v_{3(p-j)+1,2j-1} = -\frac{3(p-j)+4}{2j-1}v_{3(p-j)+4,2j-3} - \left(\frac{-3}{2}\right)^{j-1}\frac{(2j-3)\prod_{i=0}^{j-2}(p-i)}{(2j-1)(j-1)!}.$$

Equation (2.34) with $j = p$, which is

$$v_{1,2p-1} + (2p-1)v_{0,2p} = 0, \qquad (2.36)$$

can be rewritten as

$$(-1)^p\sum_{j=0}^{p}\binom{p}{j}\left(\frac{3}{2}\right)^{p-j}\frac{(2p-2j-1)\prod_{i=0}^{j-1}(3i+1)}{\prod_{i=1}^{j}(2p-2i+1)} = 0,$$

which is equivalent to

$$\frac{\left(\frac{-3}{2}\right)^p (6p-5)\Gamma(1/6)\Gamma(1/2-p)}{5\sqrt{\pi}\,\Gamma(1/6-p)} = 0,$$

where Γ is the Euler gamma function. This equation does not hold for $p \in \mathbb{N}$, so we are again in contradiction. Then we do not obtain any polynomial solution from equation ($\bigstar$). ∎

Proposition 2.3.11. *A system of type* (2.30) *having a polynomial inverse integrating factor $V(x,y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = y$, $\dot{y} = Q(x,y)$, where*

(IV.2) $Q(x,y) = x(\delta + y)$ *with* $\delta = \pm 1$, *and we get* $V(x,y) = \delta + y$.

Proof: First we compute the solutions of degree $k \leq 2$. System $A_1 V^1 = \mathbf{0}$ is

$$\begin{pmatrix} -(a_{10}+b_{01}) & a_{00} & b_{00} \\ -1 & -b_{01} & 0 \\ & 1 & -a_{10} \\ & -1 & 0 \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \end{pmatrix} = \mathbf{0}.$$

From this system, we take $a_{10} = b_{00} = 0$ and $a_{00} \neq 0$ (otherwise, the system has a common factor). After the change $((x+b_{01})/\sqrt{|a_{00}|}, (y+a_{00})/|a_{00}|, \sqrt{|a_{00}|}) \to (x,y,t)$, we get statement (IV.2), where δ is the sign of a_{00}.

System $A_2 V^2 = \mathbf{0}$, which is

$$\begin{pmatrix} -(a_{10}+b_{01}) & a_{00} & b_{00} & & & \\ -1 & -b_{01} & 0 & 2a_{00} & b_{00} & \\ & 1 & -a_{10} & 0 & a_{00} & 2b_{00} \\ & -1 & 0 & a_{10}-b_{01} & 0 & 0 \\ & & & 2 & 0 & 0 \\ & & & 0 & 1 & b_{01}-a_{10} \\ & & & -1 & 0 & 0 \\ & & & & & 1 \end{pmatrix} \begin{pmatrix} v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2} \end{pmatrix} = \mathbf{0},$$

has no non–trivial solution.

In order to prove that there is no solution of degree $k > 2$, we write $V(x,y)$ as in (2.17). From the homogeneous equation of degree $k+1$ of equation ($\bigstar$),

$$-xV_k + xy\frac{\partial V_k}{\partial y} = 0,$$

we get $V_k(x,y) = x^{k-1}y$. The homogeneous equation of degree k is

$$x^{k-2}y\big((k-2)a_{10}x + (k-1)y\big) - xV_{k-1} + xy\frac{\partial V_{k-1}}{\partial y} = 0,$$

from which

$$V_{k-1}(x,y) = x^{k-3}y(C_{k-1} - (k-1)y) - a_{10}(k-2)x^{k-2}\log y,$$

$C_{k-1} \in \mathbb{R}$. As we are assuming $k > 2$, we take $a_{10} = 0$. Solving the homogeneous equation of degree $k-1$, we get

$$V_{k-2}(x,y) = C_{k-2}x^{k-3}y + (b_{01}(k-1) - C_{k-1}(k-2))x^{k-4}y^2 + \frac{(k-1)(k-3)}{2}x^{k-5}y^3 - a_{00}(k-1)x^{k-3}y\log y,$$

$C_{k-2} \in \mathbb{R}$. Again, as we are assuming $k > 2$, we take $a_{00} = 0$. Now from the homogeneous equation of degree $k-2$ we get

$$\begin{aligned} V_{k-3}(x,y) = {} & b_{00}(C_{k-1} - b_{01})x^{k-3} + C_{k-3}x^{k-4}y + \\ & (b_{01}C_{k-1}(k-2) - C_{k-2}(k-3) - b_{01}^2(k-1))x^{k-5}y^2 + \\ & (C_{k-1}(k-2)(k-4) - b_{01}(k-1)(2k-7))\frac{x^{k-6}y^3}{2} - \\ & (k-1)(k-3)(k-5)\frac{x^{k-7}y^4}{6} + b_{00}kx^{k-4}y\log y, \end{aligned}$$

$C_{k-3} \in \mathbb{R}$. As $k > 0$, we take $b_{00} = 0$ and then the system has a common factor. ∎

Proposition 2.3.12. *A system of type* (2.31) *having a polynomial inverse integrating factor* $V(x,y)$ *must satisfy the conditions*

$$a_{10}b_{10} - 2a_{00}b_{20} = 0, \quad a_{10}^3 + (4b_{00} + b_{20})a_{10} - 2a_{00}b_{10} = 0. \tag{2.37}$$

Then, the expression of $V(x,y)$ *is*

$$V(x,y) = a_{10}^2 + 2b_{00} + b_{10} + b_{20} + 2(b_{10} + b_{20})x - 2a_{10}y + 2b_{20}x^2 + 2y^2.$$

More precisely, by applying the conditions (2.37) *on* Q *and* V*, five families of systems arise:*

(IV.3a) $Q(x,y) = \delta + y^2$ *where* $\delta^2 = 1$*, and we get* $V = \delta + y^2$.

(IV.3b) $Q(x,y) = \delta x + y^2$ *where* $\delta^2 = 1$*, and we get* $V = \delta(2x+1) + 2y^2$.

(IV.3c) $Q(x,y) = D + \delta x^2 + y^2$ *where* $D = (4b_{00}b_{20} - b_{10}^2)/(4b_{20}|b_{20}|)$*,* $\delta^2 = 1$*, and we have* $V = 2D + \delta + 2\delta x(1+x) + 2y^2$.

(IV.3d) *$Q(x,y) = D + 1/4 + y + y^2$ where $D = b_{00}/(4a_{00}^2) \neq -1/4$, and we get $V = 4D + (2y+1)^2$.*

(IV.3e) *$Q(x,y) = -(D+1/16) - y/2 + 4(D+1/16)x^2 + xy + y^2$, where $D = b_{20}/(16a_{10}^2) \neq -1/16$, and we get $V = 16D(2x+1)^2 + (1+2x+4y)^2$.*

Proof: We first prove that a polynomial solution $V(x,y)$ must have degree $k=2$. We write $V(x,y)$ as in (2.17). Equation (★) is a polynomial equation of degree $k+1$, so we can transform it into a system of $k+2$ homogeneous equations. From the homogeneous equation of degree $k+1$,

$$-2yV_k + (b_{20}x^2 + y^2)\frac{\partial V_k}{\partial y} = 0,$$

we get $V_k(x,y) = x^{k-2}(b_{20}x^2 + y^2)$. The homogeneous equation of degree k is

$$x^{k-3}\big(a_{10}b_{20}(k-1)x^3 + (2b_{10} + b_{20}k)x^2y + a_{10}(k-3)xy^2 + (k-2)y^3\big) - 2yV_{k-1} + (b_{20}x^2 + y^2)\frac{\partial V_{k-1}}{\partial y} = 0,$$

so we get

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}(b_{20}x^2 + y^2) + x^{k-2}((b_{10} + b_{20})x - a_{10}y) - \frac{k-2}{2}x^{k-3}(b_{20}x^2 + y^2)\log(b_{20}x^2 + y^2) - \frac{a_{10}(k-2)(b_{20}x^2 + y^2)}{\sqrt{b_{20}}}x^{k-3}\arctan\left(\frac{y}{\sqrt{b_{20}}x}\right)$$

if $b_{20} > 0$,

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}(b_{20}x^2 + y^2) + x^{k-2}((b_{10} + b_{20})x - a_{10}y) - \frac{k-2}{2}x^{k-3}(b_{20}x^2 + y^2)\log(b_{20}x^2 + y^2) - \frac{a_{10}(k-2)(b_{20}x^2 + y^2)}{\sqrt{-b_{20}}}x^{k-3}\arctan\left(\frac{y}{\sqrt{-b_{20}}x}\right)$$

if $b_{20} < 0$, and

$$V_{k-1}(x,y) = C_{k-1}x^{k-3}y^2 + x^{k-2}(b_{10}x + a_{10}(k-3)y) - (k-2)x^{k-3}y^2\log y$$

if $b_{20} = 0$. In all cases, $C_{k-1} \in \mathbb{R}$. As V_{k-1} is a polynomial, we take $k=2$. Now from system $A_2V^2 = \mathbf{0}$, which is

$$
\begin{pmatrix}
-a_{10} & a_{00} & b_{00} & & & \\
0 & 0 & b_{10} & 2a_{00} & b_{00} & \\
-2 & 1 & -a_{10} & 0 & a_{00} & 2b_{00} \\
 & 0 & b_{20} & a_{10} & b_{10} & 0 \\
 & -2 & 0 & 2 & 0 & 2b_{10} \\
 & & -1 & 0 & 1 & -a_{10} \\
 & & & 0 & b_{20} & 0 \\
 & & & -2 & 0 & 2b_{20} \\
 & & & & -1 & 0
\end{pmatrix}
\begin{pmatrix}
v_{0,0} \\ v_{1,0} \\ v_{0,1} \\ v_{2,0} \\ v_{1,1} \\ v_{0,2}
\end{pmatrix}
= \mathbf{0},
$$

we get the solution

$$V(x, y) = a_{10}^2 + 2b_{00} + b_{10} + b_{20} + 2(b_{10} + b_{20})x - 2a_{10}y + 2b_{20}x^2 + 2y^2$$

and the conditions $a_{10}b_{10} - 2a_{00}b_{20} = 0$, $a_{10}^3 + (4b_{00} + b_{20})a_{10} - 2a_{00}b_{10} = 0$. Next we distinguish five cases, depending on the values of the parameters:

1. If $a_{10} = a_{00} = b_{20} = b_{10} = 0$ and $b_{00} \neq 0$, then we get (IV.3a) by the change $(y/\sqrt{|b_{00}|}, \sqrt{|b_{00}|}t) \to (y, t)$.

2. If $a_{10} = a_{00} = b_{20} = 0$ and $b_{10} \neq 0$, then we get (IV.3b) by the change $(x + b_{00}/b_{10}, y/\sqrt{|b_{10}|}, \sqrt{|b_{10}|}t) \to (x, y, t)$.

3. If $a_{10} = a_{00} = 0$ and $b_{20} \neq 0$, then we get (IV.3c) by using the change $(x + b_{10}/(2b_{20}), y/\sqrt{|b_{20}|}, \sqrt{|b_{20}|}t) \to (x, y, t)$.

4. If $a_{10} = b_{20} = b_{10} = 0$ and $a_{00} \neq 0$, then we get (IV.3d) by the change $(-y/2a_{00} - 1/2, -2a_{00}t) \to (y, t)$.

5. If $a_{10} \neq 0$, $b_{10} = 2a_{00}b_{20}/a_{10}$ and $b_{00} = -(a_{10}^2 + b_{20})/4 + a_{00}^2 b_{20}/a_{10}^2$, then by the change $(x + a_{00}/a_{10}, -x/2 - y/(2a_{10}) - a_{00}/a_{10}, -2a_{10}t) \to (x, y, t)$ we get (IV.3e).

■

2.3.4 The case $P(x, y) = y + x^2$

We consider the quadratic system

$$\dot{x} = y + x^2, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.38}$$

where $d, a, b, l, m, n \in \mathbb{R}$. If $n \neq 0$ then this system becomes

$$\dot{x} = a_{00} + a_{10}x + y + x^2, \quad \dot{y} = \frac{b_{00}}{4} - \frac{b_{10}}{2}x - \frac{b_{20}}{4}x^2 + y^2, \tag{2.39}$$

by the affine change $nx \to x$, $mnx/2 + n^2y + (m + 2bn)/4 \to y$, $t/n \to t$, where $a_{00} = -(m + 2bn)/4$, $a_{10} = -m/2$, $b_{00} = (4dn - b^2)n^2 - m(m + 4bn)/4$, $b_{10} = m^2/2 + n(bm - 2an)$ and $b_{20} = m(m - 2) - 4ln$.

The set of conditions on the coefficients of system (2.38) with $n = 0$ and system (2.39) in order to have a polynomial inverse integrating factor are stated in the following two propositions.

Proposition 2.3.13. *A system of type* (2.38) *with $n = 0$ having a polynomial inverse integrating factor $V(x, y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = y + x^2$, $\dot{y} = Q(x, y)$, where Q is one of the polynomials below.*

(III.1) $Q(x, y) = b_{00} + b_{10}x - 2xy$ *where* $b_{00} = (54d - 9al + l^3)/2$ *and* $b_{10} = 3(6a - l^2)/2$. *The system is Hamiltonian.*

(III.2) *Assume* $2b(m - 1) + l(m + 2) = 0$, $l^3m + 2al(m - 1)^2 + 4d(m - 1)^3 = 0$ *and* $m \neq -2$. *We distinguish four cases depending on the value of* m.

(III.2a) *If* $m = 0$ *and* $a \neq 0$, *then* $Q(x, y) = \delta x$ *where* $\delta^2 = 1$, *and we get* $V(x, y) = \delta + 2(y + x^2)$.

(III.2b) *If* $m = 1$, *then* $Q(x, y) = b_{00} + b_{10}x + xy$ *where* $b_{00} = -27d - 9ab + 2b^3$, $b_{10} = 9a + 3b^2$, *and we get*

$$V(x, y) = b_{00}^2 + b_{00}x(2b_{10} + 3y + x^2) + (b_{10} + y)(b_{10}x^2 - y^2).$$

(III.2c) *If* $m = 2$, *then* $Q(x, y) = x(b_{10} + 2y)$ *where* $b_{10} = 4a + 6l^2$, *and we get* $V(x, y) = (b_{10} + 2y)^2$.

(III.2d) *If* $m \neq -2, 0, 1, 2$, *then* $Q(x, y) = x(b_{10} + my)$ *where* $b_{10} = 4a(m - 1)^2 + 3l^2m$, *and we have* $V(x, y) = (b_{10} + my)(b_{10} + 2y - (m - 2)x^2)$.

(III.3) $Q(x, y) = 1 + xy/2$, *and we get* $V(x, y) = (2x - y^2)(2 + 3xy - y^3)$.

(III.4) $Q(x, y) = 1 + b_{10}x + 4xy$ *where* $b_{10} = 2^{2/3}3(3a + l^2)/(54d + 9al + 2l^3)^{2/3} \in \mathbb{R}$, *and we have*

$$V(x, y) = 1 + 2x(b_{10} + 3y - x^2) + (b_{10} + 2y - 2x^2)(b_{10}x^2 - (y - x^2)^2).$$

(III.5) $Q(x, y) = 1 + 8xy$, *and we get*

$$V(x, y) = \left[(3x^2 - y)^2 - 2x\right]\left[1 - 2(3x^2 - y)(3x - (3x^2 - y)^2)\right].$$

Proof: The systems with $n = 0$ having a polynomial inverse integrating factor of degree $k \leq 4$ are obtained solving the linear systems $A_iV^i = \mathbf{0}$, $i = 1, \ldots, 4$. The results are stated in (III.1) and (III.2), where an affine change of variables has been applied in each case.

Assume $k > 4$. We write $V(x, y)$ as in (2.17). By Lemma 2.3.1, we get

$$V_k(x,y) = x^{2-mp}\left(\frac{l}{m-1}x + y\right)^{p+1} \tag{2.40}$$

and $m = 1-(k-3)/p \neq 1$, where $p \in \{-1, 1, 2, \ldots, k-1\}$.

If $m = 0$ then as $k > 4$ we get $a = d = 0$ and $b = l$ after straightforward computations and the system has a common factor. Then, from now on we assume $m \neq 0, 1$.

The homogeneous equation of degree $k - j$ of (★), $j = -1, \ldots, k$, is

$$\begin{aligned}(lx+my)x\frac{\partial V_{k-j-1}}{\partial y} - (m+2)xV_{k-j-1} + y\frac{\partial V_{k-j}}{\partial x} + (ax+by)\frac{\partial V_{k-j}}{\partial y} - \\ bV_{k-j} + d\frac{\partial V_{k-j+1}}{\partial y} = 0,\end{aligned} \tag{2.41}$$

taking $V_i \equiv 0$ if $i \notin \{0, \ldots, k\}$. From (2.40), we can write $V_k(x, y) = x^{k-p-1}y^{p+1} + \cdots$, where the dots mean lower order terms in y. If $V_{k-1}(x, y) = v_s x^{k-s-1}y^s + \cdots$, then equation (2.41) for $j = 0$ is

$$(m(s-1)-2)v_s x^{k-s}y^s + (k-p-1)x^{k-p-2}y^{p+2} + bp\, x^{k-p-1}y^{p+1} + \cdots = 0.$$

If $k = p+1$ and $b \neq 0$, then $s = p+1$. If $p = 4$, then we obtain (III.3) after the change $((x+l)/\gamma, (-2lx+y-l^2)/\gamma^2, \gamma t) \to (x, y, t)$, where $\gamma = -|d + l^3/2|^{1/3}$. If $p \neq 4$, then after some computations we are under the conditions of (III.2). If $k = p+1$ and $b = 0$, then from straightforward computations we obtain $d = l = 0$, so we are in (III.2) again. So from now on we assume $k \neq p+1$.

We claim that the degree of V_{k-j} in y is $p+j+1$ if $k-p-1-2j \geq 0$. We prove this claim using the induction principle. From the computations above, the degree of V_k in y is $p+1$ and the degree of V_{k-1} in y is $p+2$. Next we write $V_{k-j+2}(x,y) = v_2 x^{k-p+3-2j}y^{p+j-1} + \cdots$, $V_{k-j+1}(x,y) = v_1 x^{k-p+1-2j}y^{p+j} + \cdots$ and $V_{k-j}(x,y) = v_0 x^{k-s-j}y^s + \cdots$, $s \in \mathbb{Z}$. From the equality (2.41), as $k-p+1-2j > 0$ we get

$$(m(s-1)-2)v_0 x^{k-s-j+1}y^s + (k-p+1-2j)v_1 x^{k-p-2j}y^{p+j+1} + \cdots = 0,$$

so $s = p+j+1$ as we wanted. Then we can write

$$V_{k-j}(x,y) = v_{p+j+1}^{k-p-1-2j}x^{k-p-1-2j}y^{p+j+1} + v_{p+j}^{k-p-2j}x^{k-p-2j}y^{p+j} + \cdots \tag{2.42}$$

We note that $v_{p+1}^{k-p-1} \neq 0$. From equation (2.41), the equations associated to the terms of degree $p+j+2$, $p+j+1$, $p+j$ and $p+j-1$ in y are, respectively,

$$(k-p-1-2j)v_{p+j+1}^{k-p-1-2j} - (q+1)(j+1)v_{p+j+2}^{k-p-3-2j} = 0, \tag{2.43}$$

$$(k-p-2j)v_{p+j}^{k-p-2j}-((q+1)j+1)v_{p+j+1}^{k-p-2-2j}+b(p+j)v_{p+j+1}^{k-p-1-2j}+ \\ l(p+j+2)v_{p+j+2}^{k-p-3-2j}=0, \tag{2.44}$$

$$(k-p+1-2j)v_{p+j-1}^{k-p+1-2j}-((q+1)j-q+1)v_{p+j}^{k-p-1-2j}+b(p+j-1)v_{p+j}^{k-p-2j}+ \\ l(p+j+1)v_{p+j+1}^{k-p-2-2j}+a(p+j+1)v_{p+j+1}^{k-p-1-2j}=0 \tag{2.45}$$

and

$$(k-p+2-2j)v_{p+j-2}^{k-p+2-2j}-((q+1)j-2q+1)v_{p+j-1}^{k-p-2j}+b(p+j-2)v_{p+j-1}^{k-p+1-2j}+ \\ l(p+j)v_{p+j}^{k-p-1-2j}+a(p+j)v_{p+j}^{k-p-2j}+d(p+j)v_{p+j+1}^{k-p-1-2j}=0. \tag{2.46}$$

If $k-p$ is even, then from equation (2.43) we obtain $v_{p+1}^{k-p-1}=0$, a contradiction. So in what follows we assume that $k-p$ is odd. Let $C=(k-p-1)/2\in\mathbb{N}$. From equations (2.43)–(2.46) we obtain

$$v_{p+j+1}^{k-p-1-2j}=\left(\frac{2}{q+1}\right)^{j}\binom{C}{j}, \tag{2.47}$$

for $0\le j\le C$;

$$v_{p+j}^{k-p-2j}=\sum_{i=1}^{j}\frac{b(p+j-i)\prod_{s=1}^{i-1}(k-p-2(j-s))}{\prod_{s=1}^{i}((q+1)(j-s)+1)}v_{p+j+1-i}^{k-p+1-2(j+1-i)}+ \\ \sum_{i=0}^{j}\frac{l(p+j+1-i)\prod_{s=1}^{i}(k-p-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)+1)}v_{p+j+1-i}^{k-p+1-2(j+1-i)}, \tag{2.48}$$

for $0\le j\le C$;

$$v_{p+j-1}^{k-p+1-2j}=\sum_{i=0}^{j-1}\frac{a(p+j-i)\prod_{s=1}^{i}(k-p+1-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)-(q-1))}v_{p+j-i}^{k-p+1-2(j-i)}+ \\ \sum_{i=0}^{j}\frac{l(p+j-i)\prod_{s=1}^{i}(k-p+1-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)-(q-1))}v_{p+j-i}^{k-p-2(j-i)}+ \\ \sum_{i=1}^{j}\frac{b(p+j-i-1)\prod_{s=1}^{i-1}(k-p+1-2(j-s))}{\prod_{s=1}^{i}((q+1)(j-s)-(q-1))}v_{p+j-i}^{k-p-2(j-i)}, \tag{2.49}$$

for $0 \leq j \leq C+1$; and

$$v_{p+j-2}^{k-p+2-2j} = \sum_{i=0}^{j-2} \frac{d(p+j-i-1)\prod_{s=1}^{i}(k-p+2-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)-(2q-1))} v_{p+j-1-i}^{k-p+1-2(j-1-i)} +$$
$$\sum_{i=0}^{j-1} \frac{a(p+j-i-1)\prod_{s=1}^{i}(k-p+2-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)-(2q-1))} v_{p+j-1-i}^{k-p-2(j-1-i)} +$$
$$\sum_{i=0}^{j} \frac{l(p+j-i-1)\prod_{s=1}^{i}(k-p+2-2(j-s))}{\prod_{s=1}^{i+1}((q+1)(j-s)-(2q-1))} v_{p+j-1-i}^{k-p-1-2(j-1-i)} +$$
$$\sum_{i=1}^{j} \frac{b(p+j-i-2)\prod_{s=1}^{i-1}(k-p+2-2(j-s))}{\prod_{s=1}^{i}((q+1)(j-s)-(2q-1))} v_{p+j-1-i}^{k-p-1-2(j-1-i)}, \qquad (2.50)$$

for $0 \leq j \leq C+1$. We note that, as $m \neq 0, 1$, the only value of m for which some denominators in the above expressions could vanish is $m = -1$. In this case, straightforward computations show that we are under the conditions of (III.2).

Equation (2.44) with $j = C$ and equation (2.46) with $j = C+1$ are, respectively,

$$v_{C+p}^{1} + b(C+p)v_{C+p+1}^{0} = 0 \qquad (2.51)$$

and

$$v_{C+p-1}^{1} + b(C+p-1)v_{C+p}^{0} + d(C+p+1)v_{C+p+1}^{0} = 0. \qquad (2.52)$$

From these two equations we will obtain either the conditions of (III.2) or two new families of quadratic systems having a polynomial inverse integrating factor. Equation (2.51) becomes, after some computations,

$$\sum_{i=0}^{C} \Big[\big((p+C-i)((q+1)(C-1-i)+1)b + (p+C+1-i)(2i+1)l\big)$$
$$\left(\frac{2}{q+1}\right)^{C-i} \binom{C}{i} \prod_{s=1}^{i}(2s-1) \prod_{s=i+1}^{C} ((q+1)(C-1-s)+1)\Big] = 0,$$

where we have dropped the coefficient $v_{p+1}^{k-p-1} \neq 0$. Factorizing this expression with the help of Mathematica software (see [49]) we obtain

$$(k+p-2)C!\big[2bq + l(q-3)\big] \prod_{i=2}^{C}(2i(q+1)-(q-1)) = 0.$$

The factor $2bq + l(q-3) = 2b(m-1) + l(m+2)$ is the only one which can vanish in this expression, so we must take $2b(m-1) + l(m+2) = 0$. We note that this is the first condition of (III.2).

Equation (2.52) becomes

$$
\begin{aligned}
0 = & \sum_{i=0}^{C-1} \frac{d(C+p-i)\prod_{s=1}^{i}(2s+1)}{\prod_{s=1}^{i+1}((q+1)(C+1-s)-(2q-1))} v_{C+p-i}^{2(i+1)} + \\
& \sum_{i=0}^{C} \frac{a(C+p-i)\prod_{s=1}^{i}(2s+1)}{\prod_{s=1}^{i+1}((q+1)(C+1-s)-(2q-1))} v_{C+p-i}^{2i+1} + \\
& \sum_{i=0}^{C+1} \frac{l(C+p-i)\prod_{s=1}^{i}(2s+1)}{\prod_{s=1}^{i+1}((q+1)(C+1-s)-(2q-1))} v_{C+p-i}^{2i} + \\
& \sum_{i=1}^{C+1} \frac{b(C+p-1-i)\prod_{s=1}^{i-1}(2s+1)}{\prod_{s=1}^{i}((q+1)(C+1-s)-(2q-1))} v_{C+p-i}^{2i} + \\
& \sum_{i=0}^{C} \frac{a\,b(C+p-1)(C+p+1-i)2^i i!}{\prod_{s=1}^{i+1}((q+1)(C+1-s)-(q-1))} v_{C+p+1-i}^{2i} + \\
& \sum_{i=0}^{C+1} \frac{b\,l(C+p-1)(C+p+1-i)2^i i!}{\prod_{s=1}^{i+1}((q+1)(C+1-s)-(q-1))} v_{C+p+1-i}^{2i-1} + \\
& \sum_{i=1}^{C+1} \frac{b^2(C+p-1)(C+p-i)2^{i-1}(i-1)!}{\prod_{s=1}^{i}((q+1)(C+1-s)-(q-1))} v_{C+p+1-i}^{2i-1} + \\
& d(C+p+1)\left(\frac{2}{q+1}\right)^C,
\end{aligned} \tag{2.53}
$$

where the condition $b = -l(q-3)/(2q)$ is to be applied. Using equations (2.47), (2.48) and (2.49) in (2.53), we obtain an equation which is a linear combination of d, al and l^3 equaled to zero. Factorizing this equation with the help of Mathematica software we obtain, after removing the trivially non–zero terms,

$$
\frac{(q-3)\prod_{i=2}^{C}(2i(q+1)-3(q-1))}{\prod_{i=1}^{C}(i(q+1)-(2q-1))}\big(4d(k-3)^3-2alp(k-3)^2+l^3p^2(k-3-p)\big)=0.
$$

If $q=3$ then $m=-2$ and $b=0$, so the system is Hamiltonian. The product in the numerator vanishes if and only if $p=-1$ and either $k=6$ or $k=10$. The product in the denominator vanishes if and only if $m=-1$, but this case has been discarded before. Otherwise, we get $4d(k-3)^3-2alp(k-3)^2+l^3p^2(k-3-p)=0$, which is $4d(m-1)^3+2al(m-1)^2+l^3m=0$, and then we are in (III.2).

If $k=6$ and $p=-1$, then $m=4$ and $b=-l$. If $54d+(9a+2l^2)l=0$, then we are in (III.2). Otherwise, we get a new solution, which is shown in (III.4), after the change $((x-6l)/\gamma, (lx/3+y-l^2/36)/\gamma^2, \gamma t) \to (x,y,t)$, where $\gamma^3 = d+al/6+l^3/27 \neq 0$.

If $k = 10$ and $p = -1$, then $m = 8$ and $b = -5l/7$. If $d = 0$, then we are in (III.2), so we take $d \neq 0$. In order to have a new solution, we must take $a = l = 0$ (and then $b = 0$). We obtain statement (III.5) after the change $(x/d^{1/3}, y/d^{2/3}, d^{1/3}t) \to (x, y, t)$. ■

Proposition 2.3.14. *A system of type* (2.39) *having a polynomial inverse integrating factor $V(x, y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = y + x^2$, $\dot{y} = Q(x, y)$, where $Q(x, y)$ is one of the polynomials below.*

(III.6) $Q(x, y) = -b_{10}/4 - b_{10}x + y + 2xy - y^2/b_{10}$ *where* $b_{10} = 2/(3 + 4a_{00} - a_{10}^2) \in \mathbb{R} \setminus \{0\}$, *and we get* $V(x, y) = (b_{10} - 2y)^2$.

(III.7) $Q(x, y) = y(2x + y)$, *and we get* $V(x, y) = y^2$.

Proof: We can write $V(x, y) = V_0(x) + V_1(x)y + y^2$ using Lemma 2.3.1. Then equation (★) is a polynomial equation of degree 2 in y. It can be transformed into the system

$$\begin{aligned}
&-4(a_{10} + 2)V_0(x) + 4p_2(x)V_0'(x) + q_2(x)V_1(x) = 0,\\
&q_2(x) - 4V_0(x) + 2V_0'(x) - 2(a_{10} + 2)V_1(x) + 2p_2(x)V_1'(x) = 0,\\
&a_{10} + 2x + V_1(x) - V_1'(x) = 0,
\end{aligned}$$

where $p_2(x) = a_{00} + a_{10}x + x^2$ and $q_2(x) = b_{00} - 2b_{10}x - b_{20}x^2$. Solving the second and third equations we obtain $V_0(x)$ and $V_1(x)$. From the first equation we get the solutions stated in (III.6) and (III.7), depending on the value of $3 + 4a_{00} - a_{10}^2$ and after the change $(-b_{10}(x + (a_{10} + 1)/2), b_{10}^2(-x + y + a_{00} - (a_{10} + 1)^2/4), -b_{10}t) \to (x, y, t)$ for (III.6) and the same change taking $b_{10} = -1$ for (III.7). ■

2.3.5 The case $P(x, y) = x^2$

We consider the quadratic system

$$\dot{x} = x^2, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.54}$$

where $d, a, b, l, m, n \in \mathbb{R}$.

If $b = d = n = 0$, then system (2.54) has a common factor. If $b = n = 0$ and $d \neq 0$, then system (2.54) is transformed into one of the following three systems, depending on the value of m.

If $m = 0$ then

$$\dot{x} = x^2, \quad \dot{y} = 1 + \delta x, \tag{2.55}$$

where $\delta = 0, 1$, by the affine change $(ax/d, (-lx + y)/a, dt/a) \to (x, y, t)$ if $a \neq 0$, or the affine change $(y - lx)/d \to y$ if $a = 0$.

If $m = 1$ then

$$\dot{x} = x^2, \quad \dot{y} = 1 + \delta x^2 + xy, \tag{2.56}$$

where $\delta = -1, 0, 1$, by the affine change $(y + a)/d \to y$ if $l = 0$, or the affine change $(\sqrt{|dl|}x/d, (y + a)/\sqrt{|dl|}, dt/\sqrt{|dl|}) \to (x, y, t)$ if $l \neq 0$.

If $m \neq 0, 1$ then

$$\dot{x} = x^2, \quad \dot{y} = 1 + mxy, \tag{2.57}$$

by the affine change $(x/d, lx/(m - 1) + y + a/m, dt) \to (x, y, t)$.

If $n = 0$ and $b \neq 0$, then system (2.54) is transformed into one of the following two systems, depending on the value of the expression $b^2l - (m - 1)(ab - dm)$.

If $b^2l - (m - 1)(ab - dm) = 0$ then

$$\dot{x} = x^2, \quad \dot{y} = (1 + mx)y, \tag{2.58}$$

by the affine change $(x/b, (ab - dm)x/b^2 + y + d/b, bt) \to (x, y, t)$.

If $b^2l - (m - 1)(ab - dm) \neq 0$ then

$$\dot{x} = x^2, \quad \dot{y} = y + x^2 + mxy, \tag{2.59}$$

by the affine change $(x/b, B((ab - dm)x/b^2 + y + d/b), bt) \to (x, y, t)$, where $B = b/(b^2l - (m - 1)(ab - dm))$.

If $n \neq 0$, then system (2.54) becomes

$$\dot{x} = x^2, \quad \dot{y} = -\frac{b_{00}}{4} - \frac{b_{10}}{2}x - \frac{b_{20}}{4}x^2 + y^2, \tag{2.60}$$

by the affine change $mx/2 + ny + b/2 \to y$, where $b_{00} = b^2 - 4dn$, $b_{10} = bm - 2an$ and $b_{20} = m(m - 2) - 4ln$.

The set of conditions on the coefficients of the systems (2.55)–(2.60) above to have a polynomial inverse integrating factor are stated in the following three propositions.

Proposition 2.3.15. *The following statements hold.*

(VII.1) *System* (2.57) *with* $m = -2$ *is Hamiltonian.*

(VII.2) *System* (2.57) *with* $m = -1$ *has the polynomial inverse integrating factor* $V(x, y) = x$.

(VII.3) *System* (2.57) *with* $m \neq -2, -1, 0, 1$ *has the polynomial inverse integrating factor* $V(x, y) = x(1 + (m + 1)xy)$.

(VII.4) *System* (2.55) *has the polynomial inverse integrating factor* $V(x,y)=x^2$.

(VII.5) *System* (2.56) *has the polynomial inverse integrating factor* $V(x,y)=x^3$.

Proof: The solutions follow from straightforward computations of the linear systems $A_i V^i = 0$, for $i = 1, 2, 3$. ■

Proposition 2.3.16. *The following two statements hold.*

(VII.6) *System* (2.58) *has the polynomial inverse integrating factor* $V(x,y) = x^2 y$.

(VII.7) *System* (2.59) *has a polynomial inverse integrating factor (of degree* $k > 3$*) if and only if* $m = k-2$. *Its expression is given by*

$$V(x,y) = x^2 y + (k-4)! \sum_{i=0}^{k-4} \frac{(-x)^{k-i}}{i!}.$$

Moreover, system (2.59) *has no polynomial inverse integrating factors of degree* $k \leq 3$.

Proof: Statement (VII.6) follows easily from the computation of $A_3 V^3 = 0$ for system (2.58). Moreover, straightforward computations show that there is no solution of degree $k \leq 3$ for system (2.59).

We write a polynomial inverse integrating factor $V(x,y)$ of (2.59) as in (2.17). The homogeneous equation of degree $k+1$ of equation (★) is

$$(k-m-2)xV_k + x(x+(m-1)y)\frac{\partial V_k}{\partial y} = 0.$$

We get $V_k(x,y) = x^{k-1+\frac{k-3}{m-1}}(x+(m-1)y)^{1-\frac{k-3}{m-1}}$. Let $p+1$ be the degree of V_k in y. Then, $m = 1 - \frac{k-3}{p} \in \mathbb{Q}$, where $p \in \{-1, 1, 2, \ldots, k-1\}$. Next we find the maximum degree in y of V. For that purpose, we write it as a polynomial of degree $s \geq 0$ in y:

$$V(x,y) = \sum_{i=0}^{s} W_i(x) y^i.$$

We can write equation (★) as a polynomial equation in y. Then, all the coefficients of (★) (which depend on x) must vanish. The coefficient of y^{s+1} is zero and the coefficient of y^s is

$$((1+mx)(s-1) - 2x)W_s(x) + x^2 W_s'(x) = 0.$$

Solving this equation we get $W_s(x) = e^{(s-1)/x}x^{3-s+(k-3)(s-1)/p}$. As $W_s(x)$ is a polynomial, we take $s = 1$. Then, $V(x,y) = x^2y + W_0(x)$, and $W_0(x)$ is a polynomial of degree $k > 3$. As $V_k(x,y)$ does not have any term in y, we have $p = -1$, and then $m = k-2$. Now we can solve the whole equation (★), which is

$$x^4 - (1+kx)W_0(x) + x^2W_0'(x) = 0,$$

to get the solution in (VII.7). ■

Proposition 2.3.17. *System* (2.60) *has a polynomial inverse integrating factor* $V(x,y)$ *if and only if one of the following statements hold.*

(VII.8) $b_{00} = b_{10} = b_{20} = 0$. *Then* $V(x,y) = (x-y)^2$.

(VII.9) $b_{00} = b_{10} = 0$ *and* $b_{20} \neq 0$. *Then* $V(x,y) = x(b_{20}x^2 + 4xy - 4y^2)$.

(VII.10) $b_{00} \neq 0$ *and* $b_{10} = b_{20} = 0$. *Then* $V(x,y) = x^2(b_{00} - 4y^2)$.

(VII.11) $b_{20} = (k-2)(k-4) > 0$, $b_{00} > 0$ *and* $b_{10} = (k-2p)\sqrt{b_{00}} \in \mathbb{R}$, *with* $k > 4$ *and* $p \in \{2,\ldots,k-2\}$. *Then* $V(x,y) = x^2\left[p_1^1(x)y + p_2^1(x)\right]\left[p_1^2(x)y + p_2^2(x)\right]$, *where* $p_1^1(x)$ *and* $p_1^2(x)$ *are Laguerre polynomials of respective degree* $p-2$ *and* $k-p-2$ *and the expressions of the polynomials* $p_2^1(x)$ *and* $p_2^2(x)$ *are obtained from* $p_1^1(x)$ *and* $p_1^2(x)$.

We remark that under these conditions the system is simplified to

$$\dot{x} = x^2, \quad \dot{y} = -1 - (k-2p)x - (k-4)xy + y^2$$

applying the change $(2x/\sqrt{b_{00}}, ((k-4)x + 2y)/\sqrt{b_{00}}, \sqrt{b_{00}}t/2) \to (x,y,t)$.

(VII.12) $b_{20} = (k-2)(k-4) > 0$, $b_{00} < 0$ *and* $b_{10} = 0$, *with* $k > 4$ *even. Then* $V(x,y) = x^2\big(f_0(x) + f_1(x)y + f_2(x)y^2\big)$, *where* $f_2(x)$ *is a polynomial of degree* $k-4$ *and the expressions of the polynomials* $f_0(x)$ *and* $f_1(x)$ *can be obtained from* $f_2(x)$.

We remark that under these conditions the system is simplified to

$$\dot{x} = x^2, \quad \dot{y} = 1 - (k-4)xy + y^2$$

by the change $(2x/\sqrt{-b_{00}}, ((k-4)x + 2y)/\sqrt{-b_{00}}, \sqrt{-b_{00}}t/2) \to (x,y,t)$.

Proof: Let $V(x,y)$ be a polynomial inverse integrating factor of (2.60) of degree k. If $k \leq 4$, then straightforward computations show that there is no solution of degree 1; there is a solution of degree 2 if and only if $b_{00} = b_{10} = b_{20} = 0$; there is a solution of degree 3 if and only if $b_{00} = b_{10} = 0$ and $b_{20} \neq 0$; and there is a solution of degree 4 if and only if $b_{00} \neq 0$ and $b_{10} = b_{20} = 0$. The respective expressions of $V(x, y)$ are shown in the proposition. From now on, we assume $k > 4$ and $b_{00}^2 + b_{10}^2 > 0$.

By Lemma 2.3.1, we can write V as a polynomial of degree 2 in y:

$$V(x,y) = W_0(x) + W_1(x)y + W_2(x)y^2,$$

where $W_i(x)$ is a polynomial of degree $k-i$, $i = 0, 1, 2$. We rewrite equation (★) as the differential system (depending on x)

$$\begin{aligned} -8W_0(x) + 4x^2W_0'(x) - (b_{00} + 2b_{10}x + b_{20}x^2)W_1(x) &= 0, \\ 4W_0(x) + 4xW_1(x) - 2x^2W_1'(x) + (b_{00} + 2b_{10}x + b_{20}x^2)W_2(x) &= 0, \\ W_1(x) + 2xW_2(x) - x^2W_2'(x) &= 0. \end{aligned} \tag{2.61}$$

From the second and the third equations, we obtain the expressions of $W_0(x)$ and $W_1(x)$ in terms of $W_2(x)$ and its derivatives:

$$W_1(x) = -2x\, W_2(x) + x^2\, W_2'(x),$$

$$W_0(x) = -\frac{1}{4}\big((b_{00} + 2b_{10}x + (b_{20}+4)x^2)W_2(x) - 4x^3W_2'(x) + 2x^4W_2''(x)\big).$$

Observe that if $W_2(x)$ is a polynomial of degree $k-2$, then $W_1(x)$ and $W_0(x)$ are polynomials of degrees $k-1$ and k, respectively. From the first equation of (2.61) we get

$$(2b_{00} + 3b_{10}x + b_{20}x^2)W_2(x) - x(b_{00} + 2b_{10}x + b_{20}x^2)W_2'(x) + x^5W_2'''(x) = 0. \tag{2.62}$$

As $W_2(x)$ is a polynomial of degree $k-2$, we write $W_2(x) = \sum_{i=0}^{k-2} a_i x^i$. Then equation (2.62) becomes

$$\sum_{i=0}^{k-2} \big((i-2)b_{00} + (2i-3)b_{10}x + (i-1)(b_{20} - i(i-2))x^2\big)a_i x^i = 0. \tag{2.63}$$

The equation corresponding to x^k is $(k-3)(b_{20} - (k-4)(k-2)) = 0$. As $k > 4$, we take $b_{20} = (k-4)(k-2) > 0$. From the equations corresponding to x^0, x^1 and x^2,

$$\begin{aligned} 2b_{00}a_0 &= 0, \\ b_{00}a_1 + 3b_{10}a_0 &= 0, \\ b_{10}a_1 + b_{20}a_0 &= 0, \end{aligned}$$

we get $a_0 = a_1 = 0$. So, we have

$$V(x,y) = x^2\big(\tilde{W}_0(x) + \tilde{W}_1(x)y + \tilde{W}_2(x)y^2\big),$$

for some polynomials $\tilde{W}_i(x)$ of degree $k-2-i$, $i=0,1,2$.

After all these simplifications, equation (2.63) can be written as a $(k-3)\times(k-3)$ homogeneous linear system with unknowns $a_2,\ldots,a_{k-2}$. In order to obtain a non–trivial solution, the determinant of the matrix M_k:

$$\begin{pmatrix} b_{10} & b_{00} & & & & & \\ (k-4)(k-2) & 3b_{10} & 2b_{00} & & & & \\ & 2(k-1)(k-5) & 5b_{10} & 3b_{00} & & & \\ & & 3k(k-5) & 7b_{10} & 4b_{00} & & \\ & & & \cdots & \cdots & \cdots & \\ & & & & 4(k-5)(k-4) & (2k-9)b_{10} & (k-4)b_{00} \\ & & & & & (k-4)(2k-7) & (2k-7)b_{10} \end{pmatrix},$$

which is the matrix of the linear system, must vanish. If $b_{00}=0$ then the determinant vanishes if and only if $b_{10}=0$, but this is a contradiction with the hypotheses. So for the rest of the proof we take $b_{00}\neq 0$.

As $x=0$ is an invariant algebraic curve of system (2.60) with cofactor x and the divergence of the system is $2x+2y$, we will find conditions on the coefficients of system (2.60) for $\tilde{W}_0(x)+\tilde{W}_1(x)y+\tilde{W}_2(x)y^2$ to be an invariant algebraic curve of cofactor $2y$.

The following lemma gives the expression of $\det(M_k)$. We define $\mathcal{Q}(x)$ the integer quotient of x and 2 and $\mathcal{M}(k)=\mathrm{mod}\,(k,2)$.

Lemma 2.3.18. *The expression of the determinant of M_k is*

$$\det(M_k)=b_{10}^{\mathcal{M}(k-3)}\sum_{i=0}^{\mathcal{Q}(k-3)} c_i^k b_{10}^{2i}(-b_{00})^{\mathcal{Q}(k-3)-i}, \tag{2.64}$$

where $c_i^k\in\mathbb{N}$ for all i.

Proof: We prove the lemma by using the induction principle. If $k=5$ then $\det(M_5)=3(b_{10}^2-b_{00})$. If $k=6$ then $\det(M_6)=15b_{10}(b_{10}^2-4b_{00})$. If $k>6$, then solving the determinant of M_k by the last row and the last column, we get the recursive expression

$$\det(M_k)=(2k-7)\,b_{10}\det(M_{k-1})+(k-4)^2(2k-7)(-b_{00})\det(M_{k-2}).$$

Observe that the constants appearing in the above expression are natural numbers. Applying the induction principle, we obtain

$$\begin{aligned}\det(M_k)=&\,(2k-7)b_{10}^{\mathcal{M}(k)+1}\sum_{i=0}^{\mathcal{Q}(k-4)} c_i^{k-1}b_{10}^{2i}(-b_{00})^{\mathcal{Q}(k-4)-i}\\ &+(k-4)^2(2k-7)b_{10}^{\mathcal{M}(k-1)}\sum_{i=0}^{\mathcal{Q}(k-5)} c_i^{k-2}b_{10}^{2i}(-b_{00})^{\mathcal{Q}(k-3)-i},\end{aligned}$$

for some positive integers c_i^{k-2}, c_i^{k-1}. After some computations, this expression is rewritten as in (2.64). The coefficients c_i^k come from sums and products of natural numbers, so they are also natural numbers. ∎

If $b_{10} \neq 0$ and $b_{00} < 0$, then the sum in the expression of $\det(M_k)$ is a polynomial in b_{10}^2 with positive coefficients and then there is no solution of $\det(M_k) = 0$ for b_{10}. So, for $b_{00} < 0$ the only solution of $\det(M_k) = 0$ is $b_{10} = 0$ when k is even. By Lemma 2.3.19 we prove that there is a polynomial solution V in this case.

If $b_{00} > 0$ then $\det(M_k) = 0$ is a polynomial equation of degree $k-3$ in b_{10}. By Lemma 2.3.20, we find $k-3$ values of b_{10} for which there exist two invariant algebraic curves $f_1(x,y) = 0$ and $f_2(x,y) = 0$, both linear in y and with cofactors $k_1(x,y)$ and $k_2(x,y)$, respectively, such that $\deg(f_1) + \deg(f_2) = k-2$ and $k_1(x,y) + k_2(x,y) = 2y$.

Lemma 2.3.19. *System* (2.60) *with* $b_{00} < 0$*,* $b_{10} = 0$*,* $b_{20} = 4(p-1)(p-2)$ *and* $p = k/2 \in \mathbb{N} \setminus \{1,2\}$*, has an invariant algebraic curve* $f(x,y) = 0$ *of degree* $k-2$ *and cofactor* $2y$*.*

Proof: Under the hypotheses of the lemma, system (2.60) writes as

$$\dot{x} = x^2, \quad \dot{y} = -\frac{b_{00}}{4} - (p-1)(p-2)x^2 + y^2. \tag{2.65}$$

If $f(x,y) = f_0(x) + f_1(x)y + f_2(x)y^2 = 0$ is an invariant algebraic curve of system (2.65) of degree $k-2$ and cofactor $2y$, then

$$x^2 \frac{\partial f}{\partial x} + \left(-\frac{b_{00}}{4} - (p-1)(p-2)x^2 + y^2 \right) \frac{\partial f}{\partial y} = 2yf,$$

so we obtain

$$f_1(x) = x^2 f_2'(x), \quad f_0(x) = -\left(\frac{b_{00}}{4} + (p-1)(p-2)x^2 \right) f_2(x) + \frac{x^2}{2} f_1'(x)$$

and

$$x^2 f_0'(x) - \left(\frac{b_{00}}{4} + (p-1)(p-2)x^2 \right) f_1(x) = 0.$$

This last equation gives us an expression for $f_2(x)$,

$$f_2(x) = \sum_{i=0}^{p-2} \frac{(2-p)_i \, b_{00}^i \, x^{2(p-2-i)}}{4^i \, i! \, (4-2p)_i \, (5/2-p)_i},$$

where $(a)_i = a(a+1)\cdots(a+i-1)$. From the other equations, we obtain the expressions of $f_0(x)$ and $f_1(x)$, and then we have the expression of the function $f(x,y)$. ∎

Under the conditions of Lemma 2.3.19, the cofactor of the polynomial f is $2y$, so $V(x,y) = x^2 f(x,y)$ is a polynomial inverse integrating factor of system (2.60). The following lemma, which is based in Theorem 2 of [15], finishes the proof of the proposition.

Lemma 2.3.20. *System (2.60) with $b_{20} = (k-2)(k-4)$, $k \in \mathbb{N}$, $k > 4$, $b_{00} > 0$, $b_{10} = (k-2p)\sqrt{b_{00}}$ and $p \in \{2, \dots, k-2\}$, has two invariant algebraic curves, of degrees $p-1$ and $k-p-1$, respectively. These invariant algebraic curves are of the form $p_1(x)y + p_2(x)$, with $(p_1, p_2) = 1$, and the sum of their respective cofactors is $2y$. Moreover, $p_1(x)$ is a generalized Laguerre polynomial of degree $p-2$ for the curve of degree $p-1$ and of degree $k-p-2$ for the curve of degree $k-p-1$.*

Proof: Assume that $h(x,y) = p_1(x)y + p_2(x)$ is an invariant algebraic curve of system (2.60) and let $T(x) + a_2 y = a_0 + a_1 x + a_2 y \in \mathbb{C}[x,y]$ be its cofactor. Let $\eta \geq 0$ be the degree of $p_1(x)$. Then the following equation must hold:

$$\dot{x}\frac{\partial h}{\partial x} + \dot{y}\frac{\partial h}{\partial y} - (T(x) + a_2 y)h = 0.$$

Writing this differential equation as a system of equations we get

$$\begin{aligned} (a_2 - 1)p_1(x) &= 0, \\ x^2 p_1'(x) - T(x)p_1(x) - p_2(x) &= 0, \\ x^2 p_2'(x) + N(x)p_1(x) - T(x)p_2(x) &= 0, \end{aligned}$$

where $N(x) = -b_{00}/4 - b_{10}x/2 - (k-2)(k-4)x^2/4$. It follows that $a_2 = 1$. Observe that the expression of $p_2(x)$ can be obtained from the second equation. From the second and the third equations we get $T(x)^2 + N(x) = \lambda x^2$, $\lambda = (2\eta+1)a_1 - \eta(\eta+1)$ (see Lemma 4 in [15]) and

$$(\lambda - T'(x))p_1(x) + 2(x - T(x))p_1'(x) + x^2 p_1''(x) = 0 \tag{2.66}$$

(see Proposition 3 in [15]). Moreover, $4a_0^2 - b_{00} = 0$, $4a_1^2 - 4\lambda - (k-2)(k-4) = 0$ and $4a_0 a_1 - b_{10} = 0$. Then, $a_0 = \pm\sqrt{b_{00}}/2$, $a_1 = \big(2\eta + 1 \pm (k-3)\big)/2$ and

$$b_{10} = \pm(2\eta + 1 \pm (k-3))\sqrt{b_{00}}.$$

Both symbols $\pm$ in this expression are independent each other. Now we consider equation (2.66) with $\eta = p-2$, $a_0 = -\sqrt{b_{00}}/2$ and $a_1 = -(k-2p)/2$ and the same equation (2.66) with $\eta = k-p-2$, $a_0 = \sqrt{b_{00}}/2$ and $a_1 = (k-2p)/2$. According to Proposition 3 in [15], we get two invariant algebraic curves $f_1 = 0$ (of degree $p-1$) and $f_2 = 0$ (of degree $k-p-1$), both linear in y. The respective cofactors are $-\sqrt{b_{00}}/2 - (k-2p)x/2 + y$ and $\sqrt{b_{00}}/2 + (k-2p)x/2 + y$. Its sum is $2y$. Furthermore, in both cases $b_{10} = (k-2p)\sqrt{b_{00}}$. ∎

If $b_{00} > 0$ then there are $k-3$ values of b_{10} for which there exist two invariant algebraic curves $f_1 = 0$ and $f_2 = 0$, with respective cofactors k_1 and k_2, such that $k_1 + k_2 = 2y$. So, under the conditions of the lemma, $V = x^2 f_1 f_2$ is a polynomial inverse integrating factor of system (2.60), and there are no more polynomial solutions than the ones we have found. ∎

2.3.6 The case $P(x, y) = 1 + x^2$

We consider the quadratic system

$$\dot{x} = 1 + x^2, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.67}$$

where $d, a, b, l, m, n \in \mathbb{R}$.

If $n = 0$, then system (2.67) is transformed into one of the following systems, depending on the values of the parameters b and m.

If $m = b = 0$ then

$$\dot{x} = 1 + x^2, \quad \dot{y} = b_{00} + ax, \tag{2.68}$$

where $b_{00} = d - l$, by the affine change $y - lx \to y$.

If $m = 0$ and $b \neq 0$, then

$$\dot{x} = 1 + x^2, \quad \dot{y} = b_{10}x + by, \tag{2.69}$$

where $b_{10} = a + bl$, by the affine change $y - lx + (d - l)/b \to y$.

If $m = 1$ then

$$\dot{x} = 1 + x^2, \quad \dot{y} = |b|y + \delta x^2 + xy, \tag{2.70}$$

where $\delta = 0, 1$, by the change $(b/|b|x, b/|b|((ab - d)(x - b)/((b^2 + 1)l) + (y + a)/l), b/|b|t) \to (x, y, t)$ if $l \neq 0$ and the change $(b/|b|x, b/|b|((ab - d)(x - b)/(b^2 + 1) + y + a), b/|b|t) \to (x, y, t)$ if $l = 0$.

If $m \neq 0, 1$ and $(b^2 + m)l - (m - 1)(ab - dm) = 0$, then

$$\dot{x} = 1 + x^2, \quad \dot{y} = (b + mx)y, \tag{2.71}$$

by the affine change $Ax + y + (a - Ab)/m \to y$, where $A = l/(m - 1)$.

If $m \neq 0, 1$ and $(b^2 + m)l - (m - 1)(ab - dm) \neq 0$, then

$$\dot{x} = 1 + x^2, \quad \dot{y} = 1 + by + mxy, \tag{2.72}$$

by the affine change $B(Ax + y + (a - Ab)/m) \to y$, where $A = l/(m - 1)$ and $B = m(m - 1)/((b^2 + m)l - (m - 1)(ab - dm))$.

If $n \neq 0$ then system (2.67) becomes

$$\dot{x} = 1 + x^2, \quad \dot{y} = -\frac{b_{00}}{4} - \frac{b_{10}}{2}x - \frac{b_{20}}{4}x^2 + y^2, \tag{2.73}$$

where $b_{00} = b^2 - 2m - 4dn$, $b_{10} = bm - 2an$ and $b_{20} = m(m-2) - 4ln$, by the affine change $mx/2 + ny + b/2 \to y$.

The case $n = b = 0$, $m = -2$, for which system (2.67) is Hamiltonian, is not to be considered in the above subcases. The set of conditions on the coefficients of systems (2.68)–(2.73) in order to have a polynomial inverse integrating factor are stated in the following two propositions.

Proposition 2.3.21. *The following statements hold.*

(VI.1) *System* (2.67) *with $n = b = 0$ and $m = -2$ is Hamiltonian. Moreover, it can be written as*

$$\dot{x} = 1 + x^2, \quad \dot{y} = \delta - 2xy,$$

where $\delta = 0, 1$, by either the change $(2lx - 6y + 3a)/(2(l - 3d)) \to y$ if $l \neq 3d$, or by the change $2lx - 6y + 3a \to y$ if $l = 3d$.

(VI.2) *System* (2.68) *has the polynomial inverse integrating factor $V(x, y) = 1 + x^2$.*

(VI.3) *System* (2.69) *has a polynomial inverse integrating factor if and only if $b_{10} = 0$ and $b \neq 0$. Its expression is $V(x, y) = (1 + x^2)y$.*

(VI.4) *System* (2.70) *has a polynomial inverse integrating factor if and only if $\delta = 0$. Its expression is $V(x, y) = (1 + x^2)y$.*

(VI.5) *System* (2.71) *has the polynomial inverse integrating factor $V(x, y) = (1 + x^2)y$.*

System (2.72) *has a polynomial inverse integrating factor (of degree $k > 3$) if and only if one of the following two statements hold.*

(VI.6) *$b = 0$ and $m = -2p$, $p \in \mathbb{N} \setminus \{1\}$. Then,*

$$V(x, y) = \left(\frac{H(x, y)}{1 + x^2}\right)^{p-1} \left({}_2F_1(1/2, 1 - p, 3/2, 1)^2 + H(x, y)^2\right).$$

We note that $k = 2p^2 + p + 3 \geq 13$. The function H is a polynomial first integral of the system. Its expression is

$$H(x, y) = (1 + x^2)^p y - \sum_{i=0}^{p-1} \binom{p-1}{i} \frac{x^{2i+1}}{2i + 1}.$$

(VI.7) *$m = k - 2$ and $b \neq 0$. Then*

$$V(x,y) = (1+x^2)\left(y - \frac{i}{(2i)^{k-2}}\sum_{j=0}^{k-2}\binom{k-2}{j}\frac{(i-x)^j(i+x)^{k-2-j}}{k-2(j+1)-ib}\right),$$

where $i = \sqrt{-1}$. Moreover, $V(x,y)$ is a polynomial function. We note that system (2.72) *has no polynomial inverse integrating factors of degree $k \leq 3$.*

Proof: The cases (VI.2) to (VI.5) are obtained solving the respective linear systems $A_i V^i = \mathbf{0}$, $i = 1, 2, 3$. We just have to consider system (2.72). If $k \leq 3$ then straightforward computations show that there is no polynomial solution V, so we assume $k > 3$. We write $V(x,y)$ as a polynomial in the variable y:

$$V(x,y) = \sum_{i=0}^{s} W_i(x) y^i,$$

$s \in \mathbb{N} \cup \{0\}$. We can now write equation ($\bigstar$) as a polynomial equation in y. Then, all the coefficients of the new equation, which depend on x, must vanish. The coefficient of y^s is

$$\big(b(s-1) + (m(s-1)-2)x\big)W_s(x) + (1+x^2)W_s'(x) = 0.$$

From this equation we get

$$W_s(x) = e^{b(s-1)\arctan x}(1+x^2)^{(2+m-ms)/2}.$$

As $W_s(x)$ is a polynomial in x, we take $b(s-1) = 0$. Let us assume first that $b = 0$. In this case, we get the solution

$$V(x,y) = (1+x^2)^{m/2+1} F\left((1+x^2)^{-m/2}y - x\ {}_2F_1(1/2, m/2+1, 3/2, -x^2)\right),$$

where F is an arbitrary function. The *hypergeometric* function ${}_2F_1(a,b,c,z)$ is defined as

$${}_2F_1(a_1,a_2,a_3,z) = \sum_{i\geq 0}\frac{(a_1)_i(a_2)_i z^i}{(a_3)_i i!}, \tag{2.74}$$

where $(a)_i = \Gamma(a+i)/\Gamma(a)$ is the Pochhammer symbol. Since the degree of V in y is s, F must be a polynomial of degree s. Then we must take $m = -2p$, $p \in \mathbb{N}$. It follows that

$${}_2F_1(1/2, 1-p, 3/2, -x^2) = \sum_{i=0}^{p-1}\binom{p-1}{i}\frac{x^{2i}}{2i+1}.$$

We also take $p \neq 1$, otherwise we have $n = b = 0$, $m = -2$ and the system is Hamiltonian. We take the polynomial function F of degree $p+1$ in y

$$F(H(x,y)) = H(x,y)^{p-1}\big({}_2F_1(1/2, 1-p, 3/2, 1)^2 + H(x,y)^2\big),$$

where ${}_2F_1(1/2, 1-p, 3/2, 1)^2 \neq 0$ and $H(x,y) = (1+x^2)^p y - x\,{}_2F_1(1/2, 1-p, 3/2, -x^2)$ is a polynomial first integral of the system. We claim that

$$V(x,y) = \left(\frac{H(x,y)}{1+x^2}\right)^{p-1} \big({}_2F_1(1/2, 1-p, 3/2, 1)^2 + H(x,y)^2\big)$$

is a polynomial function.

Let $f(p,t) = {}_2F_1(1/2, 1-p, 3/2, -t)$. To prove the claim, we have to prove that

$$g_0(p,t) = f(p,t)^{p-1}\big(f(p,-1)^2 + tf(p,t)^2\big)$$

has a zero of multiplicity at least p at $t = -1$. That is,

$$\frac{\partial^q g_0}{\partial t^q}(p,-1) = 0, \quad q = 0, \ldots, p-1.$$

Observe that $g_0(p,-1) = 0$. We derive $g_0(p,t)$ with respect to t. Using the equality $f(p,t) + 2tf_t(p,t) = (1+t)^{p-1}$, we have

$$\frac{\partial g_0}{\partial t}(p,t) = f(p,t)^{p-2}\big((p-1)f_t(p,t)\big[f(p,-1)^2 + tf(p,t)^2\big] + f(p,t)^2(1+t)^{p-1}\big),$$

and then $\frac{\partial g_0}{\partial t}(p,-1) = 0$. We define now

$$g_1(p,t) = f(p,t)^{p-2}\big(f(p,-1)^2 + tf(p,t)^2\big).$$

Observe that $g_1(p,-1) = 0$, which implies that $g_0(p,t)$ has a zero of multiplicity at least 2. Also if $\frac{\partial g_1}{\partial t}(p,-1) = 0$ then the $g_0(p,t)$ has a zero of multiplicity at least 3 at $t = -1$. But

$$\frac{\partial g_1}{\partial t}(p,t) = f(p,t)^{p-3}\big((p-2)f_t(p,t)\big(f(p,-1)^2 + tf(p,t)^2\big) + f(p,t)^2(1+t)^{p-1}\big),$$

and then $\frac{\partial g_1}{\partial t}(p,-1) = 0$. In order to prove that $g_0(p,t)$ has a zero of multiplicity at least p at $t = -1$, it is sufficient to prove that $g_1(p,t)$ has a zero of multiplicity at least $p-1$ at $t = -1$. We can iterate this method and define the functions

$$g_q(p,t) = f(p,t)^{p-1-q}(p,t)\big(f(p,-1)^2 + tf(p,t)^2\big), \quad q = 0, \ldots, p-1.$$

For $q = 0, 1$ the functions have been defined above. As $g_q(p,-1) = 0$ for all q, the claim is proved, and so the function $V(x,y)$ defined above is a polynomial inverse integrating factor of our system. Also, it has $1+x^2$ as a common factor. It is easy to see that the degree of $V(x,y)$ is $2p^2+p+3$ and that its degree in y is $p+1$.

Assume now $b \neq 0$ and $s=1$. Then, $V(x,y)$ can be written as

$$V(x,y) = W_0(x) + W_1(x)y.$$

We write equation (★) as a polynomial equation in y. The coefficient of y is

$$-2xW_1(x) + (1+x^2)W_1'(x) = 0.$$

Then, $W_1(x) = 1+x^2$ and (★) becomes

$$1 + x^2 - (b + (m+2)x)W_0(x) + (1+x^2)W_0'(x) = 0. \tag{2.75}$$

As $W_0(x)$ is a polynomial of degree k, we can write $W_0(x) = v_k x^k + \cdots$, with $v_k \neq 0$. Then, (2.75) is $(k-m-2)v_k x^{k+1} + \cdots = 0$. As $v_k \neq 0$, we take $m = k-2$. With all these restrictions, we solve equation (★) to obtain the expression of $W_0(x)$:

$$W_0(x) = -e^{b \arctan x}(1+x^2)^{k/2} \int e^{-b \arctan x}(1+x^2)^{-k/2}\, dx.$$

After some computations, we obtain

$$W_0(x) = \frac{1+x^2}{2^{k-2}} \sum_{j=0}^{k-2} \binom{k-2}{j} \frac{(i-x)^j (i+x)^{k-2-j}}{i^{k-1}(k - bi - 2(j+1))}, \tag{2.76}$$

where $i = \sqrt{-1}$. Observe that the degree of $W_0(x)$ is k. It only remains to prove that it is a real polynomial.

If k is even, then the term of the sum for $j = (k-2)/2$ in the expression (2.76) is

$$\binom{k-2}{(k-2)/2} \frac{(-1-x^2)^{(k-2)/2}}{-bi^k},$$

and it is a real polynomial. For any $k > 3$, the terms of the sum for $j = l$ and $j = k-2-l$ in the expression (2.76) are, respectively:

$$\begin{aligned} &\binom{k-2}{l} \frac{(i-x)^l (i+x)^{k-2-l}}{i^{k-1}(k-2l-2-ib)} \\ &= \quad \binom{k-2}{l} / (i^{k-1}(k-2l-2-ib)) \sum_{p=0}^{l} \sum_{q=0}^{k-2-l} (-1)^{l-p}\, i^{p+q} x^{k-2-p-q} \end{aligned}$$

and

$$
\begin{aligned}
&-\binom{k-2}{k-2-l}\frac{(i-x)^{k-2-l}(i+x)^l}{i^{k-1}(k-2l-2+ib)}\\
&= -\binom{k-2}{l}/(i^{k-1}(k-2l-2+ib))\sum_{p=0}^{l}\sum_{q=0}^{k-2-l}(-1)^{k-2-l-q}\, i^{p+q}x^{k-2-p-q}.
\end{aligned}
$$

By adding these two expressions to get

$$
\begin{aligned}
&(-1)^l\binom{k-2}{l}\sum_{p=0}^{l}\sum_{q=0}^{k-2-l}\frac{x^{k-2-p-q}}{i^{k-1-p-q}}\left(\frac{(-1)^p}{k-2l-2-bi}-\frac{(-1)^{k-2-q}}{k-2l-2+bi}\right)\\
&= (-1)^l\binom{k-2}{l}\sum_{p=0}^{l}\sum_{q=0}^{k-2-l}\frac{x^{k-2-p-q}}{i^{k-1-p-q}}\left(\frac{(k-2l-2)((-1)^p-(-1)^{k-2-q})}{(k-2l-2)^2+b^2}+\right.\\
&\qquad\left.\frac{bi((-1)^p+(-1)^{k-2-q})}{(k-2l-2)^2+b^2}\right). \qquad (2.77)
\end{aligned}
$$

If $(k-2-q)+p$ is even, then $(-1)^p-(-1)^{k-2-q}=0$, so the expression (2.77) becomes

$$
(-1)^l\binom{k-2}{l}\sum_{p=0}^{l}\sum_{q=0}^{k-2-l}\frac{x^{k-2-p-q}}{i^{k-2-p-q}}\frac{\pm 2b}{(k-2l-2)^2+b^2},
$$

which is a real expression provided that i is powered to an even number. If $(k-2-q)+p$ is odd then $(-1)^p+(-1)^{k-2-q}=0$ and the expression (2.77) becomes

$$
(-1)^l\binom{k-2}{l}\sum_{p=0}^{l}\sum_{q=0}^{k-2-l}\frac{x^{k-2-p-q}}{i^{k-1-p-q}}\frac{\pm(k-2l-2)}{(k-2l-2)^2+b^2},
$$

which is a real expression provided that i is again powered to an even number. Summarizing, $W_0(x)$ is a real polynomial and the proof of the proposition is finished. ∎

Proposition 2.3.22. *System* (2.73) *has a polynomial inverse integrating factor if and only if one of the following statements hold.*

(VI.8) $b_{00}=-4$, $b_{10}=b_{20}=0$. *Then* $V(x,y)=(x-y)^2$.

(VI.9) $b_{00}+4\neq 0$, $b_{10}=b_{20}=0$. *Then* $V(x,y)=(1+x^2)(b_{00}-4y^2)$.

(VI.10) $b_{20} = (k-2)(k-4)$, $b_{10} = (k-2p)\sqrt{b_{00} + (k-2p)^2 - (k-2)(k-4)}$, $b_{00} + (k-2p)^2 - (k-2)(k-4) \geq 0$ *with* $k > 4$ *and* $p \in \{2, \ldots, k-2\}$. *Then* $V(x,y) = (1+x^2)\big(p_1^1(x)y + p_2^1(x)\big)\big(p_1^2(x)y + p_2^2(x)\big)$, *where* $p_1^1(x)$ *and* $p_1^2(x)$ *are Jacobi polynomials of respective degree* $p-2$ *and* $k-p-2$ *and the expression of* p_2^1 *and* p_2^2 *can be obtained from* p_1^1 *and* p_1^2, *respectively.*

Proof: Let $V(x,y)$ be a polynomial inverse integrating factor of (2.73) of degree k. If $k \leq 4$, straightforward computations show that there is no solution of degree 1 or 3; there is a solution of degree 2 if and only if $b_{10} = b_{20} = 0$ and $b_{00} + 4 = 0$; and there is a solution of degree 4 if and only if $b_{10} = b_{20} = 0$ and $b_{00} + 4 \neq 0$. The respective expressions of $V(x,y)$ are shown in Proposition 2.3.22. From now on, we assume $k > 4$. By Lemma 2.3.1, we can write V as $V(x,y) = W_0(x) + W_1(x)y + W_2(x)y^2$. Equation ($\bigstar$) can be written as a system of equations:

$$\begin{aligned}
-8W_0(x) + 4(1+x^2)W_0'(x) - (b_{00} + 2b_{10}x + b_{20}x^2)W_1(x) &= 0,\\
4W_0(x) + 4xW_1(x) - 2(1+x^2)W_1'(x) + (b_{00} + 2b_{10}x + b_{20}x^2)W_2(x) &= 0,\\
W_1(x) + 2xW_2(x) - (1+x^2)W_2'(x) &= 0.
\end{aligned} \tag{2.78}$$

We obtain expressions for $W_0(x)$ and $W_1(x)$ in terms of $W_2(x)$ and its derivatives:

$$\begin{aligned}
&W_1(x) = -2x\,W_2(x) + (1+x^2)\,W_2'(x),\\
&W_0(x) = -\frac{1}{4}\big((b_{00} + 4 + 2b_{10}x + (b_{20}-4)x^2)W_2(x) + 4x(1+x^2)W_2'(x) -\\
&\quad 2(1+x^2)^2W_2''(x)\big).
\end{aligned}$$

Observe that if $W_2(x)$ is a polynomial of degree $k-2$, then $W_1(x)$ and $W_0(x)$ are polynomials of degrees $k-1$ and k, respectively. We substitute $W_1(x)$ and $W_0(x)$ in the remaining equation of system (2.78) to get the differential equation

$$\begin{aligned}
&(-b_{10} + (2b_{00} + 8 - b_{20})x + 3b_{10}x^2 + b_{20}x^3)W_2(x) -\\
&\quad (1+x^2)(b_{00} + 4 + 2b_{10}x + b_{20}x^2)W_2'(x) + (1+x^2)^3W_2'''(x) = 0.
\end{aligned} \tag{2.79}$$

This is a differential equation with unknown $W_2(x)$. We want to obtain from this equation a polynomial solution of degree $k-2$, so we write $W_2(x) = \sum_{i=0}^{k-2} a_i x^i$. Equation (2.79) can be written as a polynomial equation of degree $k+1$ in x, and then we can transform it into a $(k+2)\times(k-1)$ homogeneous linear system. The equation corresponding to x^{k+1} is $(k-3)(b_{20} - (k-2)(k-4)) = 0$. As $k > 4$, we take $b_{20} = (k-2)(k-4) > 0$.

We have a homogeneous linear system with $k-1$ unknowns $a_0, \ldots, a_{k-2}$ and $k+1$ equations if we exclude the equation corresponding to x^{k+1}. In order to have a non–trivial solution, all the $(k-1)$–minors of the matrix of the system M_k, that is, the determinants of the matrices obtained by taking all the $(k-1)\times(k-1)$–submatrices of M_k, must be zero. The matrix of the system is

$$M_k = \left(\begin{array}{cccccccc}
\xi_{k-1} & \rho_{k-2} & 0 & \chi_{k-1} & & & & \\
\vartheta_{k-2} & \xi_{k-2} & 2\rho_{k-3} & 0 & \chi_{k-2} & & & \\
\xi_{k+1} & \vartheta_{k-3} & \xi_{k-3} & 3\rho_{k-4} & 0 & \chi_{k-3} & & \\
\kappa_{k-4} & \xi_k & \vartheta_{k-4} & \xi_{k-4} & 4\rho_{k-5} & 0 & \chi_{k-4} & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & \kappa_4 & \xi_8 & \vartheta_4 & \xi_4 & (k-4)\rho_3 & 0 & \chi_4 \\
 & & \kappa_3 & \xi_7 & \vartheta_3 & \xi_3 & (k-3)\rho_2 & 0 \\
 & & & \kappa_2 & \xi_6 & \vartheta_2 & \xi_2 & (k-2)\rho_1 \\
 & & & & \kappa_1 & \xi_5 & \vartheta_1 & \xi_1 \\
 & & & & & \kappa_0 & \xi_4 & \vartheta_0 \\
 & & & & & & \kappa_{-1} & \xi_3
\end{array}\right),$$

where

$$\begin{array}{ll}
\kappa_{k-i} = (i-5)b_{20} + \chi_{k-i+6}, & \vartheta_{k-i} = (i-1)b_{20} + (i-4)\rho_{k-i}, \\
\chi_{k-i} = -i(i+1)(i+2), & \rho_{k-i} = b_{00} - \nu_{k-i}, \\
\xi_{k-i} = (2i-1)b_{10}, & \nu_{k-i} = 3i^2 - 15i + 14.
\end{array}$$

The following lemma is related to the roots of the $(k-1)$–minors of M_k.

Lemma 2.3.23. *There are at most $k-2$ values of b_{10} for which all the $(k-1)$–minors of M_k vanish. Moreover, if all $(k-1)$–minors vanish for a non–zero value b_{10}, then they also vanish for $-b_{10}$.*

Proof: We first prove by induction that the $(k-1)$–minors of M_k have degree $k-2$ or $k-1$ in b_{10}. If $k=5$, then it is easy to check that the degree in b_{10} is 3 or 4 for all the 4–minors. If $k>5$, then we compute the degree of every $(k-1)$–minor of M_k distinguishing three cases:

1. If the minor contains the last row of M_k, then we solve it through this row. The degree in b_{10} of the minor is, using the induction principle, $k-2$ or $k-1$.

2. If the minor does not contain the last row of M_k but the k^{th}–row, then we solve it through this row. Once, again, applying the induction principle, the degree in b_{10} of the minor is $k-2$ or $k-1$.

3. The minor which does not contain the last two rows of M_k has degree $k-2$ in b_{10}.

Then the $(k-1)$–minors of M_k have degree $k-2$ or $k-1$, and as a consequence the first part of the lemma is proved.

Suppose that all the $(k-1)$–minors vanish for a certain value $b_{10} \neq 0$. We change the sign of all the components of the even rows $(2^{nd}, 4^{th}, \ldots)$ and the odd columns $(1^{st}, 3^{rd}, \ldots)$ of M_k. This fact does not change the values for which the $(k-1)$–minors vanish. Now we change b_{10} by $-b_{10}$ in this new matrix to obtain M_k again. So all the $(k-1)$–minors vanish also for $-b_{10}$. ■

System (2.73) has the invariant algebraic curves $x \pm i = 0$. The sum of their cofactors is $2x$, and the divergence of (2.73) is $2x+2y$. We shall find a polynomial inverse integrating factor of the form

$$V(x,y) = (1+x^2)\big(\tilde{W}_0(x) + \tilde{W}_1(x)y + \tilde{W}_2(x)y^2\big),$$

where $\tilde{W}_i(x)$ are polynomials of degree $k-2-i$, $i = 0,1,2$, such that $\tilde{W}_0(x) + \tilde{W}_1(x)y + \tilde{W}_2(x)y^2$ is an invariant algebraic curve with cofactor $2y$.

Let $p \in \{2,\ldots,k-2\}$ and $b_{10}^{(p)} = (k-2p)\sqrt{b_{00} + (k-2p)^2 - (k-2)(k-4)}$. We note that $b_{10}^{(p)} = -b_{10}^{(k-p)}$ for $p = 2,\ldots,[k/2]$. The following lemma is based on Theorem 2 in [15].

Lemma 2.3.24. *If $b_{10} = b_{10}^{(p)}$ for a certain $p \in \{2,\ldots,k-2\}$, then system* (2.73) *has two invariant algebraic curves, of respective degrees $p-1$ and $k-p-1$, and of the form $h(x,y) = p_1(x)y + p_2(x)$, $(p_1,p_2) = 1$. Moreover, $p_1(x)$ is a Jacobi polynomial, of degree $p-2$ for the curve of degree $p-1$ and of degree $k-p-2$ for the curve of degree $k-p-1$, and the expression of $p_2(x)$ can be obtained from $p_1(x)$ and the cofactor of $h(x,y)$. The product of both curves is an invariant algebraic curve of degree $k-2$ and cofactor $2y$.*

Proof: We assume that $h(x,y) = p_1(x)y + p_2(x)$ is an invariant algebraic curve of (2.73). Let $T(x) + a_2 y = a_0 + a_1 x + a_2 y \in \mathbb{C}[x,y]$ be its cofactor. Then,

$$\dot{x}\frac{\partial h}{\partial x} + \dot{y}\frac{\partial h}{\partial y} - (T(x) + a_2 y)h = 0.$$

Writing this differential equation as a system of equations, we get

$$\begin{aligned} (a_2-1)p_1(x) &= 0, \\ (1+x^2)p_1'(x) - T(x)p_1(x) - p_2(x) &= 0, \\ (1+x^2)p_2'(x) + N(x)p_1(x) - T(x)p_2(x) &= 0, \end{aligned}$$

where $N(x) = -b_{00}/4 - b_{10}x/2 - (k-2)(k-4)x^2/4$. From the first equation, it follows that $a_2 = 1$. The expression of $p_2(x)$ can be obtained explicitly from the second equation. From the second and the third equations we get $T(x)^2 + N(x) =$

$\lambda(1+x^2)$, where $\lambda = (2\eta+1)a_1 - \eta(\eta+1)$, $\eta = \deg p_1 \geq 0$ (see Lemma 4 of [15]), and

$$(\lambda - T'(x))p_1(x) + 2(x - T(x))p_1'(x) + (1+x^2)p_1''(x) = 0 \tag{2.80}$$

(see Proposition 3 of [15]). Moreover, $4a_0^2 - b_{00} = 4\lambda$, $4a_1^2 - (k-2)(k-4) - 4\lambda = 0$ and $4a_0a_1 - b_{10} = 0$. So, $a_0 = \pm\sqrt{b_{00} + 2[2\eta^2 + (2\eta+1)(1 \pm (k-3))]}/2$, $a_1 = (2\eta + 1 \pm (k-3))/2$ and

$$b_{10} = \pm(2\eta + 1 \pm (k-3))\sqrt{b_{00} + 2[2\eta^2 + (2\eta+1)(1 \pm (k-3))]}.$$

By the change $z = ix$, $i = \sqrt{-1}$, equation (2.80) can be transformed into

$$(a_1 - \lambda)p_1(z) - 2((1-a_1)z - a_0 i)p_1'(z) + (1 - z^2)p_1''(z) = 0.$$

Taking $\alpha = -a_1 - a_0 i$, $\beta = -a_1 + a_0 i$, the solution of this equation is the Jacobi polynomial $P_\eta^{(\alpha,\beta)}(z)$. We solve this equation for $a_1 = 2\eta - k + 4$ and $\eta = p-2, k-p-2$, $p \in \{2, \ldots, k-2\}$. For the first value of η, we take $a_0 = -\sqrt{b_{00} + (k-2p)^2 - (k-2)(k-4)}/2$; for the second one, we take $a_0 = \sqrt{b_{00} + (k-2p)^2 - (k-2)(k-4)}/2$. In both cases, $b_{10} = b_{10}^{(p)}$. So we obtain two invariant algebraic curves, of degrees $p-1$ and $k-p-1$, respectively. Their product is $f(x,y) = 0$, of degree $k-2$ and cofactor $2y$. ■

Remark 2.3.25. From the properties of the Jacobi polynomials, the curve $f = 0$ may have $i = \sqrt{-1}$ as a factor when we change z by ix. In this case, we change f by $i \cdot f$, which is a real polynomial of degree $k-2$. So, the polynomial $V(x,y) = (1+x^2)f(x,y)$ is a real polynomial inverse integrating factor of degree k for our system. □

We have obtained $k-3$ values of b_{10} for which all the $(k-1)$–minors of M_k vanish. We claim that there are no more of such values. To prove the claim, we distinguish two cases. If k is even, then $b_{10} = b_{10}^{(k/2)} = 0$ is one of these $k-3$ values. If there exists another value $b_{10}^{(*)} \neq b_{10}^{(i)}$, $i = 2, \ldots, k-2$, $b_{10}^{(*)} \neq 0$, for which all the $(k-1)$–minors vanish, then, by Lemma 2.3.23, $-b_{10}^{(*)}$ is another one. So there are $k-1$ of such values, in contradiction with this lemma.

If k is odd and there exists another value $b_{10}^{(*)} \neq b_{10}^{(i)}$, $i = 2, \ldots, k-2$, $b_{10}^{(*)} \neq 0$, for which all the $k-1$–minors vanish, then, by Lemma 2.3.23, $-b_{10}^{(*)}$ is another one. So there are $k-1$ of such values, again in contradiction with the lemma, and then the only value of b_{10} which is not in contradiction with the Lemma 2.3.23 is $b_{10}^{(*)} = 0$. We next prove that not all minors vanish for $b_{10} = 0$.

If $k = 5$, then $b_{10}^{(2)} = \sqrt{b_{00} - 2}$ and $b_{10}^{(3)} = -\sqrt{b_{00} - 2}$. The 4–minor corresponding to the 4 first rows of M_5 is $3(b_{10}^2 - b_{00} + 2)(35b_{10}^2 + 12(b_{00}+2)(b_{00}+7/4))$

and the one corresponding to the last 4 rows is $9(b_{10}^2 - b_{00} + 2)^2$. They both vanish only for $b_{10} = b_{10}^{(2)}, b_{10}^{(3)}$.

Assume that $k > 5$ is odd. Let M^1 be the $(k-1)$–minor corresponding to the first $k-1$ rows of M_k, and M^2 be the $(k-1)$–minor corresponding to the last $k-1$ rows of M_k:

$$M^1 = \left| \begin{array}{ccccccccccc}
0 & \rho_{k-2} & 0 & \chi_{k-1} & & & & & & & \\
\vartheta_{k-2} & 0 & 2\rho_{k-3} & 0 & \chi_{k-2} & & & & & & \\
0 & \vartheta_{k-3} & 0 & 3\rho_{k-4} & 0 & \chi_{k-3} & & & & & \\
\kappa_{k-4} & 0 & \vartheta_{k-4} & 0 & 4\rho_{k-5} & 0 & \chi_{k-4} & & & & \\
 & & & & \cdots & \cdots & \cdots & & & & \\
 & & & & \kappa_4 & 0 & \vartheta_4 & 0 & (k-4)\rho_3 & 0 & \chi_4 \\
 & & & & & \kappa_3 & 0 & \vartheta_3 & 0 & (k-3)\rho_2 & 0 \\
 & & & & & & \kappa_2 & 0 & \vartheta_2 & 0 & (k-2)\rho_1 \\
 & & & & & & & \kappa_1 & 0 & \vartheta_1 & 0
\end{array} \right|,$$

$$M^2 = \left| \begin{array}{ccccccccccc}
0 & \vartheta_{k-3} & 0 & 3\rho_{k-4} & 0 & \chi_{k-3} & & & & & \\
\kappa_{k-4} & 0 & \vartheta_{k-4} & 0 & 4\rho_{k-5} & 0 & \chi_{k-4} & & & & \\
 & \kappa_{k-5} & 0 & \vartheta_{k-5} & 0 & 5\rho_{k-6} & 0 & \chi_{k-5} & & & \\
 & & \kappa_{k-6} & 0 & \vartheta_{k-6} & 0 & 6\rho_{k-7} & 0 & \chi_{k-6} & & \\
 & & & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & & & \kappa_4 & 0 & \vartheta_4 & 0 & (k-4)\rho_3 & 0 & \chi_4 \\
 & & & & & \kappa_3 & 0 & \vartheta_3 & 0 & (k-3)\rho_2 & 0 \\
 & & & & & & \kappa_2 & 0 & \vartheta_2 & 0 & (k-2)\rho_1 \\
 & & & & & & & \kappa_1 & 0 & \vartheta_1 & 0 \\
 & & & & & & & & \kappa_0 & 0 & \vartheta_0 \\
 & & & & & & & & & \kappa_{-1} & 0
\end{array} \right|.$$

The degree of M^1 and M^2 in b_{10} is $k-1$. We note that the $k-3$ values $b_{10}^{(2)}, \ldots, b_{10}^{(k-2)}$ vanish for both M^1 and M^2. From the second column of M^2, as $\kappa_{k-5} = 0$ we get that $M^2 = 0$ for $Cb_{10}^2 + (b_{00} - B_0) = 0$, where $C \in \mathbb{R} \setminus \{0\}$ and $B_0 = 2(k^2 - 6k + 6)$. Observe that $B_0 \neq (k-2)(k-4) - (k-2p)^2$ because $k > 5$. Now we consider the equation

$$\begin{aligned}
&(-1)^{\left[\frac{i}{2}+1\right]} \left[\frac{i}{2} + 2\right] \xi_{i+3} + (-1)^{\left[\frac{i}{2}\right]} \left[\frac{i}{2} + 1\right] (k-i-1)\rho_i + \\
&\quad (-1)^{\left[\frac{i}{2}-1\right]} \left[\frac{i}{2}\right] \vartheta_{i-1} + (-1)^{\left[\frac{i}{2}-2\right]} \left[\frac{i}{2} - 1\right] \kappa_{i-3} = 0,
\end{aligned}$$

where $[x]$ is the greatest integer less than or equal to x. This equation corresponds to a linear combination of the even or odd rows of M^1 (for respective i even or odd) equaled to zero, taking $b_{10} = 0$. We obtain, from this equation,

$$b_{00} = k^2 - 6k + 4 + \frac{(k-2)(k-4)}{1 - k + i - 2[\frac{i}{2}]}.$$

If i is even, then $b_{00} = B_1 = k^2 - 7k + 9 - 3/(k-1)$; if i is odd, then $b_{00} = B_2 = k^2 - 7k + 8$. These values of b_{00} vanish M^1 for $b_{10} = 0$, and they are different from B_0 because $k > 5$. So the expressions of M^1 and M^2 are, up to a non–zero constant,

$$\left(C_1 b_{10}^2 - (b_{00} - B_1)(b_{00} - B_2)\right) \prod_{i=1}^{k-3} \left(b_{10}^2 - (k-2p)^2 (b_{00} - (k-2)(k-4) + (k-2p)^2)\right)$$

and

$$\big(C_2 b_{10}^2 - (b_{00} - B_0)\big) \prod_{i=1}^{k-3} \big(b_{10}^2 - (k-2p)^2 (b_{00} - (k-2)(k-4) + (k-2p)^2)\big),$$

where $C_1, C_2 \neq 0$. Then, there are no more than $k-3$ values of b_{10} for which all the $(k-1)$–minors of M_k vanish. ∎

2.3.7 The case $P(x, y) = -1 + x^2$

We consider the quadratic system

$$\dot{x} = -1 + x^2, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2, \tag{2.81}$$

where $d, a, b, l, m, n \in \mathbb{R}$.

If $n = 0$ then system (2.81) is transformed into one of the following systems, depending on the values of the parameters b and m.

If $m = b = 0$ then

$$\dot{x} = -1 + x^2, \quad \dot{y} = Q(x), \tag{2.82}$$

where either $Q(x) = b_{00} + x$ and $b_{00} = (d + l)/a \neq \pm 1$ if $a \neq 0$ by the affine change $(y - lx)/a \to y$, or $Q(x) = 1$, if $a = 0$ and $d + l \neq 0$, by the affine change $(y - lx)/(d + l) \to y$. In the case $a = d + l = 0$, the system has a common factor.

If $m = 0$ and $b \neq 0$, then

$$\dot{x} = -1 + x^2, \quad \dot{y} = b_{10} x + by, \tag{2.83}$$

where $b_{10} = a + bl$, by the affine change $y - lx + (d + l)/b \to y$.

If $m = 1$ and $b^2 \neq 1$, then

$$\dot{x} = -1 + x^2, \quad \dot{y} = |b| y + \delta x^2 + xy, \tag{2.84}$$

$\delta = 0, 1$, either by the affine change $(b/|b|x, b/|b|((x - b)(ab - d)/((b^2 - 1)l) + (y - a)/l), b/|b|t) \to (x, y, t)$ if $l \neq 0$, or the change $(b/|b|x, b/|b|((x - b)(ab - d)/(b^2 - 1) + y - a), b/|b|t) \to (x, y, t)$ if $l = 0$.

If $m = 1$ and $b = \pm 1$, then

$$\dot{x} = -1 + x^2, \quad \dot{y} = b_{00} + y + \delta x^2 + xy, \tag{2.85}$$

where $\delta = 0, 1$ and $b_{00} = (d \mp a)/l$ (if $l \neq 0$) or $b_{00} = d \mp a$ (if $l = 0$), by the affine change $(\pm x, (a \pm y)/l, \pm t) \to (x, y, t)$ if $l \neq 0$ or the change $(\pm x, a \pm y, \pm t) \to (x, y, t)$ if $l = 0$.

If $m \neq 0, 1$ and $(b^2 - m)l - (m-1)(ab - dm) = 0$, then

$$\dot{x} = -1 + x^2, \quad \dot{y} = (b + mx)y, \tag{2.86}$$

by the affine change $Ax + y + (a - Ab)/m \to y$, where $A = l/(m-1)$.

If $m \neq 0, 1$ and $(b^2 - m)l - (m-1)(ab - dm) \neq 0$, then

$$\dot{x} = -1 + x^2, \quad \dot{y} = 1 + by + mxy, \tag{2.87}$$

by the affine change $B(Ax + y + (a - Ab)/m) \to y$, where $A = l/(m-1)$ and $B = m(m-1)/((b^2 - m)l - (m-1)(ab - dm))$. Moreover, b can be assumed to be positive; otherwise, we change de sign of x, y and t.

If $n \neq 0$ then system (2.81) becomes

$$\dot{x} = -1 + x^2, \quad \dot{y} = -\frac{b_{00}}{4} - \frac{b_{10}}{2}x - \frac{b_{20}}{4}x^2 + y^2, \tag{2.88}$$

by the affine change $mx/2 + ny + b/2 \to y$, where $b_{00} = b^2 + 2m - 4dn$, $b_{10} = bm - 2an$ and $b_{20} = m(m-2) - 4ln$.

The case $n = b = 0$, $m = -2$, for which system (2.81) is Hamiltonian, is not considered in the subcases above. The set of conditions on the coefficients of systems (2.82)–(2.88) to have a polynomial inverse integrating factor are stated in the following two propositions.

Proposition 2.3.26. *The following statements hold.*

(V.1) *System* (2.81) *with $n = b = 0$ and $m = -2$ is Hamiltonian. Moreover, it can be written as*

$$\dot{x} = -1 + x^2, \quad \dot{y} = \delta - 2xy,$$

where $\delta = 0, 1$, either by the change $-(2lx - 6y + 3a)/(2(l + 3d)) \to y$ if $l \neq -3d$, or by the change $2lx - 6y + 3a \to y$ if $l = -3d$.

(V.2) *System* (2.82) *has the polynomial inverse integrating factor $V(x, y) = -1 + x^2$.*

System (2.83) *has a polynomial inverse integrating factor if and only if one of the following two statements hold.*

(V.3) *$b_{10} = 0$. Then $V(x, y) = (-1 + x^2)y$.*

(V.4) *$b_{10} \neq 0$ and $b = \pm 2$. Then $V(x, y) = (x \mp 1)^2$.*

System (2.84) *has a polynomial inverse integrating factor if and only if one of the two following statements hold.*

(V.5) $|b| \neq 1, 3$ *and* $\delta = 0$. *Then* $V(x,y) = (-1+x^2)y$.

(V.6) $|b| = 3$. *Then* $V(x,y) = (x-1)^3$.

(V.7) *System* (2.85) *has the polynomial inverse integrating factor* $V(x,y) = (-1+x^2)(x-1)$.

(V.8) *System* (2.86) *has the polynomial inverse integrating factor* $V(x,y) = (-1+x^2)y$. *We note that if* $m = -1$ *and* $b = \pm 1$, *then we also have the polynomial inverse integrating factor* $V(x,y) = x \mp 1$ *of degree* 1.

(V.9) *If* $b = (m+2) \neq 0, \pm 1$, *then system* (2.87) *has the polynomial inverse integrating factor* $V(x,y) = (x-1)(1+(m+1)(1+x)y)$.

System (2.87) *has a polynomial inverse integrating factor of degree* $k > 3$ *if and only if one of the following four statements hold.*

(V.10) $m = k-2$ *and* $b \neq k-2j$, $j = 1, \ldots, [(k-1)/2]$. *Then,*

$$V(x,y) = (-1+x^2)\left(y - \sum_{i=0}^{k-2} \frac{\prod_{j=0}^{i-1}(k-2-j)}{\prod_{j=1}^{i+1}(k-2j-b)}(1+x)^i \right).$$

In the following three cases, the solution $V(x,y)$ *is given by*

$$V(x,y) = (1+x)^{2-q+r}(1-x)^{1-r} F\left[-(\sqrt{-1})^{q+1}(1+x)^{q-r-1}\big((1-x)^r y - 2^{r-1} p \, {}_2F_1\left(1-r, q-1-r, q-r, (1+x)/2\right)\big)\right],$$

for certain p, q, r *and where* F *a complex polynomial of degree* $p+1$ *without independent term and* V *is real. The expression of the hypergeometric function* ${}_2F_1$ *is given below for each family.*

(V.11) $m = 1-q$, $b = q-1-2r$, $q \in \mathbb{Q}\backslash\mathbb{N}$, $p > 1$ *and* $r = 2, \ldots, (k-p-2)/p \in \mathbb{N}$. *In this case*

$$\begin{aligned} &{}_2F_1(1-r, q-r-1, q-r, (1+x)/2) = \\ &\quad \frac{q-r-1}{2^{r-1}} \sum_{i=0}^{r-1} \left(\frac{(1+x)^i(1-x)^{r-i-1}}{q-r+i-1} \prod_{j=1}^{i} \frac{r-j}{q-r+j-2} \right). \end{aligned}$$

(V.12) $m = 1-q$, $b = q-1-2r$, $q \in \mathbb{Q} \setminus \mathbb{N}$, $p > 1$ *and* $q-r-1 = 2, \ldots, (k-p-2)/p \in \mathbb{N}$. *In this case*

$$\begin{aligned} &{}_2F_1(1-r, q-r-1, q-r, (1+x)/2) = \\ &\quad -(q-r-1)\left[\frac{(-2)^{q-r-1}}{(1+x)^{q-r-1}(q-2)} \prod_{j=1}^{q-r-2} \frac{q-r-1-j}{r+j-1} + \right. \\ &\quad \left. \sum_{i=0}^{q-r-2} \left(\frac{(1-x)^{r+i}}{(1+x)^{i+1}} \frac{2^{1-r}}{r+i} \prod_{j=1}^{i} \frac{q-r-1-j}{r+j-1} \right) \right]. \end{aligned}$$

(V.13) *$m = 4-k$, $b = k-2i-2$, with $p = 1$, $k > 7$ and $i \in \{3,\dots,k-5\}$. In this case*

$$ {}_2F_1(2-i, k-3-i, k-2-i, (1+x)/2) = \sum_{j=0}^{i-2} \left(\frac{(k-3-i)(1+x)^j}{2^j j!(k-3-i+j)} \prod_{s=1}^{j} (s-i+1) \right). $$

Proof: The solutions stated in (V.1) to (V.9) follow from straightforward computations. No more solutions of degree $k \leq 3$ are obtained, so we assume $k > 3$ and we consider system (2.87). Let $V(x,y) = \sum_{i=0}^{k} V_i(x,y)$ be a polynomial inverse integrating factor of degree $k > 3$, where V_i is a homogeneous polynomial of degree i in x and y. From equation (★) we obtain the system

$$ (k-m-2)V_k + (m-1)y\frac{\partial V_k}{\partial y} = 0, \tag{2.89} $$

$$ (k-m-3)xV_{k-1} + (m-1)xy\frac{\partial V_{k-1}}{\partial y} = bV_k - by\frac{\partial V_k}{\partial y}, \tag{2.90} $$

$$ (j-m-2)x^2V_j + (m-1)x^2y\frac{\partial V_j}{\partial y} = bxV_{j+1} - bxy\frac{\partial V_{j+1}}{\partial y} + (j+2)V_{j+2} - (x+y)\frac{\partial V_{j+2}}{\partial y}, \tag{2.91} $$

$$ bV_0 + \frac{\partial V_1}{\partial x} - \frac{\partial V_1}{\partial y} = 0, \tag{2.92} $$

where $j = k-2,\dots,0$ in (2.91). From equation (2.89) we get

$$ V_k(x,y) = x^{k-1+\frac{k-3}{m-1}} y^{1-\frac{k-3}{m-1}}. $$

Then we take $1 - \frac{k-3}{m-1} = p+1 \in \mathbb{N} \cup \{0\}$. Set $q = (k-3)/p \in \mathbb{Q}$. Then, we get $m = 1 - \frac{k-3}{p} = 1-q$, for $p \in \{-1,1,2,3,\dots,k-1\}$, $q \in \mathbb{Q} \setminus \{0,1\}$, and $V_k(x,y) = x^{k-p-1}y^{p+1}$.

Assume $p = -1$. Then, $m = k-2$. We claim that V_j does not depend on y if $j > 3$ and that the degree in y of V is 1. We prove the claim by using the induction principle. Easily we can check that $V_k(x,y) = x^k$ and $V_{k-1}(x,y) = -bx^{k-1}$. Now assume that V_{j+2} and V_{j+1} depend only on x, for $3 \leq j \leq k-2$. Then equation (2.91) is

$$ -(k-j)x^2V_j + (k-3)x^2y\frac{\partial V_j}{\partial y} = bxV_{j+1} + (j+2)V_{j+2}. $$

From this equation, we obtain

$$ V_j(x,y) = -\frac{bxV_{j+1}(x) + (j+2)V_{j+2}(x)}{(k-j)x^2} + f_j(x)y^{\frac{k-j}{k-3}}, $$

where $f_j(x)$ is an arbitrary function. If $j > 3$, then $\frac{k-j}{k-3} < 1$, so we must take $f_j \equiv 0$, and then V_j does not depend on y. The degree of V_3 in y is, at most, 1.

In a similar way we obtain that V_2 does not depend on y and that the degree in y of V_1 is 1. So V can be written as

$$V(x,y) = W(x) + (-1+x^2)y,$$

where $W(x)$ is a polynomial of degree k in x. It remains to find the expression of this polynomial. From (★) we write

$$(-1+x^2) + (b+kx)W(x) + (-1+x^2)W'(x) = 0.$$

Then, after some computations, we obtain

$$W(x) = (-1+x^2)\sum_{i=0}^{k-2}(-1)^i \frac{\prod_{j=0}^{i-1}(k-2-j)}{\prod_{j=1}^{i+1}(b-k+2j)}(1+x)^i,$$

assuming $b \neq k-2j$, $j = 1,\ldots,k-1$. Otherwise, no solution is obtained. The polynomial V is

$$V(x,y) = (-1+x^2)\left(y - \sum_{i=0}^{k-2} \frac{\prod_{j=0}^{i-1}(k-2-j)}{\prod_{j=1}^{i+1}(k-2j-b)}(1+x)^i\right).$$

So we get (V.10). We note that we can restrict j to the interval $\{1,\ldots,[(k-1)/2]\}$.

Next we assume $p > 0$. We first prove by induction that the degree in y of V is $p+1$. The degrees of V_k and V_{k-1} in y are not bigger than $p+1$. Let $V_{j+2}(x,y) = c_0(x)y^{p+1} + \cdots$; $V_{j+1}(x,y) = b_0(x)y^{p+1} + \cdots$; and $V_j(x,y) = a_0(x)y^{p+s} + \cdots$, where $a_0 \not\equiv 0$, b_0, c_0 are polynomials and $s \in \mathbb{Z}$. Equation (2.91) can be written as

$$((q+j-3) - q(p+s))a_0(x)x^2y^{p+s} + (bpb_0x + (p-j-1)c_0)y^{p+1} + \cdots = 0.$$

If $p+s > p+1$ then we obtain $p+s = p+1-\frac{k-j}{k-3}p < p+1$ because $a_s \not\equiv 0$, and we have a contradiction. As the degree of V_k is $p+1$, the degree of V in y is $p+1$.

Next we write V as a polynomial in y: $V(x,y) = \sum_{i=0}^{p+1} W_i(x)y^i$. Then, we have

$$W_{p+1}(x) = (1-x)^{(k-p-1-bp)/2}(1+x)^{(k-p-1+bp)/2}.$$

This is a polynomial, so we $(k-p-1-bp)/2 = i \in \{0,\ldots,k-p-1\}$, and the expression of b can be rewritten as

$$b = q - 2r - 1,$$

where $i = 0,\ldots,k-p-1$ and $r = (i-1)/p$. We classify the existence of a solution V and its degree depending on the parameters:

1. $q = 1$. Then $m = 0$, but we are assuming $m \neq 0, 1$.

2. $q = 2$. Then $m = -1$. In this case, because $p > 0$, the expression of V_{k-2} contains a logarithm, and then the solution is not polynomial.

3. $q = 3$. Then $i = p + 1$, so $m = -2$ and $b = 0$, and then the system is Hamiltonian.

4. $q = 4$. Then $m = -3$ and $b = \pm 1$, so $b^2 = (m + 2)^2$ and there is a solution of degree 3.

5. If $p > 1$, $q \in \mathbb{N} \setminus \{1, 2, 3, 4\}$ and then there exists a solution of degree k with parameters (k, p, i), then $k - 3 = pq \in \mathbb{N}$ and there exists a solution of degree $p + 3 < k$ with parameters $(p + 3, 1, i')$, for $i' - 1 = (i - 1)/p$. If $p|(i - 1)$, then $i' \in \mathbb{N}$. If $p|(k - 2 - i)$, then $(k - 3 - (i - 1))/p = q - i' \in \mathbb{N}$. In both cases, $i' \in \mathbb{N}$. So the case (k, p, i) can be considered as the case $(p + 3, 1, i')$.

   Assume that $p > 1$ and $q \notin \mathbb{N}$. In this case, there is a solution of degree k if and only if either r or $q - r - 1$ belong to the set $\{2, \ldots, (k - p - 2)/p\} \subset \mathbb{N}$.

6. $p = 1$, $k > 7$ and $i \in \{3, \ldots, k - 5\}$. Then, there is a solution of degree k.

The general solution V, when it does exist, is

$$V(x, y) = (1 + x)^{2-q+r}(1 - x)^{1-r} F\left[-(\sqrt{-1})^{q+1}(1 + x)^{q-r-1}\big((1 - x)^r y - \right.$$
$$\left. 2^{r-1} p \ {}_2F_1\left(1 - r, q - 1 - r, q - r, (1 + x)/2\right)\big)\right],$$

where F is an arbitrary function and ${}_2F_1$ is the hypergeometric function defined in (2.74). The function F must be a polynomial of degree $p + 1$, because V has degree $p + 1$ in y: $F(Y) = a_0 + a_1 Y + \cdots + a_{p+1} Y^{p+1}$.

We next prove cases 5 and 6.

Case 5 If $i \in \{2p + 1, \ldots, k - p - 1\}$, $p|(i - 1)$ and $r \in \mathbb{N} \setminus \{1\}$, then the hypergeometric function in V becomes

$${}_2F_1(1 - r, q - r - 1, q - r, (1 + x)/2) =$$
$$\frac{q - r - 1}{2^{r-1}} \sum_{i=0}^{r-1} \left(\frac{(1 + x)^i (1 - x)^{r-i-1}}{q - r + i - 1} \prod_{j=1}^{i} \frac{r - j}{q - r + j - 2} \right).$$

If $i \in \{0, \ldots, k-3p-2\}$, $p|(k-2-i)$ and $r \in \mathbb{N}$, $r < q-2$, then the hypergeometric function in V becomes

$$
\begin{aligned}
&{}_2F_1(1-r, q-r-1, q-r, (1+x)/2) = \\
&\quad -(q-r-1)\left[\frac{(-2)^{q-r-1}}{(1+x)^{q-r-1}(q-2)}\prod_{j=1}^{q-r-2}\frac{q-r-1-j}{r+j-1}+\right. \\
&\quad \left.\sum_{i=0}^{q-r-2}\left(\frac{(1-x)^{r+i}}{(1+x)^{i+1}}\frac{2^{1-r}}{r+i}\prod_{j=1}^{i}\frac{q-r-1-j}{r+j-1}\right)\right].
\end{aligned}
$$

In both cases, solving an under–determined linear system we find a set of values a_i, $i = 0, \ldots, p+1$, for which V is a polynomial. Moreover, $a_0 = 0$ and the rest of the unknowns can be written as a linear combination of a_{p+1}.

Case 6 The existence of solutions is related to the value of $i \in \{0, \ldots, k-2\}$. If $i = 0$ or $i = 1$, then the hypergeometric function becomes ${}_2F_1(2, k-3, k-2, (1+x)/2)$ or ${}_2F_1(1, k-4, k-3, (1+x)/2)$ (respectively). Then no polynomial solution is obtained. If $i = 2$ or $i = k-4$, then there exists a solution of degree 3. If $i = k-3$, then $b = m$. We have the hypergeometric function ${}_2F_1(5-k, 5-k, 6-k, 2/(1+x))$, so there is no polynomial solution. If $i = k-2$, then the hypergeometric function becomes ${}_2F_1(-1, 4-k, 0, (1+x)/2)$, so there is no solution. If $3 \le i \le k-5$, then the hypergeometric function becomes

$$
{}_2F_1(2-i, k-3-i, k-2-i, (1+x)/2) = \sum_{j=0}^{i-2}\left(\frac{(k-3-i)(1+x)^j}{2^j j!(k-3-i+j)}\prod_{r=1}^{j}(r-i+1)\right).
$$

Solving an under–determined linear system we find a set of values a_i, for $i = 0, \ldots, p+1$, for which V is a polynomial. Moreover, $a_0 = 0$ and the rest of the unknowns can be written as a linear combination of a_{p+1}. ■

Proposition 2.3.27. *System* (2.88) *has a polynomial inverse integrating factor if and only if one of the following statements hold.*

(V.14) $b_{00} = 4$, $b_{10} = b_{20} = 0$. *Then* $V(x,y) = (x-y)^2$.

(V.15) $b_{00} = b_{20} + 4$, $b_{10} = -b_{20} \neq 0$. *Then* $V(x,y) = (x \pm 1)(b_{20}(x-1)^2 + 4(x-y)(y-1))$.

(V.16) $b_{00} - 4 \neq 0$, $b_{10} = b_{20} = 0$. *Then* $V(x,y) = (-1+x^2)(b_{00} - 4y^2)$.

(V.17) $b_{20} = (k-2)(k-4)$ *and* $b_{10} = (k-2p)\sqrt{b_{00} - (k-2p)^2 + (k-2)(k-4)}$, $b_{00} - (k-2p)^2 + (k-2)(k-4) \ge 0$, *with* $k > 4$, $p \in \{2, \ldots, k-2\}$. *Then* $V(x,y) = (-1+x^2)\left(p_1^1(x)y + p_2^1(x)\right)\left(p_1^2(x)y + p_2^2(x)\right)$, *where* $p_1^1(x)$ *and* $p_1^2(x)$ *are the Jacobi polynomials of degree* $p-2$ *and* $k-p-2$, *respectively, and the expressions of* p_2^1 *and* p_2^2 *can be obtained from* p_1^1 *and* p_1^2, *respectively.*

Proof: If $k \leq 4$ then straightforward computations show that there is a solution of degree 2 if and only if $b_{10} = b_{20} = 0$, $b_{00} - 4 = 0$; there is a solution of degree 3 if and only if $b_{00} = b_{20} + 4$, $b_{10} = \mp b_{20} \neq 0$, and we can assume $b_{10} = -b_{20}$ (otherwise we change the sign of x, y and t); and there is a solution of degree 4 if and only if $b_{10} = b_{20} = 0$, $b_{00} - 4 \neq 0$. The respective expressions of $V(x, y)$ are shown in Proposition 2.3.27.

The proof of this proposition follows the same steps as the proof of Proposition 2.3.22. From now on, we assume $k > 4$. Let $V(x, y)$ be a polynomial inverse integrating factor of system (2.88) of degree $k > 4$. By Lemma 2.3.1, we can write V as a polynomial of degree 2 in y:

$$V(x, y) = V_0(x) + V_1(x)y + V_2(x)y^2.$$

Equation (★) can be rewritten as the system

$$\begin{array}{rcl} -8V_0(x) + 4(-1 + x^2)V_0'(x) - (b_{00} + 2b_{10}x + b_{20}x^2)V_1(x) & = & 0, \\ 4V_0(x) + 4xV_1(x) - 2(-1 + x^2)V_1'(x) + (b_{00} + 2b_{10}x + b_{20}x^2)V_2(x) & = & 0, \\ V_1(x) + 2xV_2(x) - (-1 + x^2)V_2'(x) & = & 0. \end{array} \tag{2.93}$$

We obtain expressions for $V_0(x)$ and $V_1(x)$ in terms of $V_2(x)$ and its derivatives:

$$\begin{aligned} V_1(x) &= -2x\, V_2(x) + (-1 + x^2)\, V_2'(x), \\ V_0(x) &= -\frac{1}{4}\big((b_{00} - 4 + 2b_{10}x + (b_{20} - 4)x^2)V_2(x) + 4x(-1 + x^2)V_2'(x) - \\ &\quad 2(-1 + x^2)^2 V_2''(x)\big). \end{aligned}$$

Observe that if $V_2(x)$ is a polynomial of degree $k - 2$, then $V_1(x)$ and $V_0(x)$ are polynomials of degrees $k - 1$ and k, respectively. We substitute $V_1(x)$ and $V_0(x)$ in (2.93) to get

$$\begin{aligned} &(b_{10} + (2b_{00} - 8 + b_{20})x + 3b_{10}x^2 + b_{20}x^3)V_2(x) - \\ &\quad (-1 + x^2)(b_{00} - 4 + 2b_{10}x + b_{20}x^2)V_2'(x) + (-1 + x^2)^3 V_2'''(x) = 0. \end{aligned} \tag{2.94}$$

This is a differential equation with unknown $V_2(x)$, which must be a polynomial solution of degree $k - 2$, so we write $V_2(x) = \sum_{i=0}^{k-2} a_i x^i$.

Equation (2.94) can be written as a polynomial equation of degree $k + 1$ in x, and then we can transform it into a $(k + 2) \times (k - 1)$ homogeneous linear system. The equation corresponding to x^{k+1} is $(k - 3)(b_{20} - (k - 2)(k - 4)) = 0$. As $k > 4$, we take $b_{20} = (k - 2)(k - 4) > 0$.

We have a homogeneous linear system with $k - 1$ unknowns $a_0, \ldots, a_{k-2}$ and $k + 1$ equations if we exclude the equation corresponding to x^{k+1}. In order to have a non–trivial solution, all the $(k-1)$–minors of the matrix M_k of the system, given by

$$M_k = \begin{pmatrix}
-\xi_{k-1} & \rho_{k-2} & 0 & \chi_{k-1} & & & & \\
\vartheta_{k-2} & -\xi_{k-2} & 2\rho_{k-3} & 0 & \chi_{k-2} & & & \\
\xi_{k+1} & \vartheta_{k-3} & -\xi_{k-3} & 3\rho_{k-4} & 0 & \chi_{k-3} & & \\
\kappa_{k-4} & \xi_k & \vartheta_{k-4} & -\xi_{k-4} & 4\rho_{k-5} & 0 & \chi_{k-4} & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & \kappa_4 & \xi_8 & \vartheta_4 & -\xi_4 & (k-4)\rho_3 & 0 & \chi_4 \\
 & & \kappa_3 & \xi_7 & \vartheta_3 & -\xi_3 & (k-3)\rho_2 & 0 \\
 & & & \kappa_2 & \xi_6 & \vartheta_2 & -\xi_2 & (k-2)\rho_1 \\
 & & & & \kappa_1 & \xi_5 & \vartheta_1 & -\xi_1 \\
 & & & & & \kappa_0 & \xi_4 & \vartheta_0 \\
 & & & & & & \kappa_{-1} & \xi_3
\end{pmatrix},$$

where

$$\begin{array}{ll}
\kappa_{k-i} = (i-5)b_{20} - \chi_{k-i+6}, & \vartheta_{k-i} = -(i-1)b_{20} - (i-4)\rho_{k-i}, \\
\chi_{k-i} = i(i+1)(i+2), & \rho_{k-i} = -b_{00} - \nu_{k-i}, \\
\xi_{k-i} = (2i-1)b_{10}, & \nu_{k-i} = 3i^2 - 15i + 14.
\end{array}$$

must be zero. The following lemma is related to the roots of the $(k-1)$–minors of M_k. Its proof follows the same steps than the proof of Lemma 2.3.23.

Lemma 2.3.28. *There are at most $k-2$ values of b_{10} for which all the $(k-1)$–minors of M_k vanish. Moreover, if all the $(k-1)$–minors vanish for a non–zero value b_{10}, then they also vanish for $-b_{10}$.*

System (2.88) has the invariant algebraic curves $x \pm 1 = 0$. The sum of their cofactors is $2x$ and the divergence of (2.88) is $2x+2y$. We will find a polynomial inverse integrating factor of the form

$$V(x,y) = (-1+x^2)\big(\tilde{V}_0(x) + \tilde{V}_1(x)y + \tilde{V}_2(x)y^2\big),$$

where $\tilde{V}_i(x)$ are polynomials of degree $k-2-i$, $i=0,1,2$, such that $\tilde{V}_0(x) + \tilde{V}_1(x)y + \tilde{V}_2(x)y^2$ is formed by invariant algebraic curves and the sum of their corresponding cofactors is $2y$.

Let $p \in \{2,\ldots,k-2\}$ and $b_{10}^{(p)} = (k-2p)\sqrt{b_{00} - (k-2p)^2 + (k-2)(k-4)}$. We note that $b_{10}^{(p)} = -b_{10}^{(k-p)}$ for $p = 2,\ldots,[k/2]$. The following lemma is based on Theorem 2 of [15].

Lemma 2.3.29. *If $b_{10} = b_{10}^{(p)}$ where $p \in \{2,\ldots,k-2\}$, then system (2.88) has two invariant algebraic curves of respective degrees $p-1$ and $k-p-1$, and of the form $h(x,y) = p_1(x)y + p_2(x)$, $(p_1,p_2)=1$. For each curve $h(x,y)$, $p_1(x)$ is* *a Jacobi polynomial, of degree $p-2$ for the curve of degree $p-1$ and of degree $k-p-2$ for the curve of degree $k-p-1$. The expression of $p_2(x)$ can be obtained from $p_1(x)$ and the cofactor of $h(x,y)$. The product of both curves is an invariant algebraic curve of degree $k-2$ and cofactor $2y$.*

Proof: We assume that $h(x,y) = p_1(x)y + p_2(x)$ is an invariant algebraic curve of (2.88). Let $T(x) + a_2 y = a_0 + a_1 x + a_2 y \in \mathbb{C}[x,y]$ be its cofactor. Then

$$\dot{x}\frac{\partial h}{\partial x} + \dot{y}\frac{\partial h}{\partial y} - (T(x) + a_2 y)h = 0.$$

Writing this differential equation as a system of equations, we get

$$\begin{aligned}
(a_2 - 1)p_1(x) &= 0,\\
(-1+x^2)p_1'(x) - T(x)p_1(x) - p_2(x) &= 0,\\
(-1+x^2)p_2'(x) + N(x)p_1(x) - T(x)p_2(x) &= 0,
\end{aligned}$$

where $N(x) = -b_{00}/4 - b_{10}x/2 - (k-2)(k-4)x^2/4$. From the first equation it follows easily that $a_2 = 1$. The expression of $p_2(x)$ can be obtained explicitly from the second equation. From the second and the third equations we get $T(x)^2 + N(x) = \lambda(-1+x^2)$, where $\lambda = (2\eta+1)m - \eta(\eta+1)$, $\eta = \deg p_1 \in \{0,\dots,k-4\}$ (see Lemma 4 of [15] for a proof), and

$$(T'(x) - \lambda)p_1(x) - 2(T(x) - x)p_1'(x) + (1-x^2)p_1''(x) = 0$$

(see Proposition 3 of [15]). Moreover, $4a_0^2 - b_{00} = -4\lambda$, $4a_1^2 - (k-2)(k-4) - 4\lambda = 0$ and $4a_0a_1 - b_{10} = 0$. So, $a_0 = \pm\sqrt{b_{00} - 2[2\eta^2 + (2\eta+1)(1 \pm (k-3))]}/2$, $a_1 = (2\eta + 1 \pm (k-3))/2$ and

$$b_{10} = \pm(2\eta + 1 \pm (k-3))\sqrt{b_{00} - 2[2\eta^2 + (2\eta+1)(1 \pm (k-3))]}.$$

We note that here the first $\pm$ corresponds to a_0, and the others correspond to a_1; they are independent each other. Taking $\alpha = -a_1 - a_0$, $\beta = -a_1 + a_0$, the solution of this equation is the Jacobi polynomial $P_\eta^{(\alpha,\beta)}(x)$. We solve this equation for $a_1 = 2\eta - k + 4$ and for $\eta = p-2, k-p-2$, $p \in \{2,\dots,k-2\}$. For the first value of η, we take $a_0 = -\sqrt{b_{00} - (k-2p)^2 + (k-2)(k-4)}/2$; for the second one, we take $a_0 = \sqrt{b_{00} - (k-2p)^2 + (k-2)(k-4)}/2$. In both cases, we obtain $b_{10} = b_{10}^{(p)}$. Then we obtain two invariant algebraic curves, one of degree $p-1$ and one of degree $k-p-1$; their product is an invariant algebraic curve $f(x,y) = 0$ of degree $k-2$ and cofactor $2y$. ∎

The curve $f(x,y) = 0$ has degree $k-2$ and cofactor $2y$. Then, the polynomial $V(x,y) = (-1+x^2)f(x,y)$ is a polynomial inverse integrating factor of degree k for our system.

We have obtained $k-3$ values of b_{10} for which all the $(k-1)$–minors of M_k vanish. By Lemma 2.3.28, there are at most $k-2$ values of b_{10} for which all these $(k-1)$–minors vanish. The arguments used in Proposition 2.3.22 show that there are no more families for which we have a polynomial inverse integrating factor.

We note that if $p \in \{2, k-2\}$ and $b_{00} = b_{20} + 4$, then $b_{10} = \pm b_{20} \neq 0$. In this case we already have a polynomial inverse integrating factor of degree 2 or 3.

There are exactly $k-3$ values of b_{10} which vanish all the $(k-1)$–minors of M_k, and then there are $k-3$ families of system (2.88) for which there is a polynomial inverse integrating factor of degree k. The proof is finished. ∎

2.3.8 The cases $P(x,y) = r + xy$

We deal in this section with systems (I) and (II). These two families of systems correspond to the quadratic systems

$$\dot{x} = P(x,y) = r + xy, \quad \dot{y} = Q(x,y) = d + ax + by + lx^2 + mxy + ny^2, \tag{2.95}$$

where $r \in \{0,1\}$ and $d,a,b,l,m,n \in \mathbb{R}$. First we study the subfamilies of systems (2.95) which have a polynomial inverse integrating factor of degree $k \leq 5$. We split the results into two propositions depending on the value of r. The proofs follow from straightforward computations of the linear systems $A_i V^i = \mathbf{0}$, for $i = 1, \ldots, 5$, followed by affine changes and possibly rescaling time.

Proposition 2.3.30. *A system of type* (2.95) *with* $r = 0$ *having a polynomial inverse integrating factor* $V(x,y)$ *of degree* $k \leq 5$ *can be written, after an affine change of variables and a rescaling of the time if it is necessary, as* $\dot{x} = xy$, $\dot{y} = Q(x,y)$, *where* Q *is one of the polynomials below.*

(II.1) $Q(x,y) = d + ax + lx^2 - y^2/2$ *with* $d^2 + a^2 + l^2 \neq 0$. *In this case the system is Hamiltonian. We distinguish four subcases in order to give a simpler expression of* $Q(x,y)$.

- **(II.1a)** *If* $al \neq 0$, *then* $Q(x,y) = b_{00} + x + \delta x^2 - y^2/2$ *where* $\delta = \pm 1$ *and* $b_{00} = d|l|/a^2 \in \mathbb{R}$. *This expression is obtained by the change* $(|l|x/a, \sqrt{|l|}y/a, at/\sqrt{|l|}) \to (x,y,t)$.
- **(II.1b)** *If* $a = 0$ *and* $l \neq 0$, *then* $Q(x,y) = \sigma + \delta x^2 - y^2/2$ *where* $\delta = \pm 1$ *and* $\sigma = -1, 0, 1$. *We have used the change* $(\sqrt{|l/d|}x, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x,y,t)$ *if* $d \neq 0$, *or the change* $\sqrt{|l|}x \to x$ *if* $d = 0$.

(II.1c) *If* $a \neq 0$ *and* $l = 0$, *then* $Q(x, y) = \sigma + x - y^2/2$ *where* $\sigma = -1, 0, 1$. *We have used the change* $(ax/|d|, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$ *if* $d \neq 0$, *or the change* $ax \to x$ *if* $d = 0$.

(II.1d) *If* $a = l = 0$ *and* $d \neq 0$, *then* $Q(x, y) = \delta - y^2/2$ *where* $\delta = -1, 1$. *This expression is obtained by the change* $(y/\sqrt{|d|}, \sqrt{|d|}t) \to (y, t)$.

(II.2) $Q(x, y) = d + ax + lx^2$, *with* $d \neq 0$. *Then,* $V(x, y) = x$. *We distinguish two subcases in order to give a simpler expression of* $Q(x, y)$.

(II.2a) *If* $a \neq 0$ *then* $Q(x, y) = \delta + x + b_{20}x^2$ *where* $\delta = \pm 1$ *and* $b_{20} = d|l|/a^2 \in \mathbb{R}$, *by the change* $(ax/|d|, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$.

(II.2b) *If* $a = 0$ *then* $Q(x, y) = \delta + \sigma x^2$ *where* $\delta = \pm 1$ *and* $\sigma = -1, 0, 1$, *after the change* $(\sqrt{|l/d|}x, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$ *if* $l \neq 0$, *or after the change* $(y/\sqrt{|d|}, \sqrt{|d|}t) \to (y, t)$ *if* $l = 0$.

(II.3) $Q(x, y) = d + ax + lx^2 + y^2/2$ *with* $d^2 + a^2 + l^2 \neq 0$. *Then* $V(x, y) = x^2$. *We distinguish four subcases in order to give a simpler expression of* $Q(x, y)$.

(II.3a) *If* $da \neq 0$ *then* $Q(x, y) = \delta + x + b_{20}x^2 + y^2/2$ *where* $\delta = \pm 1$ *and* $b_{20} = d|l|/a^2 \in \mathbb{R}$. *This expression is obtained by the change* $(ax/|d|, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$.

(II.3b) *If* $d = 0$ *and* $a \neq 0$, *then* $Q(x, y) = x + \sigma x^2 + y^2/2$ *where* $\sigma = -1, 0, 1$, *after the change* $(|l|x/a, \sqrt{|l|}y/a, at/\sqrt{|l|}) \to (x, y, t)$ *if* $l \neq 0$, *or after the change* $(x/a, y/a, at) \to (x, y, t)$ *if* $l = 0$.

(II.3c) *If* $d \neq 0$ *and* $a = 0$, *then* $Q(x, y) = \delta + \sigma x^2 + y^2/2$ *where* $\delta = \pm 1$ *and* $\sigma = -1, 0, 1$, *after the change* $(\sqrt{|l/d|}x, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$ *if* $l \neq 0$, *or after the change* $(x/\sqrt{|d|}, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$ *if* $l = 0$.

(II.3d) *If* $d = a = 0$ *and* $l \neq 0$, *then* $Q(x, y) = \delta x^2 + y^2/2$ *where* $\delta = -1, 1$. *This expression is obtained by the change* $\sqrt{|l|}x \to x$.

(II.4) $Q(x, y) = (1 + \delta x)(b_{00} + y)$ *where* $\delta = 0, 1$, $b_{00} \neq 0$ *and we obtain* $V(x, y) = x(b_{00} + y)$.

(II.5) $Q(x, y) = d + ax + lx^2 + y^2$, *with* $d^2 + a^2 + l^2 \neq 0$. *Then,* $V(x, y) = x^3$. *We distinguish four subcases in order to give a simpler expression of* $Q(x, y)$.

(II.5a) *If* $da \neq 0$, *then* $Q(x, y) = \delta + x + b_{20}x^2 + y^2$ *where* $\delta = \pm 1$ *and* $b_{20} = d|l|/a^2 \in \mathbb{R}$.

(II.5b) *If* $d = 0$ *and* $a \neq 0$, *then* $Q(x, y) = x + \sigma x^2 + y^2$ *where* $\sigma = -1, 0, 1$.

(II.5c) *If* $d \neq 0$ *and* $a = 0$, *then* $Q(x, y) = \delta + \sigma x^2 + y^2$ *where* $\delta = \pm 1$ *and* $\sigma = -1, 0, 1$.

(II.5d) *If* $d = a = 0$ *and* $l \neq 0$, *then* $Q(x, y) = \delta x^2 + y^2/2$ *where* $\delta = -1, 1$.

All these expressions are obtained using the same changes as in system (II.3).

(II.6) $Q(x,y) = (b_{20}x + y)(\delta/b_{20} + x) + y^2$ *where* $\delta = 0, 1$, $b_{20} \in \mathbb{R} \setminus \{0\}$ *and we get* $V(x,y) = x^2(b_{20}x + y)$.

(II.7) $Q(x,y) = x + y + y^2$ *and* $V(x,y) = x^2(x + y)$.

(II.8) $Q(x,y) = b_{00} + \delta x + y + y^2$ *where* $\delta = 0, 1$, $b_{00} \in \mathbb{R} \setminus \{0\}$ *and we get*

$$V(x,y) = x((b_{00} + \delta x)^2 + (b_{00} + \delta x)y + b_{00}y^2).$$

(II.9) $Q(x,y) = \sigma + \delta x + b_{20}x^2 + ny^2$ *where* $\sigma = -1, 0, 1$, $\delta = 0, 1$, $b_{20} \in \mathbb{R}$, $n \neq -1/2, 0, 1/2, 1$ *and we have*

$$V(x,y) = x\left(\sigma + \frac{2\delta n}{2n-1}x + \frac{b_{20}n}{n-1}x^2 + y^2\right).$$

(II.10) $Q(x,y) = d + by + ny^2$ *where* $d \neq 0$ *and* $n \neq 0, 1$. *We get* $V(x,y) = x(d + by + ny^2)$.

(II.11) $Q(x,y) = b_{20}\delta(n-1)/n + b_{20}\delta(2n-1)x/n + \delta y + b_{20}x^2 + xy + ny^2$ *where* $\delta = 0, 1$, $b_{20} \in \mathbb{R} \setminus \{0\}$, $n \neq 0, 1$, *and we get*

$$V(x,y) = x\big(\delta(b_{20}(n-1)^2 + 2b_{20}n(n-1)x + n(n-1)y) + n^2(b_{20}x^2 + xy + (n-1)y^2)\big).$$

(II.12) $Q(x,y) = x - 6x^2/25 + xy + y^2/3$ *and*

$$V(x,y) = \big(36x^2 - 30x(2y+5) + 25y^2\big)\big(216x^3 - 125y^3 - 270x^2(2y+5) + 75x(25 + 3y(2y+5))\big).$$

(II.13) $Q(x,y) = -2/3 + x - y/3 - 6x^2/25 + xy + y^2/3$ *and*

$$V(x,y) = \big(6x - 5y - 5\big)\big(36x^2 - 30x(2y+5) + 25(y-2)^2\big)\big(36x^2 - 15x(4y+7) + 25(y+1)(y-2)\big).$$

(II.14) $Q(x,y) = b_{00}\delta + \sigma(1-\delta) - \delta x + \delta y - x^2 + 3xy - y^2$ *where* $b_{00} \in \mathbb{R}$, $\sigma = \pm 1$, $\delta = 0, 1$, *and we get*

$$V(x,y) = x\big(b_{00}\delta\sigma(1-\delta) - x(\delta + x - y)\big)\big(b_{00}\delta\sigma(1-\delta) - (x-y)(\delta + x - y)\big).$$

(II.15) $Q(x,y) = b_{00} + y + \delta x^2 + 2y^2$ *where* $b_{00} \in \mathbb{R}$, $\delta = \pm 1$, *and we get*

$$V(x,y) = x\big((b_{00} + \delta x^2)^2 + (b_{00} + \delta x^2)y + 2b_{00}y^2\big).$$

Proposition 2.3.31. *A system of type (2.27) with $r = 1$ having a polynomial inverse integrating factor $V(x, y)$ of degree $k \leq 5$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = 1 + xy$, $\dot{y} = Q(x, y)$, where Q is one of the polynomials below.*

(I.1) $Q(x, y) = d + ax + lx^2 - y^2/2$. *The system is Hamiltonian. We split it into two systems:*

- **(I.1a)** $\dot{x} = 1 + a_{20}x^2 + 2xy$, $\dot{y} = x + b_{20}x^2 - 2a_{20}xy - y^2$ *where* $a_{20} = (2/a)^{2/3}d$, $b_{20} = (2l - 3d^2)/(2a^2)^{2/3}$ *and* $a \neq 0$.
- **(I.1b)** $\dot{x} = 1 + \sigma x^2 + 2xy$, $\dot{y} = b_{20}x^2 - \sigma xy - y^2$ *where* $\sigma = -1, 0, 1$ *and* $b_{20} = l/(2|d|) - 3/4$ *if* $d \neq 0$, *or* $b_{20} = \operatorname{sign}(l)$ *if* $d = 0$.

(I.2) $Q(x, y) = y(\delta + b_{11}x - y)$ *where* $\delta = 0, 1$, $b_{11} \in \mathbb{R}$, *and we get* $V(x, y) = y$.

(I.3) $Q(x, y) = 1 + b_{11} + b_{11}x + y + b_{11}xy$ *where* $b_{11} \in \mathbb{R}$, *and we get* $V(x, y) = 1 + x + xy$.

(I.4) $Q(x, y) = b_{00} + b_{10}x + \delta y + y^2$ *where* $b_{00}, b_{10} \in \mathbb{R}$, $\delta = 0, 1$, *and we have*

$$V(x, y) = -b_{10} - b_{00}\delta + (b_{00}^2 - b_{10}\delta)x - (b_{00} + \delta)y + 2b_{00}b_{10}x^2 + (b_{00}\delta - 3b_{10})xy - 2\delta y^2 + b_{10}^2x^3 + b_{10}\delta x^2y + b_{10}xy^2 - y^3.$$

(I.5) $Q(x, y) = d + ax + \delta y + lx^2 + b_{11}xy + ny^2$ *where* $b_{11} \in \mathbb{R}$, $n \neq -1, -1/2, 0, 1$,

$$d = \frac{\delta(n+1)}{(2n+1)^2} - \frac{b_{11}(n+1)}{(n-1)(2n+1)}, \quad a = \frac{b_{11}\delta(n+1)(2n-1)}{(n-1)(2n+1)^2},$$
$$l = \frac{b_{11}^2 n(n+1)}{(n-1)(2n+1)^2}$$

$\delta = 0, 1$. *We get*

$$V(x, y) = \big((n^2-1)\delta + b_{11}n(n+1)x + n(n-1)(2n+1)y\big) \big((n-1)(2n+1) + (n^2-1)\delta x + b_{11}n(n+1)x^2 + (n^2-1)(2n+1)xy\big).$$

(I.6) $Q(x, y) = b_{00} + y + y^2/2$ *where* $b_{00} \in \mathbb{R} \setminus \{3/8\}$, *and we get*

$$V(x, y) = \big(2b_{00} + 2y + y^2\big)\big(2b_{00}(2x-1) - (y+2)^2\big).$$

(I.7) $Q(x, y) = b_{00} + \delta x^2 + y^2/2$ *where* $\delta = \pm 1$, $b_{00} \in \mathbb{R}$, *and we get*

$$V(x, y) = 8\delta(1 + xy)^2 - (2b_{00} + 2\delta x^2 + y^2)^2.$$

(I.8) $Q(x,y) = \delta + y^2/3$ *where* $\delta = \pm 1$*, and we get*

$$V(x,y) = \big(3\delta + y^2\big)\big(9x - 9\delta y - 2y^3\big).$$

(I.9) $Q(x,y) = b_{00} + y + y^2/3$ *where* $b_{00} \neq 12/25$*, and we get*

$$V(x,y) = \big(3b_{00} + 3y + y^2\big)\big(9b_{00}^2 x + 9b_{00}(6x - y - 5) - 2(y+3)^3\big).$$

(I.10) $Q(x,y) = 3b_{01}^2/25 + x + b_{01}y - b_{01}x^2/5 + 2y^2$ *where* $b_{01} \in \mathbb{R}$*, and we get*

$$V(x,y) = \big(3b_{01} + 10y - 5x^2\big)\big(5 + 3b_{01}x + 15xy - 5x^3\big).$$

In the rest of this subsection we assume $k > 5$. We consider the quadratic differential system

$$\dot{x} = P(x,y) = r + xy, \quad \dot{y} = Q(x,y) = d + ax + by + lx^2 + mxy + ny^2, \tag{2.96}$$

with $r \in \{0,1\}$ and $d, a, b, l, m, n \in \mathbb{R}$. Assume $l^2 + m^2 \neq 0$, otherwise interchanging x and y we are in cases (III) to (X), which have already been studied. Assume that $V(x,y) = \sum_{j=0}^{k} V_j(x,y)$ is a polynomial inverse integrating factor of system (2.96) of degree $k > 5$, with $V_0 \in \mathbb{R}$ and $V_j(x,y)$ a homogeneous polynomial of degree j, for $j = 1, \ldots, k$. By using the equation of definition ($\bigstar$) of the inverse integrating factor (multiplied by x) and the Euler's formula

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} = \sum_{j=1}^{k} jV_j,$$

we obtain the equation

$$P\left(\sum_{j=1}^{k} jV_j - y\frac{\partial V}{\partial y}\right) + xQ\frac{\partial V}{\partial y} - \operatorname{div}(P,Q)xV = 0,$$

where $\operatorname{div}(P,Q) = \partial P/\partial x + \partial Q/\partial y = b + mx + (2n-1)y$; or equivalently,

$$\sum_{j=1}^{k}(jr - bx + xN_j)V_j + \sum_{j=1}^{k}\big(R + xS + xT\big)\frac{\partial V_j}{\partial y} = \operatorname{div}(P,Q)xV_0, \tag{2.97}$$

where $N_j = -mx + (j - 2n - 1)y$, $R = dx - ry$, $S = ax + by$ and $T = lx^2 + mxy + (n-1)y^2$. Equation (2.97) can be written as a system of differential equations, one equation for each degree in x and y. Next we write the equations of degree $k+2$, $k+1$, $j+2$ (for $j = k-2, \ldots, 0$) and 1:

$$xN_kV_k + xT\frac{\partial V_k}{\partial y} = 0, \tag{2.98}$$

$$xN_{k-1}V_{k-1} + xT\frac{\partial V_{k-1}}{\partial y} = bxV_k - xS\frac{\partial V_k}{\partial y}, \tag{2.99}$$

$$xN_jV_j + xT\frac{\partial V_j}{\partial y} = bxV_{j+1} - xS\frac{\partial V_{j+1}}{\partial y} - r(j+2)V_{j+2} - R\frac{\partial V_{j+2}}{\partial y}, \tag{2.100}$$

$$rV_1 + R\frac{\partial V_1}{\partial y} = bV_0x. \tag{2.101}$$

We denote by E_j the equation (2.100), for $j = k-2, \dots, 0$. We solve this system of differential equations recursively, starting by equation (2.98) and finishing by equation (2.101). We main follow the results 2.3.2, 2.3.3 and 2.3.4 given in Section 2.3. These results can be found in [9].

First we study equation (2.98).

Proposition 2.3.32. *Equation* (2.98) *has a polynomial solution* V_k *if and only if one of the following statements hold.*

(a) $l \neq 0$, $m = 0$ *and* $n = 1 - (k-3)/(2(p-1))$ *with* $p \in \mathbb{N} \cup \{0\}$, $2p \leq k$, $p \neq 1$. *Then* $V_k(x,y) = x^{k-2p}T^p$. *Under these conditions system* (2.96) *is*

$$\dot{x} = r + xy, \quad \dot{y} = d + ax + by + lx^2 + \left(1 - \frac{k-3}{2(p-1)}\right)y^2. \tag{2.102}$$

(b) $m \neq 0$, $l = (p-1)(p+q-1)(2p+q-2)m^2/(q^2(k-3))$ *and* $n = 1 - (k-3)/(2p+q-2)$ *where* $p \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$, $2p+q \leq k$, $2p+q \neq 2$. *Then* $V_k(x,y) = x^{k-2p-q}F^pG^{p+q}$. *Under these conditions system* (2.96) *is*

$$\dot{x} = r + xy, \quad \dot{y} = d + ax + by + \frac{(p-1)(p+q-1)(2p+q-2)}{q^2(k-3)}x^2 +$$
$$mxy + \left(1 - \frac{k-3}{2p+q-2}\right)y^2. \tag{2.103}$$

Proof: We write $V_k = T^pW$, where $p \in \mathbb{N} \cup \{0\}$, $2p \leq k$ and W is a homogeneous polynomial of degree $k-2p$ such that $T \nmid W$. Equation (2.98) becomes

$$FW + T\frac{\partial W}{\partial y} = 0, \tag{2.104}$$

where $F = m(p-1)x + \bar{n}y$, $\bar{n} = k - 3 + 2(p-1)(n-1)$. Observe that if $p = 0$ then $W \equiv V_k$, $F \equiv N_k$ and (2.104) is just (2.98). If $F \equiv 0$ and $p = 1$, then we get $k = 3$, but we are assuming $k > 5$. If $F \equiv 0$ and $p \neq 1$, then $m = 0$, $l \neq 0$ and

$$n = 1 - \frac{k-3}{2(p-1)}.$$

As $T \not\equiv 0$, by (2.104) we have $W_y \equiv 0$, and then $W = x^{k-2p}$. The expression of V_k is given by $V_k(x,y) = x^{k-2p}T^p$ and case (a) follows.

If $F \not\equiv 0$, then as $T \nmid W$ we have $T = FG$, where $G = \alpha x + \beta y$, $\alpha, \beta \in \mathbb{R}$. Moreover, $G|W$. From $T = FG$ we compute α, β and l:

$$\alpha = \frac{k-3+(p-1)(n-1)}{\bar{n}^2}m, \quad \beta = \frac{n-1}{\bar{n}}, \quad l = \alpha m(p-1).$$

The expressions of α and β above are well defined because if $\bar{n} = 0$, then $n = 1$ and $k = 3$, which is not possible. Equation (2.104) becomes

$$W + G\frac{\partial W}{\partial y} = 0.$$

As $G|W$, we have $W = G^q\bar{W}$, where $q \in \mathbb{N}$, $2p + q \leq k$ and $\bar{W}$ is a certain homogeneous polynomial of degree $k - 2p - q$ such that $G \nmid \bar{W}$. We obtain the equation

$$(1 + q\beta)\bar{W} + G\frac{\partial \bar{W}}{\partial y} = 0.$$

As $G \nmid \bar{W}$, we must take $1 + q\beta = 0$. If $2p + q - 2 = 0$, that is $p = 0$ and $q = 2$, then $k = 3$. So $2p + q - 2 \neq 0$, and then

$$n = 1 - \frac{k-3}{2p+q-2}.$$

The expressions of α, β and l can be rewritten as

$$\begin{aligned} \alpha &= \frac{(p+q-1)(2p+q-2)}{q^2(k-3)}m, \quad \beta = -\frac{1}{q}, \\ l &= \frac{(p-1)(p+q-1)(2p+q-2)}{q^2(k-3)}m^2. \end{aligned}$$

We also take $m \neq 0$. Finally, as $G \not\equiv 0$, we have $\bar{W}_y \equiv 0$, so $\bar{W} = x^{k-2p-q}$. The expression of V_k is given by $V_k(x,y) = x^{k-2p-q}F^pG^{p+q}$ and case (b) follows. ∎

We study cases (a) and (b) of Proposition 2.3.32 separately. Case (a) is the easiest one, and from it we just obtain a solution V of degree $k = 7$, as the following proposition shows.

Proposition 2.3.33. *A system of type* (2.102) *having a polynomial inverse integrating factor $V(x,y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = 1 + xy$, $\dot{y} = Q(x,y)$, where*

(I.11) $Q(x,y)=\delta x^2+3y^2$ *with* $\delta=\pm1$, *and we get*

$$V(x,y)=\big(3y-\delta x^3\big)\big(1+4xy-\delta x^4\big).$$

Proof: Equation (2.99) can be written as

$$N_{k-1}V_{k-1}+TV_{k-1,y}=T^{p-1}L, \tag{2.105}$$

where

$$L=x^{k-2p}(bT-pST_y)=\\ x^{k-2p}\left(blx^2+\frac{ap(k-3)}{p-1}xy+\frac{b(2p-1)(k-3)}{2(p-1)}y^2\right).$$

If $p=0$, then $L=bx^kT$. We apply Lemma 2.3.4 to (2.105) to get $b=0$ and $V_{k-1}\equiv 0$. Under these conditions, there already exists a quadratic system having a polynomial inverse integrating factor of degree $k\le 3$ for $r=0$, as it follows from cases (II.1), (II.2), (II.3), (II.5) and (II.9) of Proposition 2.3.30, so we take $r=1$. Equation E_{k-2} is

$$-2yV_{k-2}+TV_{k-2,y}=-kx^{k-1}.$$

By Lemma 2.3.4, we must take $k=7$. We get $V_{k-2}=V_5=-7x^4y/l$. Equation E_4 is

$$-3yV_4+TV_{4,y}=\frac{7a}{l}x^5.$$

We get $a=0$ and $V_4\equiv 0$. Solving equation E_3, we get $V_3=\frac{l^2v_{1,2}-14}{2l}x^3+v_{1,2}xy^2$, $v_{1,2}\in\mathbb{R}$. We must also take $d=0$. From equation E_2, we get $V_2\equiv 0$. From equation E_1, we get $V_1=3y/l^2$ and $v_{1,2}=12/l^2$. Solving equation E_0 we obtain $V_0=0$, and equation (2.101) gives the identity $0=0$. So we obtain (I.11) after the change $(|l|^{1/4}x, y/|l|^{1/4}, |l|^{1/4}t)\to(x,y,t)$.

Assume $p>1$. From (2.105), as $l\neq 0$ we have $T|V_{k-1}$, so there exist $j\in\mathbb{N}$ and a homogeneous polynomial W such that $T\nmid W$ and $V_{k-1}=T^jW$. Equation (2.105) is then equivalent to

$$T^j((N_{k-1}+jT_y)W+TW_y)=T^{p-1}L.$$

We must take $j=p-1$. The equation becomes, after simplifying,

$$\left(\frac{k-3}{p-1}-1\right)yW+TW_y=L.$$

Applying Lemma 2.3.4, we get $b = 0$ and

$$V_{k-1} = \frac{ap(k-3)}{k-p-2} x^{k-2p-1} T^{p-1},$$

where $k - p - 2 \neq 0$ because $k > 5$. Equation E_{k-2} is

$$N_{k-2} V_{k-2} + T V_{k-2,y} = T^{p-2} L,$$

where $L = x^{k-2p-1}(a_0 x^4 + a_1 x^3 y + a_2 x^2 y^2 + a_3 x y^3 + a_4 y^4)$,

$$\begin{aligned} a_0 &= -kl^2 r, \quad a_1 = \frac{p(k-3)(dl(k-p-2) + a^2(k-3)(p-1))}{(k-p-2)(p-1)}, \\ a_2 &= \frac{lr(k-3)(k-p)}{p-1}, \quad a_3 = -\frac{dp(k-3)^2}{2(p-1)^2}, \quad a_4 = -\frac{r(k-2p)(k-3)^2}{4(p-1)^2}. \end{aligned}$$

We have $V_{k-2} = T^{p-2} W$, where W is a homogeneous polynomial such that $T \nmid W$. Then, we must solve the equation

$$2y \frac{k-p-2}{p-1} W + T W_y = L.$$

Applying Lemma 2.3.4, we obtain $k = 3p$. Then, $n = -1/2$ and $b = m = 0$, so the system is Hamiltonian and we are in (II.1). ∎

The rest of this section is dedicated to the proof of the following proposition.

Proposition 2.3.34. *Under the hypotheses of Proposition 2.3.32(b), a system of type* (2.103) *having a polynomial inverse integrating factor $V(x,y)$ can be written, after an affine change of variables and a rescaling of the time if it is necessary, as $\dot{x} = r + xy$, $\dot{y} = Q(x,y)$, where $r \in \{0,1\}$ and $Q(x,y)$ are stated below.*

(I.12) *$r = 1$ and $Q(x,y) = -5\delta/7 - 15x^2/98 + \delta xy + y^2/5$ where $\delta = \pm 1$ and*

$$\begin{aligned} V(x,y) = \big(&26250x + 3375\delta x^3 + 36750\delta y - 9450x^2 y + 8820\delta x y^2 - \\ &2744y^3\big)\big(1531250 + 525000\delta x^2 + 50625x^4 + 245000xy - \\ &189000\delta x^3 y - 686000\delta y^2 + 264600x^2 y^2 - 164640\delta x y^3 + 38416y^4\big). \end{aligned}$$

(I.13) *$r = 1$ and $Q(x,y) = b_{00}\delta - (s-1)x^2/(2s^2) + \delta xy - y^2/(2(s-1))$ where $\delta = \pm 1$, $b_{00} \in \mathbb{R}$ and $s \in \mathbb{N}$.*

(I.14) *$r = 1$ and $Q(x,y) = d - b_{11}(q-2)x/q + y - (q-2)b_{11}^2 x^2/q^2 + b_{11} xy - y^2/(q-2)$ where $\delta = \pm 1$, $b_{11} \neq 0$ and $q \in \mathbb{N}$, $q > 3$. Moreover, d satisfies*

$$\begin{aligned} &qj^2(q-2)(q-2-j)^2 - jq(q-2-j)(q-2-2j)^2 d - \\ &(q-2)(q-2-2j)^2(j^2 - (q-2)(j+1))b_{11} = 0, \end{aligned}$$

for $j \in \{1, 2, 3, \ldots, [q-3]/2\}$.

(I.15) $r = 1$ *and* $Q(x, y) = b_{00}\delta + (q+2)x^2/q^2 + \delta xy + y^2/(q+2)$ *where* $\delta = \pm 1$, $b_{00} \in \mathbb{R}$ *and* $q \in \mathbb{N} \setminus \{1\}$.

(I.16) $r = 1$ *and* $Q(x, y) = d - b_{11}(q+2)x/q + y + (q+2)b_{11}^2 x^2/q^2 + b_{11}xy + y^2/(q+2)$ *where* $b_{11} \neq 0$. *Moreover,* d *satisfies*

$$q\,j^2(q+2)(q+2-j)^2 + q\,j(q+2-j)(q+2-2j)^2 d + \\ (q+2)(q+2-2j)^2(j^2 - (q+2)(j-1))b_{11} = 0,$$

for $j \in \{1, 2, 3, \ldots, [(q+1)/2]\}$.

(I.17) $r = 1$ *and* $Q(x, y) = 10(b_{11}+2)/9 + 70b_{11}x/27 + y + 10b_{11}^2x^2/27 + b_{11}xy - y^2/5$ *where* $b_{11} \neq 0$ *and*

$$V(x, y) = \big(60 + 10b_{11}x - 9y\big)\big(45 + 60x + 10b_{11}x^2 + 36xy\big)\big(236196xy^5 - \\ 98415(10x(6+b_{11}x) - 3)y^4 + 145800(6+b_{11}x)(10x(6+b_{11}x) - 9)y^3 - \\ 81000(6+b_{11}x)^2(10x(6+b_{11}x) - 27)y^2 - 1620000(6+b_{11}x)^3 y + \\ 50000x(2b_{11}^5x^5 + 60b_{11}^4x^4 + 9b_{11}^3(80+b_{11})x^3 + 216b_{11}^2(20+b_{11})x^2 + \\ 648b_{11}(20+3b_{11})x + 7776(2+b_{11}))\big).$$

(I.18) $r = 1$ *and* $Q(x, y) = -(2p-1)\delta/2 + p(2p-1)x^2/2 + \delta xy + y^2/(2p-1)$ *where* $\delta = \pm 1$ *and* $p \in \mathbb{N}$, $p > 2$.

(I.19) $r = 1$ *and* $Q(x, y) = (2p+q-2)\delta/(2q) + (p-1)(2p+q-2)x^2/(2q^2) + \delta xy - qy^2/(2p+q-2)$, *where* $\delta = \pm 1$ *and* $p \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{N}$, $p + q > 2$, $p \neq 1$.

(I.20) $r = 1$ *and* $Q(x, y) = \delta/6 - x^2/18 + \delta xy - 3y^2$ *where* $\delta = \pm 1$ *and*

$$V(x, y) = \big(\delta x - 6y\big)\big(3 + x(\delta x - 6y)\big)\big(54 + 3\delta x^2(3 - 2xy) + x^4\big).$$

(II.16) $r = 0$ *and* $Q(x, y) = \delta + (p-1)(p+q-1)x^2/(2q^2) + xy - y^2$ *where* $\delta = \pm 1$ *and* $p \in \mathbb{N} \cup \{0\}$, $p \neq 1$, $q \in \mathbb{N}$, $2p + q > 3$. *We have an expression for* $V(x, y)$ *in the case* $p = 0$:

$$V(x, y) = \frac{1}{x}\Big(4\delta^{2-q}q^{2(2-q)}[2\delta q^2 - x((q-1)x - 2qy)]^q - \\ 2^q[2\delta q^2 - (q-1)x(x - 2qy)][2\delta q^2 - x((q-1)x - 2qy)]\Big).$$

(II.17) $r = 0$ *and* $Q(x, y) = d = -bm(q-2)x/q + by - (q-2)m^2x^2/q^2 + mxy - y^2/(q-2)$, *where* $m \neq 0$ *and* $q \in \mathbb{N}$, $q > 3$. *In order to give a simpler expression of* Q, *we distinguish two cases, depending on the value of* b.

(II.17a) *If* $b \neq 0$ *then* $Q(x, y) = -b_{00} - (q-2)x/q + y - (q-2)x^2/q^2 + xy - y^2/(q-2)$ *where* $b_{00} = -d/b^2$. *Here we have used the change* $(mx/b, y/b, bt) \to (x, y, t)$.

(II.17b) *If* $b = 0$ *then* $Q(x, y) = \delta - (q-2)x^2/q^2 + xy - y^2/(q-2)$ *where* $\delta = \pm 1$. *This is due to the change* $(mx/\sqrt{|d|}, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$.

(II.18) $r = 0$ *and* $Q(x, y) = \delta + (q+1)(q+2)x^2/(q^2(q-1)) + xy + 3y^2/(q+2)$ *where* $\delta = \pm 1$ *and* $q \in \mathbb{N} \setminus \{1\}$.

(II.19) $r = 0$ *and* $Q(x, y) = d + ax + y + lx^2 + xy + 3y^2/(q+2)$ *where* $q \in \mathbb{N} \setminus \{1\}$ *and*

$$d = -\frac{j(q+2)(q+2-j)}{(q+2-2j)^2}, \quad a = -\frac{(q+1)(q+2)(q-4)}{3q^2(q-1)}, \quad l = \frac{(q+1)(q+2)}{q^2(q-1)},$$

for $j \in \{1, 2, 3, \dots, [(q+1)/2]\}$.

Remark 2.3.35. 1. In systems (I.13) and (I.15) we assume that $b = 0$, but this condition is not proved. In systems (I.14), (I.16) and (II.19) the condition on d is not proved. In all these cases, there is numerical evidence that the conditions hold.

2. Systems (II.17) and (II.18) are not proved to have a polynomial inverse integrating factor, as we will show in the proof.

3. We do not have an expression of $V(x, y)$ in cases (I.13)–(I.16), (I.18), (I.19) and (II.16)–(II.19). In case (I.19), using Theorem 1.4.7 it is possible to find an expression for V, as we explain in the proof.

□

Equation (2.99) can be written as

$$N_{k-1}V_{k-1} + FGV_{k-1,y} = F^{p-1}G^{p+q-1}L,$$

where $L = x^{k-2p-q}(a_0x^2 + a_1xy + a_2y^2)$,

$$\begin{aligned} a_0 &= \frac{bm(p-1)(p+q-1)(2p+q-2) - aq^2(k-3)}{q^2(k-3)}m, \\ a_1 &= \frac{a(2p+q)(k-3)}{2p+q-2}, \quad a_2 = \frac{b(2p+q-1)(k-3)}{2p+q-2}. \end{aligned}$$

As $(G, N_{k-1}) = (F, N_{k-1}) = 1$, there exist $i, j \in \mathbb{N} \cup \{0\}$ and a homogeneous polynomial W such that $V_{k-1} = F^iG^jW$. Equation (2.99) is

$$F^iG^j((N_{k-1} + iGF_y + jFG_y)W + FGW_y) = F^{p-1}G^{p+q-1}L.$$

We must take $j = p + q - 1$. If $p = 0$, then $i = 0$ and the equation becomes

$$\left(-\frac{m}{q}x + \left(\frac{k-3}{q-2} - 1\right)y\right)W + FGW_y = F^{-1}L.$$

If $p > 0$, then $i = p - 1$ and the equation becomes

$$\left(-mx + \left(\frac{2(k-3)}{2p+q-2} - 1\right)y\right)W + FGW_y = L.$$

In both subcases, two cases must be considered:

(b.1) If $k = 2p + q + 1$ then $b = 0$ from Lemma 2.3.4.

(b.2) If $k \neq 2p + q + 1$ then $a = bl(2n-1)/(mn)$ from Lemma 2.3.4. Moreover,

$$V_{k-1}(x,y) = (k-3)\left(\frac{a(2p+q)}{2k-2p-q-4}x + \frac{b(2p+q-1)}{k-2p-q-1}y\right)x^{k-2p-q}F^{p-1}G^{p+q-1},$$

where $k \neq (2p+q+4)/2$ because $k > 5$.

The case (b.1) is studied in the following proposition.

Proposition 2.3.36. *Under the conditions of case (b.1), system* (2.103) *has no polynomial inverse integrating factor.*

Proof: Assume $p = 1$. Then we must take $q > 2$, $a = 0$ and $V_{k-1}(x,y) = v_1xyG^q$, where $v_1 \in \mathbb{R}$. Equation E_{k-2} is

$$N_{k-2}V_{k-2} + FGV_{k-2,y} = G^qL,$$

where $L = (-dmx^2 + (q+2)(d-mr)xy + ry^2)$. As $q > 2$, we have $(G, N_{k-2}) = 1$, and then $V_{k-2} = G^jW$, where W is a homogeneous polynomial. We must solve the equation

$$G^j((N_{k-2} + jFG_y)W + FGW_y) = G^qL.$$

We take $j = q - 1$, so the equation becomes $-q(W - yW_y) = L$, from which we get $d = mr$, and the system has a common factor.

Assume $p \neq 1$. Then

$$V_{k-1}(x,y) = a\left((2p+q)x + \frac{q^2(2p+q-1)}{m(p-1)(p+q-1)}y\right)xF^{p-1}G^{p+q-1}.$$

Equation E_{k-2} writes as

$$N_{k-2}V_{k-2} + FGV_{k-2,y} = F^{p-2}G^{p+q-2}L,$$

where L is a certain homogeneous polynomial of degree 4. We have $V_{k-2} = F^iG^jW$, where W is a homogeneous polynomial. We obtain

$$F^iG^j((N_{k-2} + iGF_y + jFG_y)W + FGW_y) = F^{p-2}G^{p+q-2}L.$$

We must take $i = p-2$ and $j = p+q-2$. Then the equation becomes $(-2mx + 2y)W + FGW_y = L$. Applying Lemma 2.3.4 we get $d = mr$ and

$$V_{k-2}(x, y) = \left(v_0x^3 + v_1x^2y + v_2xy^2 + y^3\right) F^{p-2}G^{p+q-2},$$

for certain coefficients v_i, $i = 0, 1, 2$. If $r = 0$ then the system has a common factor, so we take $r = 1$, and then $d = m$. Equation E_{k-3} is

$$N_{k-3}V_{k-3} + FGV_{k-3,y} = F^{p-3}G^{p+q-3}L,$$

where L is a certain homogeneous polynomial of degree 5. We have $V_{k-3} = F^iG^jW$, where W is a homogeneous polynomial. We must solve the equation

$$F^iG^j((N_{k-3} + iGF_y + jFG_y)W + FGW_y) = F^{p-3}G^{p+q-3}L.$$

Now we apply Lemma 2.3.4 again. If $p = 0$ and $q = 5$, then $j = 1$. If $q = p-1 > 0$ then $j = p+q-2$. Otherwise, $i = p-3$ and $j = p+q-3$. In all cases, we get $a = 0$ and $L \equiv 0$, so $V_{k-3} \equiv 0$. Equation E_{k-4} is

$$N_{k-4}V_{k-4} + FGV_{k-4,y} = F^{p-2}G^{p+q-2}L,$$

where L is a certain homogeneous polynomial of degree 2. We have $V_{k-4} = F^iG^jW$, where W is a homogeneous polynomial. We must solve the equation

$$F^iG^j((N_{k-4} + iGF_y + jFG_y)W + FGW_y) = F^{p-2}G^{p+q-2}L.$$

If $p = 0$ and $q = 6$, then $j = 1$. In this case, we must solve the equation $-6W + FW_y = F^{-2}G^2L$, from which we obtain $m = 0$, a contradiction. If $q = 2(p-1) > 0$ then $j = 3p-5$. In this case we obtain $m(3p-2) = 0$, another contradiction. So we must take $i = p-2$ and also $j = p+q-2$ (if $p > 1$) or $j = q-2$ (if $p = 0$). If $p > 1$ then the equation becomes $-2mxW + FGW_y = L$. Applying Lemma 2.3.4, we get $m^2(p-1)(p+q-1)(2p+q+2) = 0$, which is not possible. If $p = 0$ then the degree of W is zero and the equation becomes

$$\left(-\frac{2m}{q}x - 2y\right) W = -\frac{m(q-1)}{q^3},$$

from which $W = 0$, and then $m(q-1) = 0$, which is a contradiction because $m \neq 0$ and $k = q+1 > 5$. ∎

From now on we assume case (b.2). Equation E_{k-2} is

$$N_{k-2}V_{k-2} + FGV_{k-2,y} = F^{p-2}G^{p+q-2}L,$$

where L is a certain homogeneous polynomial of degree $k-2p-q+3$. We have $V_{k-2} = F^iG^jW$, where W is a homogeneous polynomial. We must solve the equation

$$F^iG^j((N_{k-2} + iGF_y + jFG_y)W + FGW_y) = F^{p-2}G^{p+q-2}L. \tag{2.106}$$

If $p = 0$ and $k = q = 7$, then $j = 4$. In this case, we get the solution (I.12).

Assume $p = 1$. From E_{k-2} we obtain $r = 1$ and $d = m$, and from E_{k-3} we get $b = 0$. Finally, from E_{k-4} we have $(k+1)(k-q-3)m^2 = 0$. Then the only case from which we could obtain a solution is $k = q+3$, but in this case we would be in case (b.1), so there is no solution. From now on we assume $p \neq 1$.

We take $i = p-2$ and $j = p+q-2$ in (2.106). Then the equation becomes

$$\left(-2mx + 2\left(\frac{2(k-3)}{2p+q-2} - 1\right)y\right)W + FGW_y = L.$$

Applying Lemma 2.3.4, we get three subcases:

(b.2.1) If $k = 4p+2q-1$ then $r = 0$.

(b.2.2) If $k = p+q+2$ then $p = 0, 2$.

(b.2.3) If $k \neq 4p+2q-1, p+q+2$, then $r = 1$ and

$$d = \frac{(p-1)(p+q-1)(2p+q-2)}{q^2(k-2p-q-1)}b^2 + \Big(2(k-3p)((k-2p-1)(p-1)-q^2)+ \\ q(k^2-3+k(4-10p)+6p(3p-1))\Big)\frac{(2p+q-2)m}{(k-3)(k-4p-2q+1)q^2}.$$

In all cases,

$$V_{k-2}(x,y) = p_3(x,y)x^{k-2p-q-1}F^{p-2}G^{p+q-2},$$

where $p_3(x,y)$ is a homogeneous polynomial of degree 3.

In the following three propositions we deal separately with cases (b.2.1), (b.2.2) and (b.2.3). These propositions will finish our proof.

Proposition 2.3.37. *Under the conditions of (b.2.1) the only case for which system* (2.103) *has a polynomial inverse integrating factor is (II.16).*

Proof: Equation E_{k-3} writes as

$$N_{k-3}V_{k-3} + FGV_{k-3} = F^{p-3}G^{p+q-3}L,$$

where $L = x^{2p+q-2}p_5(x,y)$, for a certain homogeneous polynomial p_5 of degree 5. We take $V_{k-3} = F^iG^jW$, with $i,j \in \mathbb{N}\cup\{0\}$ and W a homogeneous polynomial. If $p=0$ and $q=5$, then we take $j=3$, but we obtain $b=d=0$ and the system is homogeneous, so there is a solution of degree 3. Taking $j=p+q-3$ and $i=p-3$, we get $(-3mx+9y)W + FGW_y = L$. If $b \neq 0$, then $d = b^2(p-1)(p+q-1)/q^2$ and we obtain a solution of degree at most 3. If $b=0$ (and $d \neq 0$), then $V_{k-3} \equiv 0$. We note that all the computations hold also for $p=0$.

The polynomials V_{k-2i-1} are identically zero for all i. We obtain the polynomials V_{k-2i}, $i = 1,\ldots,2p+q-1$, solving equations E_{k-2i} in a recursive way. The integration constant must be taken as zero in every step, except for V_3. The integration constant appeared in the computation of V_3, denoted by C_1, can be obtained from equation (2.101), which is $V_{1,y}=0$ because this equation is linear in C_1 and its coefficient is $\mathrm{sign}(d)q(p-1)/2(p+q-1) \neq 0$. The system is

$$\dot{x} = xy, \quad \dot{y} = d + \frac{(p-1)(p+q-1)}{2q^2}m^2x^2 + mxy - y^2, \tag{2.107}$$

with $d \neq 0$. The polynomials V_{k-2i} have the factor $x^{2p+q-1-i}$, for $i = 0,\ldots,2p+q-2$. As $V_{1,y}=0$, the polynomial $V(x,y)$ vanishes at $x=0$. Applying the change $(mx/\sqrt{|d|}, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x,y,t)$, we obtain (II.16).

We can compute an expression of $V(x,y)$ for system (2.107) in the case $p=0$. First we compute the expression of V_{k-2i}, for $i=1,\ldots,q-1$. By a linear change of variables, we transform our system into

$$\dot{x} = x(x-y), \quad \dot{y} = s + y((2q-3)x+y),$$

where $s^2=1$. Equation E_{k-2i}, for $0<i<q-2$, is given by

$$\frac{is^i}{2^{i-1}}\binom{q}{i}x^{q-i}((q-2)x+y)^{q-i} - 2(ix+(q-i)y)V_{k-2i} + \\ 2y((q-2)x+y)V_{k-2i,y} = 0.$$

From this equation, we get

$$\begin{aligned} V_{k-2i}(x,y) \;=\; & \frac{s^i}{2^i}\binom{q}{i}x^{q-i-1}((q-2)x+y)^{q-i} + \\ & c_ix^{q-i-1}y^{i/(q-2)}((q-2)x+y)^{q-i-i/(q-2)}. \end{aligned}$$

As $i<q-2$, we take $c_i=0$. Equation E_3 is

$$sV_{5,y} - 2((q-2)x+y)V_3 + 2y((q-2)x+y)V_{3,y} = 0,$$

from which we get

$$V_3(x,y) = x((q-2)x+y)(k_1x+C_1y),$$

where $k_1 = q(q-1)(q-2)s^{q-2}/2^{q-1}$ and $C_1 \in \mathbb{R}$. In order to compute $V_1(x,y) = v_0x + v_1y$, we must solve the linear system

$$v_0 + v_1 = C_1 s, \quad 2v_0(q-1) = k_1 s + C_1 s(q-2),$$

which is obtained from E_0. We get

$$V_1(x,y) = s\frac{C_1(q-2)+k_1}{2(q-1)}x + s\frac{C_1q-k_1}{2(q-1)}y.$$

Finally, from (2.101) we have $C_1 = k_1/q$. So we obtain the polynomial inverse integrating factor

$$V(x,y) = \frac{k_1}{q}x\big(s+(q\,x+y)((q-2)x+y)\big) + \sum_{i=0}^{q-3}\frac{s^i}{2^i}\binom{q}{i}x^{q-i-1}((q-2)x+y)^{q-i},$$

with $s^2 = 1$. Back to the initial system, which is

$$\dot{x} = xy, \quad \dot{y} = d - \frac{m^2(q-1)}{2q^2}x^2 + mxy - y^2,$$

the polynomial inverse integrating factor can be written as

$$V(x,y) = \frac{1}{x}\Big(4d^{2-q}q^{2(2-q)}[2dq^2 - mx(m(q-1)x-2qy)]^q - 2^q[2dq^2 - m(q-1)x(mx-2qy)][2dq^2 - mx(m(q-1)x-2qy)]\Big).$$

■

Proposition 2.3.38. *Under the conditions of (b.2.2) the cases for which system* (2.103) *has a polynomial inverse integrating factor are (I.13)–(I.16) and (II.17)–(II.19).*

Proof: Assume $p = 0$. If $i \neq 3$, then from E_i we get an expression for V_i in a recursive way, and without new conditions. If $i = 3$, then we must solve the equation

$$(K_1x^2 + K_2xy + K_3y^2)(x+y)^2 - (x+2y)V_3(x,y) + y(x+y)V_{3,y}(x,y) = 0,$$

for certain expressions K_1, K_2, K_3. This equation has the solution

$$V_3(x,y) = -K_2xy(x+y)\ln y + \widetilde{V}_3(x,y),$$

where $\widetilde{V}_3(x,y)$ is a homogeneous polynomial of degree 3; so we must take $K_2 = 0$, which means that either $r = 0$, or $r = 1$ and $b = 0$ (only if q is even), or $r = 1$ and

$$\begin{aligned}&qj^2(q-2)(q-2-j)^2b^2 - jq(q-2-j)(q-2-2j)^2d-\\&(q-2)(q-2-2j)^2(j^2-(q-2)(j+1))m = 0,\end{aligned} \tag{2.108}$$

$j \in \{1, 2, 3, \ldots, [q-3]/2\}$. From equation (2.101) we get the expression of C_1. The system is

$$\dot{x} = r + xy, \quad \dot{y} = d - \frac{bm(q-2)}{q}x + by - \frac{(q-2)}{q^2}m^2x^2 + mxy - \frac{y^2}{q-2},$$

taking into account the conditions given above. We note that we do not have a proof from which we get the conditions derived from $K_2 = 0$ and the expression of $V(x,y)$.

If $r = 0$, then by the change $(mx/b, y/b, bt) \to (x, y, t)$ we get (II.17). If $r = 1$ and $b = 0$, then by the change $(\sqrt{|m|}x, y/\sqrt{|m|}, \sqrt{|m|}t) \to (x, y, t)$ we get (I.13). If $r = 1$ and equation (2.108) holds, then by the change $(bx, y/b, bt) \to (x, y, t)$ we get (I.14).

Assume $p = 2$. If $i \neq 3$, then we get an expression for V_i in a recursive way. If $i = 3$, then from equation E_i we must solve the equation

$$\begin{aligned}&(K_1x^2 + K_2xy + K_3y^2)y^2-\\&\quad (x + 2(q+1)y)V_3(x,y) - y(x + (q+1)y)V_{3,y}(x,y) = 0,\end{aligned}$$

for certain expressions K_1, K_2, K_3. This equation has the solution

$$V_3(x,y) = \frac{(q+1)K_2 - 2K_3}{(q+1)^3}xy(x + (q+1)y)\ln(x + (q+1)y) + \widetilde{V}_3(x,y),$$

where $\widetilde{V}_3(x,y)$ is a homogeneous polynomial of degree 3; so we must take $(q+1)K_2 - 2K_3 = 0$, which means that either $b = 0$, or $b \neq 0$ and

$$\begin{aligned}&q\,j^2(q+2)(q+2-j)^2b^2 + q\,j(q+2-j)(q+2-2j)^2d+\\&(q+2)(q+2-2j)^2(j^2-(q+2)(j-1))mr = 0,\end{aligned} \tag{2.109}$$

$j \in \{1, 2, 3, \ldots, [(q+1)/2]\}$. The system is

$$\begin{aligned}\dot{x} &= r + xy,\\ \dot{y} &= d - \frac{bm(q+2)}{q}x + by + \frac{q+2}{q^2}m^2x^2 + mxy + \frac{3y^2}{q+2},\end{aligned}$$

taking into account the conditions given above. Again, we do not have a proof from which we get the conditions from $(q+1)K_2 - 2K_3 = 0$ and the expression of $V(x,y)$.

If $r = 0$ and $b = 0$, then by the change $(mx/\sqrt{|d|}, y/\sqrt{|d|}, \sqrt{|d|}t) \to (x, y, t)$ we get (II.18). If $r = 0$ and equation (2.109) holds, then by $(mx/b, y/b, bt) \to (x, y, t)$ we get (II.19). If $r = 1$ and $b = 0$, then by $(\sqrt{|m|}x, y/\sqrt{|m|}, \sqrt{|m|}t) \to (x, y, t)$ we get (I.15). If $r = 1$ and equation (2.109) holds, then by the change $(bx, y/b, bt) \to (x, y, t)$ we get (I.16). ∎

Proposition 2.3.39. *Under the conditions of (b.2.3) the cases for which system* (2.103) *has a polynomial inverse integrating factor are (I.17)–(I.20).*

Proof: Equation E_{k-3} writes as

$$N_{k-3}V_{k-3} + FGV_{k-3} = F^{p-3}G^{p+q-3}L,$$

where $L = x^{k-2p-q-1}p_5(x, y)$, for a certain homogeneous polynomial p_5 of degree 5. We take $V_{k-3} = F^iG^jW$, with $i, j \in \mathbb{N} \cup \{0\}$ and W a homogeneous polynomial. Taking $j = p + q - 3$ and $i = p - 3$, we get the equation

$$\left(-3mx - \left(3 - 6\frac{k-3}{2p+q-2}\right) y\right) W + FGW_y = L.$$

By applying Lemma 2.3.4, some values of k and b must be distinguished. If $k = 3p+q$ or $k = 3p+2q$, then there exists a solution of degree 3. If $k = 3(2p+q-1)$, then no solution is obtained. If $k = 3(p + q + 1)/2$, then there is a solution only in the case $p = 2$, $q = 3$, which is (I.17) after the change $(bx, y/b, bt) \to (x, y, t)$. If $k = 3(2p+q+2)/4$, then in order to obtain a solution we must take $p = 0$ and $q = 6$. But then we must take $b = 0$, otherwise there is a solution of degree 3. If $k = 3(2p + q)/2$, then we obtain $b = 0$. The last case is $b = 0$, which includes these two cases above, and from which we obtain $V_{k-3}(x, y) \equiv 0$. So from now and on we assume $b = 0$. Equation E_{k-4} is given by

$$N_{k-4}V_{k-4} + FGV_{k-4} = F^{p-2}G^{p+q-2}L,$$

where $L = x^{k-2p-q-2}p_3(x, y)$, for a certain homogeneous polynomial p_3 of degree 3. We take $V_{k-4} = F^iG^jW$, with $i, j \in \mathbb{N} \cup \{0\}$ and W a homogeneous polynomial. We distinguish two cases.

If $k = (4p + 4q + 5)/3$, then we can take $j = p + q - 3$ and $i = p - 2$. But in this case later computations show that we must take $p = 2$ and either $q = 5$ or $q = 8$. In the first case, there is a solution of degree 3. In the second one there is no solution.

Assume that $k \neq (4p + 4q + 5)/3$. Then, we must take $j = p + q - 2$ and $i = p - 2$. We obtain the equation

$$\left(-2mx - \left(4 - 4\frac{k-3}{2p+q-2}\right) y\right) W + FGW_y = L.$$

By applying Lemma 2.3.4, four subcases must be distinguished, depending on k, p and q.

Assume $k = 2p+1$, $q = 1$. The polynomials V_i, $i > 3$, can be obtained recursively for all i, without new conditions. The expression of $V_3(x,y)$ is

$$V_3(x,y) = (2C_1(2p-1)x + C_2 y)FG,$$

where $C_1 \in \mathbb{R}$ is to be determined and $C_2 \in \mathbb{R}$ depends on $V_5(x,y)$. From equation E_1 we obtain an expression for V_1, and from (2.101) we obtain

$$C_1 = -\frac{C_2(p-2)}{4(p-1)}m,$$

so we get (I.18) after the change $(\sqrt{|m|}x, y/\sqrt{|m|}, \sqrt{|m|}t) \to (x,y,t)$. We note that we cannot find the expression of $V(x,y)$.

Assume $k = 2p+2q+1$ and $p \neq q+1$. Once again, the polynomials V_i, $i > 3$, can be obtained recursively for all i, without new conditions. The expression of $V_3(x,y)$ is

$$V_3(x,y) = (a_0 C_1 xF + p_2(x,y))G,$$

where $a_0 \in \mathbb{R}$ is known, $C_1 \in \mathbb{R}$ is to be determined and $p_2(x,y)$ is a known homogeneous polynomial of degree 2. From equation E_1 we obtain an expression for V_1, and from (2.101) we obtain the expression of C_1. So we get (I.19) after the change $(\sqrt{|m|}x, y/\sqrt{|m|}, \sqrt{|m|}t) \to (x,y,t)$. Again, we cannot give the expression of $V(x,y)$. If $p > 1$, then we have a polynomial first integral (see Proposition 3.1.3) given by $H(x,y) = f_1(x,y)^q f_2(x,y)^{p-1}$, where

$$\begin{aligned} f_1(x,y) &= (2p+q-2)(q-\delta(p-1)x^2) + 2q(p-1)xy,\\ f_2(x,y) &= \delta(2p+q-2)^2(2p+q-2+\delta(p-1)x^2)+\\ &\quad 2q^2(2p+q-2)\delta xy - 4q^2(p+q-1)y^2. \end{aligned}$$

Then, applying Theorem 1.4.7 we can obtain a polynomial inverse integrating factor of the form $V(x,y) = f_1(x,y)f_2(x,y)s(x,y)$, where $s(x,y)$ is the solution of the equation

$$H(x,y) - h = s(x,y)g(x,y).$$

In this equation, $g(x,y) = (2p+q-2)\delta x - 2qy$ is contained in the level set $H(x,y) = h$, and it is the denominator of the rational inverse integrating factor $P/(\log H)_y$ (see Proposition (1.4.1)).

Assume $k = 8p+4q-5$. Then there exists a solution only in the case $p = 0, q = 3$, which is (I.20) after the change $(\sqrt{|m|}x, y/\sqrt{|m|}, \sqrt{|m|}t) \to (x,y,t)$.

Finally, assume $k = 4q+3$ and $p = q+1$. Then there exists a solution of degree 3. ∎

2.4 Algebraic limit cycles

This section is related to ($\star$) quadratic systems, which are the quadratic systems from which we have obtained a polynomial inverse integrating factor in Section 2.3. We remark that the systems from which we do not have a polynomial inverse integrating factor are subfamilies of systems (I) and (II). For systems (III)–(X) we always have an explicit polynomial V.

As we stated in Theorem 1.4.5, the limit cycles of a planar system having an inverse integrating factor $V(x, y)$ are contained in the set $V^{-1}(0)$ if they belong to the domain of definition of V. For ($\star$) quadratic systems, the polynomial inverse integrating factors contain all the limit cycles of the corresponding systems. Moreover, these limit cycles must be algebraic. The following theorem provides more information about limit cycles of quadratic systems.

Theorem 2.4.1 (see [51])**.** *If a quadratic system possesses a limit cycle, then there exists a unique singular point inside the bounded region limited by this limit cycle, and it is a focus.*

The only systems from which we can find limit cycles are (I), (II), (III) and (IV). Systems (V)–(X) do not have limit cycles, because of the expression of $\dot{x}$. From this fact, following Theorem 2.4.1 and from the study of the expressions of V of normal forms (I) to (IV) we state the following theorem.

Theorem 2.4.2. *A ($\star$) quadratic system has no algebraic limit cycles.*

Chapter 3

Quadratic systems having a polynomial inverse integrating factor

In this chapter we study some of the properties of the quadratic systems having a polynomial inverse integrating factor $V(x,y)$. We compute a first integral $H(x,y)$ for each one of the ($\star$) quadratic systems, and we study the critical remarkable values when the first integral is rational.

3.1 Classification of the first integrals

Consider the planar polynomial differential system

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{3.1}$$

where P and Q are polynomials in the variables x and y. As we know, from a polynomial inverse integrating factor V of system (3.1), we can find a first integral H defined on $\mathbb{R}^2 \setminus V^{-1}(0)$. The following proposition sets the type of this first integral.

Proposition 3.1.1 (see [10, 44])**.** *If system* (3.1) *has a rational inverse integrating factor then it has a Darboux first integral.*

The polynomial functions are included in the rational ones, so the quadratic systems we found in Chapter 2 having a polynomial inverse integrating factor have a Darboux first integral.

We distinguish in our classification three types of systems, depending on the type of their first integrals: systems having a polynomial first integral; systems having a rational first integral and not having a polynomial first integral; and systems having a Darboux first integral and not having neither polynomial nor rational first integrals. They are listed in the three propositions below.

Remark 3.1.2. 1. The system

$$\dot{x} = a_{00}+a_{10}x+a_{01}y+a_{20}x^2+a_{11}xy, \quad \dot{y} = d+ax-a_{10}y+lx^2-2a_{20}xy-a_{11}y^2/2,$$

with $\max\{\deg(\dot{x}), \deg(\dot{y})\} = 2$, is Hamiltonian and has the polynomial first integral of degree 3

$$H(x,y) = dx - a_{00}y + \frac{a}{2}x^2 - a_{10}xy - \frac{a_{01}}{2}y^2 + \frac{l}{3}x^3 - a_{20}x^2y - \frac{a_{11}}{2}xy^2.$$

This system appears in all cases of the classification of Chapter 2.

2. We have not been able to compute first integrals for some of the systems of the classification. This is either due to the expression of V, or because there is no expression for V, or because it is very difficult for us to solve the equation $\mathbf{X}H = 0$.

□

Proposition 3.1.3. *The ($\star$) quadratic systems having a polynomial inverse integrating factor and a polynomial first integral are:*

(VIII.4) *with $b = -p/q \in \mathbb{Q}^- \setminus \{-1\}$. Writing $V(x,y) = xf(x,y)$, where*

$$f(x,y) = (b-2)y + x^2,$$

we have

$$H(x,y) = x^p f(x,y)^q.$$

(III.2d) *with $m = -p/q \in \mathbb{Q}^- \setminus \{-2\}$. Writing $V(x,y) = f_1(x,y)f_2(x,y)$, where*

$$f_1(x,y) = b_{10}q + 2qy + (p+2q)x^2,$$
$$f_2(x,y) = b_{10}q - py,$$

then

$$H(x,y) = f_1(x,y)^p f_2(x,y)^{2q}.$$

(VII.3) *with $m + 1 = -p/q \in \mathbb{Q}^- \setminus \{-1, 0\}$. We have*

$$H(x,y) = x^p(q - pxy)^q.$$

(VI.5) *with $b = 0$ and $m = -p/q \in \mathbb{Q}^-$. We have*

$$H(x,y) = (1+x^2)^p y^{2q}.$$

(VI.6). *We have*

$$H(x,y) = (1+x^2)^p y - \sum_{i=0}^{p-1} \binom{p-1}{i} \frac{x^{2i+1}}{2i+1}.$$

(V.8) *with* $m=-p/q\in\mathbb{Q}^-$, $b=r/s\in\mathbb{Q}$, $q,s>0$ *and* $m\le b\le -m$. *We have*

$$H(x,y)=(1-x)^{ps-qr}(1+x)^{ps+qr}y^{2qs}.$$

(V.9) *with* $m+1=-p/q\in\mathbb{Q}^-\setminus\{-1\}$. *We have*

$$H(x,y)=(x\mp 1)^p(q-(x\pm 1)py)^q.$$

(V.11) *with* $m\in\mathbb{Q}^-$, $b\in(m,-m)\cap\mathbb{Q}$ *and* $r\in\mathbb{N}$. *We have*

$$H(x,y)=(x+1)^{k-p-2-i}\left[(x-1)^r y-\sum_{j=0}^{r-1}(-2)^{r-1-j}\binom{r-1}{j}\frac{(1+x)^j}{q-r-1+j}\right]^p.$$

(V.12) *with* $m\in\mathbb{Q}^-$, $b\in(m,-m)\cap\mathbb{Q}$ *and* $q-r-1\in\mathbb{N}$. *We have*

$$H(x,y)=(x-1)^{i-1}\left[(x+1)^{q-r-1}y-\sum_{j=0}^{q-r-1}(-2)^{q-r-2-j}\binom{q-r-2}{j}\frac{(x-1)^j}{r+j}\right]^p.$$

(II.9) *with* $2n=-p/q\in\mathbb{Q}^-$. *Writing* $V(x,y)=xf(x,y)$, *where* $f(x,y)=\sigma+\frac{2\delta n}{2n-1}x+\frac{b_{20}n}{n-1}x^2+y^2$, *we have*

$$H(x,y)=x^p f(x,y)^q.$$

(II.10) *with* $b/\sqrt{b^2-4dn}=p/q\in\mathbb{Q}\cap(-1,1)\setminus\{0\}$, $n=-r/s\in\mathbb{Q}^-$, $q,r,s\in\mathbb{N}$. *We have*

$$H(x,y)=x^{2qr}\big(bs(p-q)-2pry\big)^{(q-p)s}\big(bs(p+q)-2pry\big)^{(q+p)s}.$$

(II.10) *with* $b=0$, $n=-r/s\in\mathbb{Q}^-$. *We have*

$$H(x,y)=x^{2r}(ds-ry^2)^s.$$

(II.11) *with* $m/\sqrt{m^2-4l(n-1)}=p/q\in\mathbb{Q}\cap(-1,1)\setminus\{0\}$ *and* $n=-r/s\in\mathbb{Q}^-$, $q,r,s\in\mathbb{N}$. *Writing* $V(x,y)=xf_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=(p-q)s(\delta(r+s)+rx)-2pr(r+s)y,\\ f_2(x,y)&=(p+q)s(\delta(r+s)+rx)-2pr(r+s)y,\end{aligned}$$

we have

$$H(x,y)=x^{2qr}f_1(x,y)^{(q-p)s}f_2(x,y)^{(q+p)s}.$$

(II.16) *with* $p>1$, *then*

$$\begin{aligned}H(x,y)=&\big(2\delta q^2+(p-1)x((p+q-1)x-2qy)\big)^{p+q-1}\\ &\big(2\delta q^2+(p+q-1)x((p-1)x+2qy)\big)^{p-1}.\end{aligned}$$

(I.5) *with* $n = -p/q \in \mathbb{Q}^- \cap (-1, 0)$. *Writing* $V(x, y) = f_1(x, y)f_2(x, y)$, *where*

$$\begin{aligned} f_1(x,y) &= \delta(n^2-1) + b_{11}n(n+1)x + n(n-1)(2n+1)y, \\ f_2(x,y) &= (n-1)(2n+1)(1+(n+1)xy) + (\delta(n-1) + b_{11}nx)(n+1)x, \end{aligned}$$

we have

$$H(x, y) = f_1(x, y)^{q-p} f_2(x, y)^p.$$

(I.17). *Writing* $V(x, y) = f_1(x, y)f_2(x, y)f_3(x, y)$, *where*

$$\begin{aligned} f_1(x,y) &= 60 + 10b_{11}x - 9y, \\ f_2(x,y) &= 45 + 60x + 10b_{11}x^2 + 36xy, \\ f_3(x,y) &= 236196xy^5 - 98415(10x(6+b_{11}x) - 3)y^4 + \\ &\quad 145800(6+b_{11}x)(10x(6+b_{11}x) - 9)y^3 - \\ &\quad 81000(6+b_{11}x)^2(10x(6+b_{11}x) - 27)y^2 - 1620000(6+b_{11}x)^3y + \\ &\quad 50000x(2b_{11}^5x^5 + 60b_{11}^4x^4 + 9b_{11}^3(80^2+b_{11})x^3 + 216b_{11}^2(20^2+b_{11})x^2 + \\ &\quad 648^2b_{11}(20^2+3b_{11})x + 7776^3(2^2+b_{11})), \end{aligned}$$

we have

$$H(x, y) = f_1(x, y)^4 f_2(x, y).$$

(I.19). *Let*

$$\begin{aligned} f_1(x,y) &= (2p+q-2)(q - \delta(p-1)x^2) + 2q(p-1)xy, \\ f_2(x,y) &= \delta(2p+q-2)^2(2p+q-2+\delta(p-1)x^2) + \\ &\quad 2q^2(2p+q-2)\delta xy - 4q^2(p+q-1)y^2. \end{aligned}$$

If $p > 1$, *then*

$$H(x, y) = f_1(x, y)^q f_2(x, y)^{p-1}.$$

In the following proposition we give the $(\star)$ quadratic systems having a rational first integral, and also the expression of such functions. We also give, when they exist, the critical remarkable values associated to these first integrals. In order to compute the critical values, we must write the rational first integral $H = f/g$ as $\tilde{H} = \tilde{f}/\tilde{g} = (c_2 f + (c_1+1)g)/(f + c_1 g)$, where c_1 and c_2 are taken such that $\tilde{f}$ and $\tilde{g}$ are irreducible and $(\tilde{f}, \tilde{g}) = 1$. See Lemma 1.5.1 for more information.

Proposition 3.1.4. *The rational first integrals which rise from polynomial inverse integrating factors of* $(\star)$ *quadratic systems are:*

(IX.4) *with* $\delta = 0$. *We have*

$$H(x, y) = \frac{1+xy}{y}.$$

H has no critical remarkable values.

(VIII.3) *with $\delta = 0$. We have*

$$H(x, y) = \frac{y - x^2}{x}.$$

H has no critical remarkable values.

(VIII.4) *with $b = p/q \in \mathbb{Q}^+ \setminus \{1, 2\}$. Writing $V(x, y) = xf(x, y)$, where*

$$f(x, y) = (b - 2)y + x^2,$$

we have

$$H(x, y) = x^{-p} f(x, y)^q.$$

If $p > 1$, then $c = -c_2 - c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x = 0$, and it has exponent p.
If $q > 1$, then $c = -c_2$ is a critical remarkable value of H. The associated curve is $f(x, y) = 0$, and it has exponent q.

(VIII.8) *with $b_{00} = -p^2/q^2 \in \mathbb{Q}^-$, $q \in \mathbb{N}$. We have*

$$H(x, y) = x^p (p + 2qy)^q (p - 2qy)^{-q}.$$

If $p > 1$ or $q > 1$, then $c = -c_2$ is a critical remarkable value of H. The associated curves are $x = 0$ with exponent p and $p + 2qy = 0$ with exponent q.
If $p < -1$ or $q > 1$, then $c = -c_2 - c_1^{-1}$ is a critical remarkable value of H. The associated curves are $x = 0$ with exponent $-p$ and $p - 2qy = 0$ with exponent q.

(III.2d) *with $m = p/q \in \mathbb{Q}^+ \setminus \{1, 2\}$. Writing $V(x, y) = f_1(x, y) f_2(x, y)$, where*

$$f_1(x, y) = b_{10} + 2y - (m - 2)x^2,$$
$$f_2(x, y) = b_{10} + my,$$

then

$$H(x, y) = f_1(x, y)^p f_2(x, y)^{-2q}.$$

If $p > 1$, then $c = -c_2 - c_1^{-1}$ is a critical remarkable value of H. The associated curve is $f_1(x, y) = 0$ with exponent p.
Moreover, $c = -c_2$ is another critical remarkable value of H. The associated curve is $f_2(x, y) = 0$, and it has exponent $2q$.

(III.3). *Writing $V(x, y) = f_1(x, y) f_2(x, y)$, where*

$$f_1(x, y) = 2x - y^2,$$
$$f_2(x, y) = 2 + 3xy - y^3,$$

we have

$$H(x,y)=\frac{f_1(x,y)^3}{f_2(x,y)^2}.$$

The value $c=-c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y)=0$, and it has exponent 3.
The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_2(x,y)=0$, and it has exponent 2.

(III.5). *Writing $V(x,y)=f_1(x,y)f_2(x,y)$, where*

$$\begin{aligned}f_1(x,y)&=1-2(3x^2-y)(3x-(3x^2-y)^2),\\ f_2(x,y)&=(3x^2-y)^2-2x,\end{aligned}$$

we have

$$H(x,y)=\frac{f_1(x,y)^2}{f_2(x,y)^3}.$$

The value $c=-c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y)=0$, and it has exponent 2.
The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_2(x,y)=0$, and it has exponent 3.

(VII.3) *with $m+1=p/q\in\mathbb{Q}^+\setminus\{1\}$. We have*

$$H(x,y)=x^{-p}(q+pxy)^q.$$

If $p>1$, then $c=-c_2$ is a critical remarkable value of H. The associated curve is $x=0$ with exponent p.
If $q>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $q+pxy=0$, and it has exponent q.

(VII.4) *with $\delta=0$. We have*

$$H(x,y)=\frac{1+xy}{x}.$$

We note that this first integral is also associated to the system $\dot{x}=x^2,\dot{y}=1$, which is equivalent to (IX.4) with $b_{00}=0$. H has no critical remarkable values.

(VII.5) *with $\delta=0$. We have*

$$H(x,y)=\frac{1+2xy}{x^2}.$$

The value $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x=0$, and it has exponent 2.

(VII.8). *We have*

$$H(x,y)=\frac{xy}{x-y}.$$

H has no critical remarkable values.

(VII.9) *with $\sqrt{b_{20}+1}=p/q\in\mathbb{Q}^+$. We have*

$$H(x,y)=x^p\left(\frac{(q-p)x-2qy}{(q+p)x-2qy}\right)^q.$$

If $p>1$ or $q>1$, then $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$ with exponent p and $(q-p)x-2qy=0$ with exponent q.
If $q>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $(q+p)x-2qy=0$, and it has exponent q.

(VI.4) *with $b=0$. We have*

$$H(x,y)=\frac{y^2}{1+x^2}.$$

The value $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $y=0$, and it has exponent 2.

(VI.5) *with $b=0$ and $m=p/q\in\mathbb{Q}^+\setminus\{1\}$. We have*

$$H(x,y)=(1+x^2)^p y^{-2q}.$$

If $p>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $1+x^2=0$, with exponent p.
Moreover, $c=-c_2$ is another critical remarkable value of H. The associated curve is $y=0$, and it has exponent $2q$.

(VI.8). *We have*

$$H(x,y)=\frac{1+xy}{x-y}.$$

H has no critical remarkable values.

(V.3) *with $b=p/q\in\mathbb{Q}\setminus\{0\}$. We have*

$$H(x,y)=\left(\frac{1+x}{1-x}\right)^p y^{2q}.$$

If $p^2>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is either $1-x=0$ or $1+x=0$, with respective exponent either p or $-p$.
Moreover, $c=-c_2$ is another critical remarkable value of H. The associated

curves are $y = 0$, *with exponent* $2q$ *and, in the case* $p^2 > 1$, *the curve* $1 + x = 0$ *with exponent* p *or the curve* $1 - x = 0$ *with exponent* $-p$.

(V.5) *with* $b = p/q \in \mathbb{Q} \setminus \{0\}$. *We have*

$$H(x, y) = (1 + x)^{p-q}(1 - x)^{-p-q}y^{2q}.$$

We distinguish three cases in order to compute the critical remarkable values:

($p > q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curves are* $y = 0$, *with exponent* $2q$ *and, in the case* $p - q > 1$, *the curve* $1 + x = 0$ *with exponent* $p - q$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $1 - x = 0$, *with exponent* $p + q$.

($p < -q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curves are* $y = 0$, *with exponent* $2q$ *and, in the case* $-(p+q) > 1$, *the curve* $1 - x = 0$ *with exponent* $-(p + q)$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $1 + x = 0$, *with exponent* $q - p$.

($p^2 < q^2$) *The value* $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curves are* $1 - x = 0$, *with exponent* $p + q$ *and, in the case* $q - p > 1$, *the curve* $1 - x = 0$ *with exponent* $q - p$.
The value $c = -c_2$ *is another critical remarkable value of* H. *The associated curve is* $y = 0$, *with exponent* $2q$.

(V.5) *with* $b = 0$. *We have*

$$H(x, y) = \frac{y^2}{1 - x^2}.$$

The value $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $y = 0$, *and it has exponent* 2.

(V.6) *with* $\delta = 0$. *We have*

$$H(x, y) = \frac{x \pm 1}{(x \mp 1)^2}y.$$

The value $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $x \mp 1 = 0$, *and it has exponent* 2.

(V.8) *with* $b = r/s, m = p/q \in \mathbb{Q}$, $q, s > 0$, $r \neq 0$ *and not* $m \leq b \leq -m$. *We have*

$$H(x, y) = (1 - x)^{-qr-ps}(1 + x)^{qr-ps}y^{2qs}.$$

If $qr+ps>1$ or $ps-qr>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curves are $1-x=0$, with exponent $qr+ps$, and $1+x=0$ with exponent $ps-qr$.
If $-(qr+ps)>1$ or $qr-ps>1$, then $c=-c_2$ is another critical remarkable value of H. The associated curves are $y=0$, with exponent $2qs$, $1-x=0$ with exponent $-(qr+ps)$, and $1+x=0$ with exponent $qr-ps$.

(V.8) *with $b=0$ and $m=p/q\in\mathbb{Q}^+$. We have*

$$H(x,y)=(1-x^2)^{-p}y^{2q}.$$

If $p>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $1-x^2=0$, with exponent p.
The value $c=-c_2$ is another critical remarkable value of H. The associated curve is $y=0$, with exponent $2q$.

(V.9) *with $m+1=p/q\in\mathbb{Q}^+$. We have*

$$H(x,y)=\frac{(x\mp 1)^p}{(q+(x\pm 1)py)^q}.$$

If $q>1$, then $c=-c_2$ is a critical remarkable value of H. The associated curve is $q+p(x\pm 1)y=0$ with exponent q.
If $p>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x\mp 1=0$, and it has exponent p.

(V.14) *with $b_{00}=4$ and $b_{10}=b_{20}=0$. We have*

$$H(x,y)=\frac{1-xy}{x-y}.$$

H has no critical remarkable values.

(V.15) *with $b_{20}+1=p/q\in\mathbb{Q}^+$. We have*

$$H(x,y)=(x\pm 1)^p\left(\frac{p(x\mp 1)-q(x\pm 1-2y)}{p(x\mp 1)+q(x\pm 1-2y)}\right)^q.$$

If $p>1$ or $q>1$, then $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$ with exponent p and $(q-p)x-2qy=0$ with exponent q.
If $q>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $(q+p)x-2qy=0$, and it has exponent q.

(V.16) *with $\sqrt{b_{00}}=p/q\in\mathbb{Q}^+$. We have*

$$H(x,y)=\left(\frac{1+x}{1-x}\right)^p\left(\frac{p-2qy}{p+2qy}\right)^{2q}.$$

The value $c=-c_2$ is a critical remarkable value of H. The associated curves are $p-2qy=0$, with exponent $2q$ and, if $p>1$, the curve $1+x=0$ with exponent p. The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curves are $p+2qy=0$, with exponent $2q$ and, if $p>1$, the curve $1-x=0$ with exponent p.

(II.3c). *We have*

$$H(x,y)=\frac{-2\delta+2\sigma x^2+y^2}{x}.$$

H has no critical remarkable values.

(II.3d). *We have*

$$H(x,y)=\frac{2\delta x^2-y^2}{x}.$$

H has no critical remarkable values.

(II.5) *with $l=0$. We have*

$$H(x,y)=\frac{d+2ax+y^2}{x^2},$$

where the conditions of the subcases are to be applied.

The value $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x=0$, and it has exponent 2.

(II.8) *with $B=1/\sqrt{1-4b_{00}}=p/q\in\mathbb{Q}$. We have*

$$H(x,y)=x^{2q}\big((p-q)(b_{00}+\delta x)+2b_{00}py\big)^{p-q}\big((p+q)(b_{00}+\delta x)+2b_{00}py\big)^{-p-q}.$$

We distinguish three cases in order to compute the critical remarkable values:

($p>q$) *The value $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$, with exponent $2q$ and, in the case $p-q>1$, the curve $(p-q)(b_{00}+\delta x)+2b_{00}py=0$ with exponent $p-q$.*
The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $(p+q)(b_{00}+\delta x)+2b_{00}py=0$, with exponent $q+p$.

($p<-q$) *The value $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$, with exponent $2q$ and, in the case $-(q+p)>1$, the curve $(p+q)(b_{00}+\delta x)+2b_{00}py=0$ with exponent $-(q+p)$.*
The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $(p-q)(b_{00}+\delta x)+2b_{00}py=0$, with exponent $q-p$.

($p^2 < q^2$) *If $q+p>1$ or $q-p>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curves are $(p+q)(b_{00}+\delta x)+2b_{00}py=0$, with exponent $q+p$, and $(p-q)(b_{00}+\delta x)+2b_{00}py=0$ with exponent $q-p$. The value $c=-c_2$ is another critical remarkable value of H. The associated curve is $x=0$, with exponent $2q$.*

(II.9) *with $2n=p/q\in\mathbb{Q}^+$. Writing $V(x,y)=xf(x,y)$, where $f(x,y)=\sigma+2\delta nx/(2n-1)+b_{20}nx^2/(n-1)+y^2$, we have*

$$H(x,y)=x^pf(x,y)^{-q}.$$

If $p>1$, then $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x=0$, with exponent p.
If $q>1$, then $c=-c_2$ is another critical remarkable value of H. The associated curve is $f(x,y)=0$, with exponent q.

(II.10) *with $b=0$, $n=r/s\in\mathbb{Q}^+$. We have*

$$H(x,y)=x^{-2r}(ds+ry^2)^s.$$

The value $c=-c_2-c_1^{-1}$ is a critical remarkable value of H. The associated curve is $x=0$, and it has exponent $2r$.
If $s>1$, then $c=-c_2$ is a critical remarkable value of H. The associated curve is $ds+ry^2=0$, and it has exponent s.

(II.10) *with $b/\sqrt{b^2-4dn}=p/q\in\mathbb{Q}\setminus\{0\}$ and $n=-r/s\in\mathbb{Q}$, $q,s\in\mathbb{N}$, such that either $p^2>q^2$ or $r<0$ and $p^2<q^2$. We have*

$$H(x,y)=x^{2qr}\big(bs(p-q)-2pry\big)^{(q-p)s}\big(bs(p+q)-2pry\big)^{(q+p)s}.$$

We distinguish five cases in order to compute the critical remarkable values:

($r>0,p>q$) *The value $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$, with exponent $2qr$ and, in the case $(q+p)s>1$, the curve $f_1(x,y)=0$ with exponent $(q+p)s$.*
If $(p-q)s>1$, then $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_2(x,y)=0$, with exponent $(p-q)s$.

($r>0,p<-q$) *The value $c=-c_2$ is a critical remarkable value of H. The associated curves are $x=0$, with exponent $2qr$ and, in the case $(q-p)s>1$, the curve $f_2(x,y)=0$ with exponent $(q-p)s$.*
If $-(q+p)s>1$, then $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_1(x,y)=0$, with exponent $-(q+p)s$.

$(r<0, p>q)$ *The value* $c=-c_2-c_1^{-1}$ *is a critical remarkable value of* H. *The associated curves are* $x=0$, *with exponent* $-2qr$ *and, in the case* $(p-q)s>1$, *the curve* $f_2(x,y)=0$ *with exponent* $(p-q)s$.
If $(q+p)s>1$, *then* $c=-c_2$ *is another critical remarkable value of* H. *The associated curve is* $f_1(x,y)=0$, *with exponent* $(q+p)s$.

$(r<0, p<-q)$ *The value* $c=-c_2-c_1^{-1}$ *is a critical remarkable value of* H. *The associated curves are* $x=0$, *with exponent* $-2qr$ *and, in the case* $-(q+p)s>1$, *the curve* $f_1(x,y)=0$ *with exponent* $-(q+p)s$.
If $(q-p)s>1$, *then* $c=-c_2$ *is another critical remarkable value of* H. *The associated curve is* $f_2(x,y)=0$, *with exponent* $(q-p)s$.

$(r<0, p^2<q^2)$ *The value* $c=-c_2-c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $x=0$, *with exponent* $-2qr$.
If $(q+p)s>1$ *or* $(q-p)s>1$, *then* $c=-c_2$ *is another critical remarkable value of* H. *The associated curves are* $f_1(x,y)=0$, *with exponent* $(q+p)s$ *and* $f_2(x,y)=0$ *with exponent* $(q-p)s$.

(II.11) *with* $1/\sqrt{1-4b_{20}(n-1)}=p/q\in\mathbb{Q}\setminus\{0\}$ *and* $n=-r/s\in\mathbb{Q}$, $q,s\in\mathbb{N}$, *such that* $p^2>q^2$. *Writing* $V(x,y)=xf_1(x,y)f_2(x,y)$, *where*

$$f_1(x,y)=(p+q)s(\delta(r+s)+rx)-2pr(r+s)y,$$
$$f_2(x,y)=(p-q)s(\delta(r+s)+rx)-2pr(r+s)y,$$

we have

$$H(x,y)=x^{2qr}f_1(x,y)^{(q+p)s}f_2(x,y)^{(q-p)s}.$$

The computations of the critical remarkable values are exactly the same as in case (II.10) above.

(II.12). *Writing* $V(x,y)=f_1(x,y)f_2(x,y)$, *where*

$$f_1(x,y)=150x-36x^2+60xy-25y^2,$$
$$f_2(x,y)=1875x-1350x^2+216x^3+1125xy-540x^2y+450xy^2-125y^3,$$

we have

$$H(x,y)=\frac{f_1(x,y)^3}{f_2(x,y)^2}.$$

The value $c=-c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x,y)=0$, *and it has exponent* 3.
The value $c=-c_2-c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $f_2(x,y)=0$, *and it has exponent* 2.

(II.13). *Writing* $V(x,y) = f_1(x,y)f_2(x,y)f_3(x,y)$, *where*

$$\begin{aligned} f_1(x,y) &= 6x - 5y - 5,\\ f_2(x,y) &= 36x^2 - 15x(4y+7) + 25(y+1)(y-2),\\ f_3(x,y) &= 36x^2 - 30x(2y+5) + 25(y-2)^2, \end{aligned}$$

we have

$$H(x,y) = \frac{f_2(x,y)^2}{f_1(x,y)^2 f_3(x,y)}.$$

The value $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x,y) = 0$, *and it has exponent* 2.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $f_1(x,y) = 0$, *and it has exponent* 2.

(II.14) *with* $\delta = 1$, $1 + 4b_{00} > 0$, $b_{00} \neq 0$. *Writing* $V(x,y) = xf_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned} f_1(x,y) &= b_{00} - x(1+x-y),\\ f_2(x,y) &= b_{00} - (x-y)(1+x-y), \end{aligned}$$

we have

$$H(x,y) = \frac{x^2 f_2(x,y)}{f_1(x,y)^2}.$$

The value $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $x = 0$, *with exponent* 2.
The value $c = -c_2$ *is another critical remarkable value of* H. *The associated curve is* $f_1(x,y)$, *with exponent* 2.

(II.14) *with* $\delta = 0$. *Writing* $V(x,y) = xf_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned} f_1(x,y) &= \pm 1 - x(x-y),\\ f_2(x,y) &= \pm 1 - (x-y)^2, \end{aligned}$$

we have

$$H(x,y) = \frac{x^2 f_2(x,y)}{f_1(x,y)^2}.$$

The value $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $x = 0$, *with exponent* 2.
The value $c = -c_2$ *is another critical remarkable value of* H. *The associated curve is* $f_1(x,y)$, *with exponent* 2.

(II.15) *with* $1-8b_{00}>0$ *and* $1/\sqrt{1-8b_{00}}=p/q\in\mathbb{Q}\setminus\{0,1\}$, $q>0$. *Writing* $V(x,y)=xf_1(x,y)f_2(x,y)$, *where*

$$f_1(x,y)=(p+q)(b_{00}+\delta x^2)+4b_{00}py,$$
$$f_2(x,y)=(p-q)(b_{00}+\delta x^2)+4b_{00}py,$$

we have

$$H(x,y)=x^{-4q}f_1(x,y)^{q+p}f_2(x,y)^{q-p}.$$

We distinguish three cases in order to compute the critical remarkable values:

($p>q$) *The value* $c=-c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x,y)=0$ *with exponent* $q+p>1$.
The value $c=-c_2-c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $x=0$, *with exponent* $4q$ *and, if* $p-q>1$, *the curve* $f_2(x,y)=0$, *with exponent* $p-q$.

($p<-q$) *The value* $c=-c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x,y)=0$, *with exponent* $q-p>1$.
The value $c=-c_2-c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $x=0$, *with exponent* $4q$ *and, if* $-(q+p)>1$, *the curve* $f_1(x,y)=0$, *with exponent* $-(q+p)$.

($p^2<q^2$) *If* $q+p>1$ *or* $q-p>1$, *then* $c=-c_2$ *is a critical remarkable value of* H. *The associated curves are* $f_1(x,y)=0$, *with exponent* $q+p$ *and* $f_2(x,y)=0$ *with exponent* $q-p$.
The value $c=-c_2-c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $x=0$, *with exponent* $4q$.

(II.16) *with* $p=0$ *and* $q>3$. *We have*

$$H(x,y)=\frac{(2\delta q^2-x((q-1)x-2qy))^{q-1}}{2\delta q^2-(q-1)x(x-2qy)}.$$

The value $c=-c_2$ *is a critical remarkable value of* H. *The associated curve is* $2\delta q^2-x((q-1)x-2qy)=0$, *with exponent* $q>3$.

(I.5) *with* $n=p/q\in\mathbb{Q}\setminus[-1,0]$. *Writing* $V(x,y)=f_1(x,y)f_2(x,y)$, *where*

$$f_1(x,y)=(n^2-1)+b_{11}n(n+1)x+n(n-1)(2n+1)y,$$
$$f_2(x,y)=(n-1)(2n+1)(1+(n+1)xy)+((n-1)+b_{11}nx)(n+1)x,$$

we have

$$H(x,y)=f_1(x,y)^{p+q}f_2(x,y)^{-p}.$$

We distinguish two cases in order to compute the critical remarkable values:

($p > 0$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$, *with exponent* $p + q$.
If $p > 1$, *then* $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x, y) = 0$, *with exponent* p.

($p < -q$) *If* $-(p + q) > 1$, *then* $c = -c_2 - c_1^{-1}$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$, *with exponent* $-(p + q)$.
The value $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x, y) = 0$, *with exponent* $-p$.

(I.6) *with* $b_{00} < 1/2$ *and* $1/\sqrt{1 - 2b_{00}} = p/q \in \mathbb{Q}$. *Writing* $V(x, y) = f_1(x, y)f_2(x, y)f_3(x, y)$, *where*

$$f_1(x, y) = 2b_{00}(1 - 2x) + (y + 2)^2,$$
$$f_2(x, y) = \sqrt{1 - 2b_{00}} + (y + 1),$$
$$f_3(x, y) = \sqrt{1 - 2b_{00}} - (y + 1),$$

we have

$$H(x, y) = f_1(x, y)^{-q} f_2(x, y)^{q+p} f_3(x, y)^{q-p}.$$

We distinguish three cases in order to compute the critical remarkable values:

($p > q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x, y) = 0$ *with exponent* $q + p > 1$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $f_1(x, y) = 0$, *with exponent* q *and, if* $p - q > 1$, *the curve* $f_3(x, y) = 0$, *with exponent* $p - q$.

($p < -q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_3(x, y) = 0$, *with exponent* $q - p > 1$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $f_1(x, y) = 0$, *with exponent* q *and, if* $-(q + p) > 1$, *the curve* $f_2(x, y) = 0$, *with exponent* $-(q + p)$.

($p^2 < q^2$) *If* $q + p > 1$ *or* $q - p > 1$, *then* $c = -c_2$ *is a critical remarkable value of* H. *The associated curves are* $f_2(x, y) = 0$, *with exponent* $q + p$ *and* $f_3(x, y) = 0$ *with exponent* $q - p$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$, *with exponent* q.

(I.7) *with* $\delta = 1$, $b_{00} < -\sqrt{2}$, $\sqrt{-b_{00} - \sqrt{2}}/\sqrt{-b_{00} + \sqrt{2}} = p/q \in \mathbb{Q} \cap (-1, 1) \setminus \{0\}$. *Writing* $V(x, y) = f_1(x, y)f_2(x, y)f_3(x, y)f_4(x, y)$, *where*

$$f_{1,2}(x, y) = 2^{5/4}q \pm \sqrt{q^2 - p^2}(\sqrt{2}x - y),$$
$$f_{3,4}(x, y) = 2^{5/4}p \pm \sqrt{q^2 - p^2}(\sqrt{2}x + y),$$

we have

$$H(x,y) = \left(\frac{f_1(x,y)}{f_2(x,y)}\right)^p \left(\frac{f_3(x,y)}{f_4(x,y)}\right)^q.$$

We distinguish two cases in order to compute the critical remarkable values:

($p > 0$) *The value $c = -c_2$ is a critical remarkable value of H. The associated curves are $f_3(x,y) = 0$, with exponent q and, if $p > 1$, the curve $f_1(x,y) = 0$ with exponent p.*
The value $c = -c_2 - c_1^{-1}$ is another critical remarkable value of H. The associated curves are $f_4(x,y) = 0$, with exponent q and, if $p > 1$, the curve $f_2(x,y) = 0$ with exponent p.

($p < 0$) *The value $c = -c_2$ is a critical remarkable value of H. The associated curves are $f_3(x,y) = 0$, with exponent q and, if $-p > 1$, the curve $f_2(x,y) = 0$ with exponent $-p$.*
The value $c = -c_2 - c_1^{-1}$ is another critical remarkable value of H. The associated curves are $f_4(x,y) = 0$, with exponent q and, if $-p > 1$, the curve $f_1(x,y) = 0$ with exponent $-p$.

(I.8). *Writing $V(x,y) = f_1(x,y)f_2(x,y)$, where*

$$f_1(x,y) = 3\delta + y^2,$$
$$f_2(x,y) = 9\delta x - 9\delta y - 2y^3,$$

we have

$$H(x,y) = \frac{f_1(x,y)^3}{f_2(x,y)^2}.$$

The value $c = -c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y) = 0$, with exponent 3.
The value $c = -c_2 - c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_2(x,y) = 0$, with exponent 2.

(I.9) *with $b_{00} > 3/4$ and $9/\sqrt{9 - 12b_{00}} = p/q \in \mathbb{Q}$. Writing $V(x,y) = f_1(x,y)f_2(x,y)f_3(x,y)$, where*

$$f_1(x,y) = 3(p+3q) + 2py,$$
$$f_2(x,y) = 3(p-3q) + 2py,$$
$$f_3(x,y) = 108(13p^2 - 45q^2)p^2 - 243(p^2 - q^2)(3(p^2 - 9q^2)x - 4p^2y) + 32p^4(9+y)y^2,$$

we have

$$H(x,y) = f_1(x,y)^{3q+p} f_2(x,y)^{3q-p} f_3(x,y)^{-2q}.$$

We distinguish three cases in order to compute the critical remarkable values:

($p > 3q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$ *with exponent* $3q + p > 1$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $f_3(x, y) = 0$, *with exponent* $2q$ *and, if* $p - 3q > 1$, *the curve* $f_2(x, y) = 0$, *with exponent* $p - 3q$.

($p < -3q$) *The value* $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_2(x, y) = 0$, *with exponent* $3q - p > 1$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curves are* $f_3(x, y) = 0$, *with exponent* $2q$ *and, if* $-(3q + p) > 1$, *the curve* $f_1(x, y) = 0$, *with exponent* $-(3q + p)$.

($p^2 < 9q^2$) *If* $3q - p > 1$ *or* $3q + p > 1$, *then* $c = -c_2$ *is a critical remarkable value of* H. *The associated curves are* $f_1(x, y) = 0$, *with exponent* $3q + p$ *and* $f_2(x, y) = 0$ *with exponent* $3q - p$.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $f_3(x, y) = 0$, *with exponent* $2q$.

(I.10). *Writing* $V(x, y) = f_1(x, y)f_2(x, y)$, *where*

$$f_1(x, y) = 3b_{01} - 5x^2 + 10y,$$
$$f_2(x, y) = 5 + 3b_{01}x - 5x^3 + 15xy,$$

we have

$$H(x, y) = \frac{f_1(x, y)^3}{f_2(x, y)^2}.$$

The value $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$, *with exponent* 3.
The value $c = -c_2 - c_1^{-1}$ *is another critical remarkable value of* H. *The associated curve is* $f_2(x, y) = 0$, *with exponent* 2.

(I.11). *Writing* $V(x, y) = f_1(x, y)f_2(x, y)$, *where*

$$f_1(x, y) = 1 - \delta x^4 + 4xy,$$
$$f_2(x, y) = \delta x^3 - 3y,$$

we have

$$H(x, y) = \frac{f_1(x, y)^3}{f_2(x, y)^4}.$$

The value $c = -c_2$ *is a critical remarkable value of* H. *The associated curve is* $f_1(x, y) = 0$, *with exponent* 3.

The value $c=-c_2-c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_2(x,y)=0$, with exponent 4.

(I.12). *Writing $V(x,y)=f_1(x,y)f_2(x,y)$, where*

$$
\begin{aligned}
f_1(x,y) &= 1531250+525000\delta x^2+50625x^4+245000xy- \\
&\quad 189000\delta x^3y-686000\delta y^2+264600x^2y^2-164640\delta xy^3+38416y^4, \\
f_2(x,y) &= 26250x+3375\delta x^3+36750\delta y-9450x^2y+8820\delta xy^2-2744y^3,
\end{aligned}
$$

we have

$$H(x,y)=\frac{f_1(x,y)^3}{f_2(x,y)^4}.$$

The computations of the critical remarkable values are exactly the same as in case (I.11) above.

(I.18). *Let*

$$
\begin{aligned}
f_1(x,y) &= \delta(p(\delta x^2+2)-1)(1-2p)^2+2\delta(2p-1)xy-4(p-1)y^2, \\
f_2(x,y) &= (2p-1)(\delta px^2+1)+2pxy.
\end{aligned}
$$

Then, we have

$$H(x,y)=\frac{f_1(x,y)^p}{f_2(x,y)}.$$

The value $c=-c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y)=0$, with exponent $p>2$.

(I.19) *with $p=0$. Let*

$$
\begin{aligned}
f_1(x,y) &= (q-2)(q+\delta x^2)-2qxy, \\
f_2(x,y) &= \delta(q-2)^2(q-2-\delta x^2)+2q^2(q-2)\delta xy-4q^2(q-1)y^2.
\end{aligned}
$$

We have

$$H(x,y)=\frac{f_1(x,y)^q}{f_2(x,y)}.$$

The value $c=-c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y)=0$, with exponent $q>2$.

(I.20). *Writing $V(x,y)=f_1(x,y)f_2(x,y)f_3(x,y)$, where*

$$
\begin{aligned}
f_1(x,y) &= \delta x-6y, \\
f_2(x,y) &= 54+3\delta x^2(3-2xy)+\delta^2x^4, \\
f_3(x,y) &= 3+\delta x^2-6xy,
\end{aligned}
$$

we have

$$H(x,y) = \frac{f_1(x,y)^2 f_2(x,y)}{f_3(x,y)^3}.$$

The value $c = -c_2$ is a critical remarkable value of H. The associated curve is $f_1(x,y) = 0$, with exponent 2.
The value $c = -c_2 - c_1^{-1}$ is another critical remarkable value of H. The associated curve is $f_3(x,y) = 0$, with exponent 3.

Proposition 3.1.5. *The Darboux first integrals which rise from polynomial inverse integrating factors of $(\star)$ quadratic systems are:*

(IX.2). *We have*

$$H(x,y) = V(x,y)\, e^{-x}.$$

(IX.3). *We have*

$$H(x,y) = V(x,y)\, e^{-x-\delta x^2/2}.$$

(IX.4) *with $\delta = 1$. We have*

$$H(x,y) = \left(\frac{1+2yi}{1-2yi}\right)^i e^x.$$

(IX.4) *with $\delta = -1$. We have*

$$H(x,y) = \frac{1+2y}{1-2y}\, e^x.$$

(VIII.2). *We have*

$$H(x,y) = x^{2\delta}\, e^{-y+x^2}.$$

(VIII.3) *with $\delta = 1$. We have*

$$H(x,y) = x\, e^{-\frac{y-x^2}{x}}.$$

(VIII.4) *with $b \notin \mathbb{Q}$. Writing $V(x,y) = xf(x,y)$, where*

$$f(x,y) = (b-2)y + x^2,$$

we have

$$H(x,y) = x^{-b} f(x,y).$$

(VIII.4) *with* $b=2$. *We have*

$$H(x,y)=x\,e^{-\frac{y}{x^2}}.$$

(VIII.5). *We have*

$$H(x,y)=V(x,y)\,e^{-x}.$$

(VIII.6). *We have*

$$H(x,y)=x^{-b}y\,e^{-x}.$$

(VIII.7) *with* $\delta=1$. *We have*

$$H(x,y)=\left(\frac{x+(1-2y)i}{x-(1-2y)i}\right)^i e^{-x}.$$

(VIII.7) *with* $\delta=-1$. *We have*

$$H(x,y)=\frac{x+(1-2y)}{x-(1-2y)}\,e^{-x}.$$

(VIII.8) *with* $b_{00}>0$. *We have*

$$H(x,y)=\left(\frac{\sqrt{b_{00}}x+2yi}{\sqrt{b_{00}}x-2yi}\right)^i x^{\sqrt{b_{00}}}.$$

(VIII.8) *with* $b_{00}<0$ *and* $b_{00}\neq -p^2/q^2\in\mathbb{Q}^-$. *We have*

$$H(x,y)=\frac{\sqrt{-b_{00}}x+2y}{\sqrt{-b_{00}}x-2y}x^{\sqrt{-b_{00}}}.$$

(VIII.8) *with* $b_{00}=0$. *We have*

$$H(x,y)=x\,e^{1/y}.$$

(IV.2). *We have*

$$H(x,y)=V(x,y)\,e^{-\frac{\delta(x^2-2y)}{2}}.$$

(IV.3a), (IV.3b), (IV.3c). *We have*

$$H(x,y)=V(x,y)\,e^{-2x}.$$

(IV.3d) *with* $D>0$. *We have*

$$H(x,y)=e^{-2x}e^{-\frac{\arctan\frac{2y+1}{2\sqrt{D}}}{\sqrt{D}}}\big((2y+1)^2+4D\big).$$

(IV.3d) *with* $D<0$, $D\neq -1/4$. *We have*

$$H(x,y)=e^{-4\sqrt{-D}x}\left(1+2\sqrt{-D}+2y\right)^{2\sqrt{-D}+1}\left(1-2\sqrt{-D}+2y\right)^{2\sqrt{-D}-1}.$$

(IV.3d) *with* $D=0$. *We have*

$$H(x,y)=e^{-x}e^{\frac{1}{2y+1}}(2y+1).$$

(IV.3e) *with* $D>0$. *We have*

$$H(x,y)=e^{-2x}e^{-\frac{\arctan\left(\frac{2x+4y+1}{4\sqrt{D}(2x+1)}\right)}{2\sqrt{D}}}\left(16D(2x+1)^2+(2x+4y+1)^2\right).$$

(IV.3e) *with* $D<0$, $D\neq -1/16$. *We have*

$$\begin{aligned}H(x,y)={}&e^{-8\sqrt{-D}x}\\&((4\sqrt{-D}+1)(2x+1)+4y)^{4\sqrt{-D}+1}((4\sqrt{-D}-1)(2x+1)-4y)^{4\sqrt{-D}-1}.\end{aligned}$$

(IV.3e) *with* $D=0$. *We have*

$$H(x,y)=e^{\frac{-4x^2-8yx+2x+2}{2x+4y+1}}(2x+4y+1)^2.$$

(III.2a). *We have*

$$H(x,y)=V(x,y)\,e^{-2\delta y}.$$

(III.2c). *We have*

$$H(x,y)=V(x,y)\,e^{2\frac{b_{10}-2x^2}{b_{10}+2y}}.$$

(III.2d) *with* $m\notin\mathbb{Q}$. *Writing* $V(x,y)=f_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=b_{10}+2y-(m-2)x^2,\\f_2(x,y)&=b_{10}+my,\end{aligned}$$

we have

$$H(x,y)=f_1(x,y)^{-m}f_2(x,y)^2.$$

(III.6). *We have*

$$H(x,y)=V(x,y)\,e^{2\frac{2+2a_{00}+a_{10}+a_{10}x+2xy}{2+a_{10}+2x-2y}}.$$

(III.7). *We have*

$$H(x,y)=y\,e^{-x(x+y)/y}.$$

(VII.2). *We have*

$$H(x,y) = V(x,y)\, e^{-xy}.$$

(VII.3) *with* $m+1 \notin \mathbb{Q}$. *We have*

$$H(x,y) = V(x,y)\, x^{-m-2}.$$

(VII.4) *with* $\delta = 1$. *We have*

$$H(x,y) = V(x,y)\, e^{-2(1+xy)/x}.$$

(VII.5) *with* $\delta = \pm 1$. *We have*

$$H(x,y) = V(x,y)\, e^{-3\frac{1+2xy}{2\delta x^2}}.$$

(VII.6). *We have*

$$H(x,y) = x^{-m} y\, e^{1/x}.$$

(VII.7). *We have*

$$H(x,y) = V(x,y) x^{-k}\, e^{1/x}.$$

(VII.9) *with* $b_{20} = -1$. *We have*

$$H(x,y) = x\, e^{-2x/(x-2y)}.$$

(VII.9) *with* $\sqrt{b_{20}+1} \notin \mathbb{Q}$. *We have*

$$H(x,y) = x^{\sqrt{b_{20}+1}} \frac{(1-\sqrt{b_{20}+1})x - 2y}{(1+\sqrt{b_{20}+1})x - 2y}.$$

(VII.10) *with* $b_{00} > 0$. *We have*

$$H(x,y) = \frac{\sqrt{b_{00}} + 2y}{\sqrt{b_{00}} - 2y}\, e^{-\sqrt{b_{00}}/x}.$$

(VII.10) *with* $b_{00} < 0$. *We have*

$$H(x,y) = \left(\frac{\sqrt{-b_{00}} + 2yi}{\sqrt{-b_{00}} - 2yi} \right)^{i} e^{-\sqrt{-b_{00}}/x}.$$

(VI.2). *We have*

$$H(x,y) = (1+x^2)^a \left(\frac{1+ix}{1-ix} \right)^{-b_{00} i} e^{-2y}.$$

(VI.3). *We have*

$$H(x,y) = y^2 \left(\frac{1+ix}{1-ix}\right)^{bi}.$$

(VI.4) *with* $b \neq 0$. *We have*

$$H(x,y) = \frac{y^2}{1+x^2}\left(\frac{1+ix}{1-ix}\right)^{bi}.$$

(VI.5) *with* $b \neq 0$. *We have*

$$H(x,y) = y^2(1+x^2)^{-m}\left(\frac{1+ix}{1-ix}\right)^{bi}.$$

(VI.5) *with* $b = 0$ *and* $m \notin \mathbb{Q}$. *We have*

$$H(x,y) = y^2(1+x^2)^{-m}.$$

(VI.7). *We have*

$$H(x,y) = V(x,y)\frac{(1+ix)^{(-k+ib)/2}}{(1-ix)^{(k+ib)/2}}.$$

(VI.9) *with* $b_{00} > 0$. *We have*

$$H(x,y) = \frac{\sqrt{b_{00}}-2y}{\sqrt{b_{00}}+2y}\left(\frac{1+ix}{1-ix}\right)^{\sqrt{b_{00}}i/2}.$$

(VI.9) *with* $b_{00} < 0$, $b_{00} \neq -4$. *We have*

$$H(x,y) = \frac{\sqrt{-b_{00}}-2yi}{\sqrt{-b_{00}}+2yi}\left(\frac{1+ix}{1-ix}\right)^{\sqrt{-b_{00}}/2}.$$

(VI.9) *with* $b_{00} = 0$. *We have*

$$H(x,y) = \left(\frac{1+ix}{1-ix}\right)^{i} e^{-2/y}.$$

(V.2). *We have*

$$H(x,y) = (1-x)^{1+\sigma}(1+x)^{1-\sigma}e^{-2y},$$

where $\sigma = b_{00}$ *if* $a \neq 0$ *and* $\sigma = 0$ *if* $a = 0$.

(V.3) *with* $b \notin \mathbb{Q}$. *We have*

$$H(x,y) = \left(\frac{1+x}{1-x}\right)^b y^2.$$

(V.4). *We have*

$$H(x,y) = (1 \pm x)^{b_{10}} e^{-(\pm b_{10} + (x\pm 1)y)/(x \mp 1)}.$$

(V.5) *with* $b \notin \mathbb{Q}$. *We have*

$$H(x,y) = (1-x)^{-b-1}(1+x)^{b-1}y^2.$$

(V.6) *with* $\delta = 1$. *We have*

$$H(x,y) = (x-1)^{\delta} e^{(3\delta - 2xy - 2(2\delta x + y))/(2(x-1)^2)}.$$

(V.7). *We have*

$$H(x,y) = (x+1)^{b_{00}+\delta-1}(x-1)^{1-b_{00}+3\delta} e^{2(1-b_{00}-\delta-2y)/(x-1)}.$$

(V.8) *with* $b \notin \mathbb{Q}$ *or* $m \notin \mathbb{Q}$. *We have*

$$H(x,y) = (1-x)^{-b-m}(1+x)^{b-m}y^2.$$

(V.9) *with* $m \notin \mathbb{Q}$. *We have*

$$H(x,y) = (x \mp 1)^{-m-1}(1 + (m+1)(x \pm 1)y).$$

(V.10)–(V.13). *Let* $\lambda = -(b+m)/2$ *and* $\mu = (b-m)/2$. *We have*

$$H(x,y) = (1-x)^{\lambda}(1+x)^{\mu} y - 2^{\lambda+\mu-1}\beta((1-x)/2, \lambda, \mu),$$

where

$$\beta(z,\lambda,\mu) = \int_0^z t^{\lambda-1}(1-t)^{\mu-1}\, dt.$$

We note that, in some cases, a rational first integral can be obtained.
(V.15) *with* $b_{20} + 1 > 0$ *and* $b_{20} \notin \mathbb{Q}$. *We have*

$$H(x,y) = (x \pm 1)^{\sqrt{b_{20}+1}} \frac{\sqrt{b_{20}+1}(x \mp 1) - (x \pm 1 - 2y)}{\sqrt{b_{20}+1}(x \mp 1) + x \pm 1 - 2y}.$$

(V.15) *with* $b_{20} + 1 < 0$. *We have*

$$H(x,y) = (x \pm 1)^{\sqrt{-b_{20}-1}} \left(\frac{\sqrt{-b_{20}-1}(x \mp 1) - i(x \pm 1 - 2y)}{\sqrt{-b_{20}-1}(x \mp 1) + i(x \pm 1 - 2y)}\right)^i.$$

(V.15) *with* $b_{20} = -1$. *We have*

$$H(x, y) = (x \pm 1)e^{-2(x \mp 1)/(x \pm 1 - 2y)}.$$

(V.16) *with* $b_{00} > 0$, $\sqrt{b_{00}} \notin \mathbb{Q}^+$. *We have*

$$H(x, y) = \frac{\sqrt{b_{00}} - 2y}{\sqrt{b_{00}} + 2y} \left(\frac{1 + x}{1 - x}\right)^{\sqrt{b_{00}}/2}.$$

(V.16) *with* $b_{00} < 0$. *We have*

$$H(x, y) = \left(\frac{1 - x}{1 + x}\right)^{\sqrt{-b_{00}}/2} \left(\frac{\sqrt{-b_{00}} + 2iy}{\sqrt{-b_{00}} - 2iy}\right)^{i}.$$

(V.16) *with* $b_{00} = 0$. *We have*

$$H(x, y) = e^{-2/y}\frac{1 + x}{1 - x}.$$

(II.2a). *We have*

$$H(x, y) = x^{2\delta} e^{2x + b_{20}x^2 - y^2}.$$

(II.2b). *We have*

$$H(x, y) = x^{2\delta} e^{\sigma x^2 - y^2}.$$

(II.3a). *We have*

$$H(x, y) = xe^{(-2\delta + 2b_{20}x^2 - y^2)/(2x)}.$$

(II.3b). *We have*

$$H(x, y) = xe^{(2\sigma x^2 - y^2)/(2x)}.$$

(II.4) *with* $\delta = 1$. *We have*

$$H(x, y) = x(b_{00} + y)^{b_{00}} e^{x - y}.$$

(II.5a) *with* $b_{20} \neq 0$. *We have*

$$H(x, y) = x^{b}_{20} e^{-(\delta + 2x + y^2)/(2x^2)}.$$

(II.5b) *with* $\sigma \neq 0$. *We have*

$$H(x, y) = x^{\sigma} e^{-(2x + y^2)/(2x^2)}.$$

(II.5c) *with* $\sigma \neq 0$. *We have*

$$H(x,y) = x^{\sigma} e^{-(\delta+y^2)/(2x^2)}.$$

(II.5d). *We have*

$$H(x,y) = x^{\delta} e^{-y^2/(2x^2)}.$$

(II.6). *We have*

$$H(x,y) = x^{\delta - b_{20}} (b_{20}x + \delta y)^{b}_{20} e^{-(1+\delta y)/x}.$$

(II.7). *We have*

$$H(x,y) = \frac{x+y}{x} e^{-(1+y)/x}.$$

(II.8) *with* $B = 1/\sqrt{1-4b_{00}} \in \mathbb{R} \setminus \mathbb{Q}$. *We have*

$$H(x,y) = \frac{V(x,y)}{x^3} \left(\frac{\sqrt{1-4b_{00}}(d+\delta x) + (b_{00}+\delta x + 2b_{00}y)}{\sqrt{1-4b_{00}}(b_{00}+\delta x) - (b_{00}+\delta x + 2b_{00}y)} \right)^{B}.$$

(II.8) *with* $1 - 4b_{00} < 0$. *We have*

$$H(x,y) = \frac{V(x,y)}{x^3} \left(\frac{\sqrt{-(1-4b_{00})}(b_{00}+\delta x) + i(b_{00}+\delta x + 2b_{00}y)}{\sqrt{-(1-4b_{00})}(b_{00}+\delta x) - (b_{00}+\delta x + 2b_{00}y)} \right)^{Bi},$$

where $B = 1/\sqrt{4b_{00}-1}$.

(II.8) *with* $b_{00} = 1/4$. *Writing* $V(x,y) = xf(x,y)^2$, *where* $f(x,y) = 1 + 4\delta x + 2y$, *we have*

$$H(x,y) = \frac{f(x,y)}{x} e^{(1+4\delta x)/f(x,y)}.$$

(II.9) *with* $2n \notin \mathbb{Q}$. *Writing* $V(x,y) = xf(x,y)$, *where* $f(x,y) = \sigma/n + 2\delta x/(2n-1) + b_{20}x^2/(n-1) + y^2$, *we have*

$$H(x,y) = x^{-2n} f(x,y).$$

(II.10) *with* $4dn - b^2 > 0$. *We have*

$$H(x,y) = x^{-2n}(d + by + ny^2) \left(\frac{\sqrt{4dn-b^2} + i(b+2ny)}{\sqrt{4dn-b^2} - i(b+2ny)} \right)^{bi/\sqrt{4dn-b^2}}.$$

(II.10) *with* $4dn - b^2 < 0$ *and* $n \notin \mathbb{Q}$ *or* $b/\sqrt{b^2-4dn} \notin \mathbb{Q}$. *We have*

$$H(x,y) = x^{-2n}(d + by + ny^2) \left(\frac{\sqrt{b^2-4dn} + (b+2ny)}{\sqrt{b^2-4dn} - (b+2ny)} \right)^{b/\sqrt{b^2-4dn}}.$$

(II.10) *with* $d = b^2/(4n) \neq 0$. *We have*

$$H(x,y) = x^{-n}(b + 2ny)e^{b/(b+2ny)}.$$

(II.11) *with* $L = 4b_{20}(n-1) - 1 > 0$. *Writing* $V(x,y) = xf(x,y)\bar{f}(x,y)$, *where*

$$f(x,y) = \sqrt{L}(\delta(n-1) + nx) + i(\delta(n-1) + nx + 2n(n-1)y),$$

we have

$$H(x,y) = x^{-2n} f(x,y)^{1+i/\sqrt{L}} \bar{f}(x,y)^{1-i/\sqrt{L}}.$$

(II.11) *with* $L = 1 - 4b_{20}(n-1) > 0$. *Writing* $V(x,y) = xf_1(x,y)f_2(x,y)$, *where*

$$f_1(x,y) = \sqrt{L}(\delta(n-1) + nx) + (\delta(n-1) + nx + 2n(n-1)y),$$
$$f_2(x,y) = \sqrt{L}(\delta(n-1) + nx) - (\delta(n-1) + nx + 2n(n-1)y),$$

we have

$$H(x,y) = x^{-2n} f_1(x,y)^{1+1/\sqrt{L}} f_2(x,y)^{1-1/\sqrt{L}}.$$

(II.11) *with* $b_{20} = 1/(4(n-1))$. *Writing* $V(x,y) = xf(x,y)^2$, *where*

$$f(x,y) = \delta(n-1) + nx + 2n(n-1)y,$$

we have

$$H(x,y) = x^{-n} f(x,y)\, e^{(\delta(n-1)+nx)/f(x,y)}.$$

(II.14) *with* $\delta = 1$, $1 + 4b_{00} < 0$. *We have*

$$H(x,y) = \frac{x^2(-b_{00} + x - y + x^2 - 2xy + y^2)}{(b_{00} - x - x^2 + xy)^2}\, e^{\frac{2\arctan\frac{-2x+2y-1}{\sqrt{-4b_{00}-1}}}{\sqrt{-4b_{00}-1}}}.$$

(II.14) *with* $\delta = 1$, $b_{00} = 0$. *We have*

$$H(x,y) = \frac{(x-y)}{x-y+1}\, e^{\frac{x+1}{x^2-yx+x}}.$$

(II.14) *with* $\delta = 1$, $b_{00} = -1/4$. *We have*

$$H(x,y) = \frac{(4x^2 - 4yx + 4x + 1)}{x(2x-2y+1)}\, e^{-\frac{1}{2x-2y+1}}.$$

(II.15) *with* $1-8b_{00}>0$ *and* $1/\sqrt{1-8b_{00}}\in\mathbb{R}\setminus\mathbb{Q}$. *Writing* $V(x,y)=xf_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=(1+\sqrt{1-8b_{00}})(b_{00}+\delta x^2)+4b_{00}y,\\ f_2(x,y)&=(1-\sqrt{1-8b_{00}})(b_{00}+\delta x^2)+4b_{00}y,\end{aligned}$$

we have

$$H(x,y)=x^{-4}f_1(x,y)^{1+1/\sqrt{1-8b_{00}}}f_2(x,y)^{1-1/\sqrt{1-8b_{00}}}.$$

(II.15) *with* $1-8b_{00}<0$. *Writing* $V(x,y)=xf_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=\sqrt{8b_{00}-1}(b_{00}+\delta x^2)+i(b_{00}+\delta x^2+4b_{00}y),\\ f_2(x,y)&=\sqrt{8b_{00}-1}(b_{00}+\delta x^2)-i(b_{00}+\delta x^2+4b_{00}y),\end{aligned}$$

we have

$$H(x,y)=x^{-4}f_1(x,y)^{1+i/\sqrt{8b_{00}-1}}f_2(x,y)^{1-i/\sqrt{8b_{00}-1}}.$$

(II.15) *with* $b_{00}=1/8$. *Writing* $V(x,y)=xf(x,y)^2$, *where*

$$f(x,y)=1+8\delta x^2+4y,$$

we have

$$H(x,y)=\frac{x^2}{f(x,y)}\,e^{-(1+8\delta x^2)/f(x,y)}.$$

(I.2). *We have*

$$H(x,y)=y^2\,e^{-x(2\delta+b_{11}x-2y)}.$$

(I.3). *We have*

$$H(x,y)=V(x,y)\,e^{b_{11}x-y}.$$

(I.5) *with* $n\notin\mathbb{Q}$. *Writing* $V(x,y)=f_1(x,y)f_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=n^2-1+b_{11}n(n+1)x+n(n-1)(2n+1)y,\\ f_2(x,y)&=(n-1)(2n+1)(1+(n+1)xy)+(n-1+b_{11}nx)(n+1)x,\end{aligned}$$

we have

$$H(x,y)=f_1(x,y)^{-n-1}f_2(x,y)^n.$$

(I.6) *with* $b_{00}>1/2$. *Writing* $V(x,y)=f_1(x,y)f_2(x,y)\bar{f}_2(x,y)$, *where*

$$\begin{aligned}f_1(x,y)&=2b_{00}(1-2x)+(2+y)^2,\\ f_2(x,y)&=\sqrt{2b_{00}-1}+i(1+y),\end{aligned}$$

we have

$$H(x,y)=\frac{f_2(x,y)^{1+i/\sqrt{2b_{00}-1}}f_3(x,y)^{1-i/\sqrt{2b_{00}-1}}}{f_1(x,y)}.$$

(I.6) *with* $b_{00}<1/2$*,* $b_{00}\neq 0$ *and* $1/\sqrt{1-2b_{00}}\in\mathbb{R}\setminus\mathbb{Q}$*. Writing* $V(x,y)=f_1(x,y)f_2(x,y)f_3(x,y)$*, where*

$$\begin{aligned}f_1(x,y)&=2b_{00}(1-2x)+(2+y)^2,\\ f_2(x,y)&=\sqrt{1-2b_{00}}+(1+y),\\ f_3(x,y)&=\sqrt{1-2b_{00}}-(1+y),\end{aligned}$$

we have

$$H(x,y)=\frac{f_2(x,y)^{1+1/\sqrt{1-2b_{00}}}f_3(x,y)^{1-1/\sqrt{1-2b_{00}}}}{f_1(x,y)}.$$

(I.6) *with* $b_{00}=1/2$*. Writing* $V(x,y)=f_1(x,y)^2f_2(x,y)$*, where*

$$\begin{aligned}f_1(x,y)&=5-2x+4y+y^2,\\ f_2(x,y)&=1+y,\end{aligned}$$

we have

$$H(x,y)=\frac{f_1(x,y)}{f_2(x,y)^2}\,e^{-2/f_2(x,y)}.$$

(I.6) *with* $b_{00}=0$*. We have*

$$H(x,y)=\frac{y}{2+y}\,e^{2(3-2x+y)/(2+y)}.$$

(I.7) *with neither* $\delta=1$*,* $b_{00}<-\sqrt{2}$ *and* $\sqrt{-b_{00}-\sqrt{2}}/\sqrt{-b_{00}+\sqrt{2}}\in\mathbb{Q}$*, nor* $\delta=1$ *and* $b_{00}^2=2$*. Writing* $V(x,y)=f_1(x,y)f_2(x,y)f_3(x,y)f_4(x,y)$*, where*

$$\begin{aligned}f_{1,2}(x,y)&=\sqrt{2}\sqrt{-b_{00}-\sqrt{2\delta}}\pm(\sqrt{2\delta}x+y),\\ f_{3,4}(x,y)&=\sqrt{2}\sqrt{-b_{00}+\sqrt{2\delta}}\pm(\sqrt{2\delta}x-y),\end{aligned}$$

we have

$$H(x,y)=\left(\frac{f_1(x,y)}{f_2(x,y)}\right)^{\sqrt{-b_{00}+\sqrt{2\delta}}}\left(\frac{f_3(x,y)}{f_4(x,y)}\right)^{\sqrt{-b_{00}-\sqrt{2\delta}}}.$$

(I.7) *with* $\delta = 1$ *and* $b_{00} = \sqrt{2}$. *Writing* $V(x,y) = f_1(x,y)^2 f_2(x,y)\bar{f}_2(x,y)$, *where*

$$f_1(x,y) = \sqrt{2}x - y,$$
$$f_2(x,y) = 2^{5/4} + i(\sqrt{2}x + y),$$

we have

$$H(x,y) = \left(\frac{f_2(x,y)}{\bar{f}_2(x,y)}\right)^i e^{2^{9/4}/f_1(x,y)}.$$

(I.7) *with* $\delta = 1$ *and* $b_{00} = -\sqrt{2}$. *Writing* $V(x,y) = f_1(x,y)^2 f_2(x,y) f_3(x,y)$, *where*

$$f_1(x,y) = \sqrt{2}x + y,$$
$$f_2(x,y) = 2^{5/4} + (\sqrt{2}x - y),$$
$$f_3(x,y) = 2^{5/4} - (\sqrt{2}x - y),$$

we have

$$H(x,y) = \frac{f_2(x,y)}{f_3(x,y)} e^{2^{9/4}/f_1(x,y)}.$$

(I.9) *with* $b_{00} < 3/4$ *and* $1/\sqrt{9 - 12b_{00}} \in \mathbb{R} \setminus \mathbb{Q}$. *Writing*

$$V(x,y) = f_1(x,y) f_2(x,y) f_3(x,y),$$

where

$$f_1(x,y) = 3 + 2y + \sqrt{9 - 12b_{00}},$$
$$f_2(x,y) = 3 + 2y - \sqrt{9 - 12b_{00}},$$
$$f_3(x,y) = 9(6 + 5b_{00}) - 9(6 + b_{00})(b_{00}x - y) + 2(9 + y)y^2,$$

we have

$$H(x,y) = \frac{f_1(x,y)^{3+9/\sqrt{9-12b_{00}}} f_2(x,y)^{3-9/\sqrt{9-12b_{00}}}}{f_3(x,y)^2}.$$

(I.9) *with* $b_{00} = 0$. *Writing* $V(x,y) = y f(x,y)$, *where* $f(x,y) = (3 + y)^4$, *we have*

$$H(x,y) = \frac{f(x,y)}{y} e^{-3(3(11-6x)+15y+2y^2)/(2f(x,y)^3)}.$$

(I.9) *with* $> 3/4$. *Writing* $V(x,y) = f_1(x,y) f_2(x,y) f_3(x,y)$, *where*

$$f_1(x,y) = 3 + 2y + i\sqrt{12b_{00} - 9},$$
$$f_2(x,y) = 3 + 2y - i\sqrt{12b_{00} - 9},$$
$$f_3(x,y) = 9(6 + 5b_{00}) - 9(6 + b_{00})(b_{00}x - y) + 2(9 + y)y^2,$$

we have

$$H(x,y) = \frac{f_1(x,y)^{3-9i/\sqrt{12b_{00}-9}}\, f_2(x,y)^{3+9i/\sqrt{12b_{00}-9}}}{f_3(x,y)^2}.$$

(I.9) *with* $b_{00} = 3/4$. *Writing* $V(x,y) = f_1(x,y)^2 f_2(x,y)$, *where*

$$f_1(x,y) = 3 + 2y,$$
$$f_2(x,y) = 1404 - 729x + 972y + 288y^2 + 32y^3,$$

we have

$$H(x,y) = \frac{f_2(x,y)}{f_1(x,y)^3}\, e^{-9/f_1(x,y)}.$$

We remark that all the first integral provided in the proposition above are real, due to equality (1.9).

3.2 Phase portraits

In this section we give the phase portraits of the quadratic systems having a polynomial inverse integrating factor. As in Chapter 2, we follow the classification into the normal forms of Proposition 2.1.2.

First we introduce the basic definitions, notations and results that we need for the analysis of the local phase portraits of the finite and infinite singular points of the quadratic systems and then we define the Poincaré compactification. The results of Sections 3.2.1 and 3.2.3 can be found in [1]. The results of Section 3.2.2 can be found in [12].

3.2.1 Singular points

Consider an analytic planar system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ and its associated vector field $X = (P,Q)$. A point $p \in \mathbb{R}^2$ is a *singular point* of X if $P(p) = Q(p) = 0$. We define, for a singular point $p \in \mathbb{R}^2$, $\Delta = P_x(p)Q_y(p) - P_y(p)Q_x(p) \in \mathbb{R}$ and $T = P_x(p) + Q_y(p) \in \mathbb{R}$. They correspond, respectively, to the determinant and the trace of the Jacobian matrix $DX(p)$.

The singular point p is *non–degenerated* if $\Delta \neq 0$ and it is *degenerated* otherwise. Then, p is an isolated singular point. Moreover, p is a *saddle* if $\Delta < 0$, a *node* if $T^2 \geq 4\Delta > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), a *focus* if $4\Delta > T^2 > 0$ (*stable* if $T < 0$, *unstable* if $T > 0$), and either a *weak focus* or a *center* if $T = 0 < \Delta$ (for more details, see [1]).

The singular point p is called *hyperbolic* if the two eigenvalues of the Jacobian matrix $DX(p)$ have nonzero real part. So, the hyperbolic singular points are the non–degenerate ones except the weak foci and the centers.

A degenerate singular point p such that $T \neq 0$ is called *semi–hyperbolic*, and p is isolated in the set of all singular points. Next we summarize the results on semi–hyperbolic singular points, see Theorem 65 of [1].

Proposition 3.2.1. *Let $(0,0)$ be an isolated point of the vector field $(F(x,y), y+G(x,y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in the variables x and y. Let $y = g(x)$ be the solution of the equation $y + G(x,y) = 0$ in a neighborhood of $(0,0)$. Assume that the development of the function $f(x) = F(x, g(x))$ is of the form $f(x) = \mu x^m + \cdots$, where $m \geq 2$ and $\mu \neq 0$. When m is odd, then $(0,0)$ is either an unstable node, or a saddle depending if $\mu > 0$ or $\mu < 0$, respectively. If m is even, then $(0,0)$ is a saddle–node, i.e. the singular point is formed by the union of two hyperbolic sectors with one parabolic sector.*

The singular points which are non–degenerate or semi–hyperbolic are called *elementary.*

When $\Delta = T = 0$ but the Jacobian matrix at p is not the zero matrix and p is isolated in the set of all singular points, we say that p is *nilpotent.* Next we summarize some results on nilpotent singular points (see Theorems 66 and 67 and the simplified scheme of Section 22.3 of [1]).

Proposition 3.2.2. *Let $(0,0)$ be an isolated singular point of the vector field $(y + F(x,y),\ G(x,y))$, where F and G are analytic functions in a neighborhood of the origin starting at least with quadratic terms in the variables x and y. Let $y = f(x)$ be the solution of the equation $y + F(x,y) = 0$ in a neighborhood of $(0,0)$. Assume that the development of the function $G(x, f(x))$ is of the form $Kx^\kappa + \cdots$ and $\Phi(x) \equiv (\partial F/\partial x + \partial G/\partial y)(x, f(x)) = Lx^\lambda + \cdots$, with $K \neq 0$, $\kappa \geq 2$ and $\lambda \geq 1$. Then the following statements hold.*

(1) *If κ is even and*

  (a) *$\kappa > 2\lambda + 1$, then the origin is a saddle–node.*

  (b) *$\kappa < 2\lambda + 1$ or $\Phi \equiv 0$,, then the origin is a cusp, i.e. a singular point formed by the union of two hyperbolic sectors.*

(2) *If κ is odd and $K > 0$, then the origin is a saddle.*

(3) *If κ is odd, $K < 0$ and*

  (a) *λ even, $\kappa = 2\lambda+1$ and $L^2 + 4K(\lambda+1) \geq 0$, or λ even and $\kappa > 2\lambda+1$,, then the origin is a stable (unstable) node if $L < 0$ ($L > 0$).*

  (b) *λ odd, $\kappa = 2\lambda + 1$ and $L^2 + 4K(\lambda + 1) \geq 0$, or λ odd and $\kappa > 2\lambda + 1$, then the origin is an elliptic–saddle, i.e. a singular point formed by the union of one hyperbolic sector and one elliptic sector.*

(c) $\kappa = 2\lambda + 1$ *and* $L^2 + 4K(\lambda + 1) < 0$, *or* $\kappa < 2\lambda + 1$, *then the origin is a focus or a center, and if* $\Phi(x) \equiv 0$ *then the origin is a center.*

Finally, if the Jacobian matrix at the singular point p is identically zero, and p is isolated inside the set of all singular points, then we say that p is *linearly zero*. The study of its local phase portrait needs a special treatment using the directional blow–ups technique, see for more details [3]. But if a quadratic vector field has a finite linearly zero singular point, then it is equivalent to a homogeneous quadratic vector field doing if necessary a translation of the linearly zero singular point to the origin, and the global phase portraits of the quadratic homogeneous vector fields are well known, see for more details [51].

The definitions of hyperbolic, parabolic and elliptic sectors near a singular point can be found in [1]. Roughly speaking, in a *hyperbolic sector* there are two orbits one starting and the other ending at the singular point and all the other orbits between them and in a neighborhood of the singular point approach to the singular point and after this they go away. A sector such that all curves in a sufficiently small neighborhood of the singular point tend to it as either $t \to +\infty$ or $t \to -\infty$ is known as a *parabolic sector*. Finally, a sector containing loops to the singular point, and moreover only nested loops, is known as an *elliptic sector*.

The number of elliptic sectors and the number of hyperbolic sectors in a neighborhood of a singular point are denoted by $\mathbf{e}$ and $\mathbf{h}$, respectively. The rest of the sectors are parabolic. The (topological) *index* of a singular point p is defined as

$$i(p) = \frac{\mathbf{e} - \mathbf{h}}{2} + 1. \tag{3.2}$$

For a proof of the formula (3.2), see [1].

3.2.2 Separatrices and canonical regions

Consider the planar differential system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{3.3}$$

where P and Q are $\mathcal{C}^r$ maps, $r \geq 1$ from an open subset $U \subseteq \mathbb{R}^2$ to $\mathbb{R}$. For a differential system (3.3) the following three properties are well–known, see for more details [47].

1. For all $p \in U$ there exists an open interval $I_p \subseteq \mathbb{R}$ where the unique maximal solution $\varphi_p : I_p \to U$ of (3.3) such that $\varphi_p(0) = p$ is defined.

2. If $q = \varphi_p(t)$ and $t \in I_p$, then $I_q = I_p - t = \{r - t : r \in I_p\}$ and $\varphi_q(s) = \varphi_p(t + s)$ for all $s \in I_q$.

3. The set $D = \{(t,p) : p \in U, t \in I_p\}$ is open in $\mathbb{R}^3$ and the map $\varphi : D \to U$ defined by $\varphi(t,p) = \varphi_p(t)$ is C^r.

The map $\varphi : D \to U$ is a *local flow* of class C^r on U associated to system (3.3). It verifies:

1. $\varphi(0,p) = p$ for all $p \in U$.

2. $\varphi(t, \varphi(s,p)) = \varphi(t+s, p)$ for all $p \in U$ and for all s and t such that $s, t+s \in I_p$.

3. $\varphi_p(-t) = \varphi_p^{-1}(t)$ for all $p \in U$ such that $t, -t \in I_p$.

We consider C^r–local flows with $r \geq 0$ on $\mathbb{R}^2$. Of course, when $r = 0$ the flow is only continuous. Two such flows, φ and φ', are *C^k–equivalent*, with $k \geq 0$, if there exists a C^k diffeomorphism which takes orbits of φ onto orbits of φ' preserving sense (but not necessarily the parametrization).

Let φ be a $\mathcal{C}^r$–local flow with $r \geq 0$ on $\mathbb{R}^2$. We say that φ is *$\mathcal{C}^k$–parallel* if it is $\mathcal{C}^k$–equivalent to one of the following flows:

1. $\mathbb{R}^2$ with the flow defined by $x' = 1$, $y' = 0$.

2. $\mathbb{R}^2 \setminus \{0\}$ with the flow defined (in polar coordinates) by $r' = 0$, $\theta' = 1$.

3. $\mathbb{R}^2 \setminus \{0\}$ with the flow defined by $r' = r$, $\theta = 0$.

We call these flows as *strip*, *annular* and *spiral*, respectively.

Let $p \in \mathbb{R}^2$. We denote by $\gamma(p)$ the *orbit* of the flow φ through p, more precisely $\gamma(p) = \{\varphi_p(t) : t \in I_p\}$. The *positive semiorbit* of p is $\gamma^+(p) = \{\varphi_p(t) : t \in I_p, t \geq 0\}$. In a similar way we define the *negative semiorbit* $\gamma^-(p)$ of p.

We define the *α–limit* and the *ω–limit* of p as $(\gamma^{\pm}(p))$ and let

$$\alpha(p) = \mathrm{cl}\,(\gamma^-(p)) - \gamma^-(p), \qquad \omega(p) = \mathrm{cl}\,(\gamma^+(p)) - \gamma^+(p),$$

respectively, where cl denotes the closure of the set.

Let $\gamma(p)$ be an orbit of the flow φ. A *parallel neighborhood* of the orbit $\gamma(p)$ is an open neighborhood N of $\gamma(p)$ such that φ is $\mathcal{C}^k$–equivalent in N to a parallel flow for some $k \geq 0$.

We say that $\gamma(p)$ is a *separatrix* of φ if it is not contained in a parallel neighborhood N satisfying the following two assumptions:

1. For any $q \in N$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$.

2. $\mathrm{cl}\,(N) \setminus N$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of φ, with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

We denote by Σ the union of all separatrices of φ. Σ is a closed invariant subset of $\mathbb{R}^2$. A component of the complement of Σ in R^2, with the restricted flow, is a *canonical region* of φ.

The following lemma can be found in [41].

Lemma 3.2.3. *Every canonical region of a local flow φ on $\mathbb{R}^2$ is $\mathcal{C}^0$–parallel.*

3.2.3 The Poincaré compactification

Let X be a real planar polynomial vector field of degree n. The *Poincaré compactified vector field* $p(X)$ *corresponding to* X is an analytic vector field induced on Σ^2 as follows (see, for instance [35]). Let $\Sigma^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ (the *Poincaré sphere*) and $T_y\Sigma^2$ be the tangent plane to Σ^2 at point y. Identify $\mathbb{R}^2$ with $T_{(0,0,1)}\Sigma^2$. Consider the central projection $f : T_{(0,0,1)}\Sigma^2 \to \Sigma^2$. This map defines two copies of X on Σ^2, one in the northern hemisphere and the other in the southern hemisphere. Denote by X' the vector field $Df \circ X$ defined on Σ^2 except on its equator $\Sigma^1 = \{y \in \Sigma^2 : y_3 = 0\}$. Clearly Σ^1 is identified to the *infinity* of $\mathbb{R}^2$. Usually, when we talk about the circle of the infinity of X we simply talk about the infinity.

In order to extend X' to a vector field on Σ^2 (including Σ^1) it is necessary that X satisfies suitable conditions. If X is a real polynomial vector field of degree n, then $p(X)$ is the only analytic extension of $y_3^{n-1}X'$ to Σ^2. On $\Sigma^2 \backslash \Sigma^1$ there are two symmetric copies of X, and knowing the behavior of $p(X)$ around Σ^1, we know the behavior of X in a neighborhood of the infinity. The Poincaré compactification has the property that Σ^1 is invariant under the flow of $p(X)$. The projection of the closed northern hemisphere of Σ^2 on $y_3 = 0$ under $(y_1, y_2, y_3) \longmapsto (y_1, y_2)$ is called the *Poincaré disc,* and it is denoted by $\mathbf{D}^2$.

Two polynomial vector fields X and Y on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on Σ^2 preserving the infinity Σ^1 carrying orbits of the flow induced by $p(X)$ into orbits of the flow induced by $p(Y)$.

As Σ^2 is a differentiable manifold, for computing the expression for $p(X)$, we can consider the six local charts $U_i = \{y \in \Sigma^2 : y_i > 0\}$, and $V_i = \{y \in \Sigma^2 : y_i < 0\}$, $i = 1, 2, 3$. The diffeomorphisms $F_i : U_i \to \mathbb{R}^2$ and $G_i : V_i \to \mathbb{R}^2$ for $i = 1, 2, 3$ are the inverses of the central projections from the planes tangent at the points $(1, 0, 0), (-1, 0, 0), (0, 1, 0), (0, -1, 0), (0, 0, 1)$ and $(0, 0, -1)$, respectively. If $z = (u, v)$ is the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (so z represents different things according to the local charts under consideration), then we obtain the following expressions for $p(X)$:

$$v^n \Delta(z) \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right), -vP\left(\frac{1}{v}, \frac{u}{v}\right) \right) \text{ in } U_1, \tag{3.4}$$

$$
\begin{aligned}
& v^n\Delta(z)\left(P\left(\frac{u}{v},\frac{1}{v}\right)-uQ\left(\frac{u}{v},\frac{1}{v}\right),-vQ\left(\frac{u}{v},\frac{1}{v}\right)\right) \text{ in } U_2, \\
& \Delta(z)\left(P(u,v),Q(u,v)\right) \text{ in } U_3,
\end{aligned} \tag{3.5}
$$

where $\Delta(z) = (u^2+v^2+1)^{-\frac{1}{2}(n-1)}$. The expression for V_i is the same as that for U_i except for a multiplicative factor $(-1)^{n-1}$. In these coordinates for $i = 1, 2$, $v = 0$ always denotes the points of Σ^1. We can omit the factor $\Delta(z)$ by scaling the vector field $p(X)$. Thus, the expression of $p(X)$ becomes a polynomial vector field in each local chart.

3.2.4 Construction of the phase portraits

In our study of the phase portraits of quadratic systems having a polynomial inverse integrating factor, we follow some steps. First we compute and classify all the singular points (finite and infinite) of the system to obtain the local phase portrait at them, using the results of Subsections 3.2.1, 3.2.2 and 3.2.3 and blow-ups if necessary. Once this classification is finished, we look for the separatrices of the system. As $(\star)$ quadratic systems have no limit cycles, we just have to look for the separatrices of the hyperbolic sectors. Using the first integral associated to the system we determine the global behavior of these separatrices, and then the global phase portrait is completed and we can draw it. The pictures of the phase portraits of this work have been done using the program P4 (see [23]). The program has also been used to verify the study of the phase portraits.

We deal with systems of the form

$$
\dot{x} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy, \quad \dot{y} = d + ax + by + lx^2 + mxy + ny^2,
$$

where all the parameters are real.

The finite singular points We solve the system of equations $\dot{x} = \dot{y} = 0$. The solutions (x_0, y_0) of this system are the singular points of the phase portrait. In cases (IV) and (VI)–(IX), the maximum number of finite singular points is two. The other four systems have at most four finite singular points. The number of finite singular points and their multiplicity for systems (I) and (III) depend on a discriminant Δ. In case (I),

$$
\begin{aligned}
\Delta \;=\; & -27a^4n^2 + 2a^3b(2b^2 - 9n(d-m)) + (a^2b^2((d-m)^2 - 6ln) + \\
& 4n(d-m)(36ln - (d-m)^2)) + 2abl(8n(5(d-m)^2 + 12ln) - \\
& 9b^2(d-m)) - l(27b^4l + 4b^2(d-m)((d-m)^2 - 36ln) - \\
& 16n((d-m)^2 - 4ln)^2).
\end{aligned} \tag{3.6}
$$

In case (III),

$$
\begin{aligned}
\Delta \;=\; & 256d^3n^3 - n^2(27a^4 + 144a^2d(b-l) - 192ad^2m + 128d^2(b-l)^2) + \\
& 2n(2(b-l)^3(a^2 + 4d(b-l)) + am(b-l)(9a^2 + 40d(b-l)) - \\
& 3dm^2(a^2 + 24d(b-l))) + m^2(4a^3m + a^2(b-l)^2 + 18adm(b-l) + \\
& d(4(b-l)^3 - 27dm^2)).
\end{aligned} \tag{3.7}
$$

For the rest of the systems we deal with the discriminant of a polynomial equation of degree 2, so it gets an easier expression.

Once all the finite singular points are found, we want to know their behavior. For that purpose, we use the results given in Section 3.2.1. Different behaviors will lead to different phase portraits.

The infinite singular points In order to compute the singular points on the line of the infinity, we use the Poincaré compactification, see Subsection 3.2.3. We note that the singular points at the infinity appear by pairs, one of them on the chart U_i and the other one on V_i. Each pair is formed by two diametrally opposed infinite singular points.

In order to compute the singular points at infinity on U_1 and V_1, we must compute the singular points of type $(u,0)$ of the system

$$\dot{u} = v^2\left[Q\left(\frac{1}{v},\frac{u}{v}\right) - uP\left(\frac{1}{v},\frac{u}{v}\right)\right], \quad \dot{v} = -v^2 P\left(\frac{1}{v},\frac{u}{v}\right).$$

This is equivalent to solve the quadratic equation

$$(n - a_{11})u^2 + (m - a_{20})u + l = 0, \tag{3.8}$$

with unknown u.

To compute the singular points at infinity on U_2 and V_2, we must find the singular points of type $(u,0)$ of the system

$$\dot{u} = v^2\left[P\left(\frac{u}{v},\frac{1}{v}\right) - uQ\left(\frac{u}{v},\frac{1}{v}\right)\right], \quad \dot{v} = -v^3 Q\left(\frac{u}{v},\frac{1}{v}\right),$$

which is equivalent to solve the cubic equation

$$u(lu^2 + (m - a_{20})u + (n - a_{11})) = 0, \tag{3.9}$$

with unknown u. Observe that the point $A = (0,0)$ on the chart U_2 is always an infinite singular point, and also $A' = (0,0)$ on V_2.

Remark 3.2.4.

1. If $l = 0$, $m = a_{20}$ and $n = a_{11}$, then all the points at infinity are singular. In this case, we say that the infinity is *degenerated*.
2. A non–zero solution $(u,0)$ of equation (3.8) corresponds to the non–zero solution $(1/u,0)$ of equation (3.9). So, if the infinity is not degenerated, then there are at most three pairs of infinite singular points for each phase portrait.
3. If the infinity is degenerated, the infinite singular points of the system that is obtained by dropping the factor v (scaling time), give all the information of the phase portrait at infinity.

4. In general, when we refer to an infinite singular point in the study of the phase portraits we mean the corresponding pair of singular points.

□

The phase portraits are grouped by topological equivalence, and shown depending on the numbers of pairs of singular points at infinity. Figures **(1)**–**(11)** correspond to phase portraits of systems with degenerated infinity. Figures **(12)**–**(41)** correspond to phase portraits of systems with one pair of singular points at infinity. Figures **(42)**–**(82)** correspond to phase portraits of systems with two pairs of infinite singular points. Figures **(83)**–**(122)** correspond to phase portraits of systems with three pairs of singular points at infinity. The pictures are shown in Section 3.4.

Remark 3.2.5. In the most of the cases, the systems depend on several parameters, so the phase portrait may vary with these parameters. In these cases we compute a bifurcation diagram, from which we know the behavior of the system for all the possible values of the parameters. □

In the following subsections, we compute the phase portrait of the $(\star)$ quadratic systems. We also show the who the set $V^{-1}(0)$ is in the phase portrait.

3.2.5 Systems (IX)

These systems were studied in Subsection 2.3.1. We use the notation given in that subsection. As $\dot{x} = 1$, the systems of this section have no finite singular points, so the study of the local phase portraits is reduced to the behavior of the singular points at infinity.

(IX.1) The only singular point is the point A at infinity, and it is a non–elementary singular point of index $i(A) = 1$, without elliptic and hyperbolic sectors. The phase portrait is shown in **(12)**.

(IX.2) Again, the only singular point at infinity is the point A, which is a non–elementary singular point, with $\mathbf{e} = \mathbf{h} = 1$, so $i(A) = 1$. The phase portrait is shown in **(33)**. The set $V^{-1}(0)$ is the parabola separating the two canonical regions.

(IX.3) There are two singular points at the infinity: A (non–elementary) and the semi–hyperbolic singular point $p_1 = (0,0)$ on chart U_1. For $\delta = 1$, we have $i(A) = 2$, with $(\mathbf{e},\mathbf{h}) = (2,0)$, and p_1 is a saddle. If $\delta = -1$, then $i(A) = 0$, with $(\mathbf{e},\mathbf{h}) = (0,2)$, and p_1 is a node.

The phase portraits are shown in **(44)** and **(42)**, there is one phase portrait for each respective value of $\delta = -1, 1$. The set $V^{-1}(0)$ is, for $\delta = 1$, the straight line $y = 0$ separating the two elliptic sectors.

(IX.4) Two singular points appear at the infinity: A which is a node, and a non–elementary singular point p_1 on chart U_1. In all cases, $i(p_1) = 0$, with $(\mathbf{e}, \mathbf{h}) = (0, 2)$.

If $\delta = 1$, then there is only one canonical region. The set $V^{-1}(0)$ is, for $\delta = 0$, the straight line $y = 0$ separating two canonical regions. The other separatrices correspond to the set $H^{-1}(0)$. If $\delta = -1$, then the set $V^{-1}(0)$ is formed by the straight lines $y = \pm 1$, which divide the phase portrait in three canonical regions. The phase portraits are shown in **(47)**, **(46)** and **(44)**, one phase portrait for each respective value of $\delta = -1, 0, 1$.

3.2.6 Systems (VIII)

These systems were studied in Subsection 2.3.2. We use the notation given in that subsection.

(VIII.1) The origin is the only finite singular point, and it is a saddle. The only singular point at infinity is the point A, and it is a non–elementary singular point of index $i(A) = 2$, with $(\mathbf{e}, \mathbf{h}) = (2, 0)$. The phase portrait is shown in **(36)**. The separatrices correspond to the set $H^{-1}(0)$.

(VIII.2) There are no finite singular points. The only singular point at infinity is the point A, which is a non–elementary singular point, with $i(A) = 1$. If $\delta = -1$, then we have $(\mathbf{e}, \mathbf{h}) = (2, 2)$. If $\delta = 1$, then we have $\mathbf{e} = \mathbf{h} = 0$.

The phase portraits are shown in **(34)** and **(12)**, there is one phase portrait for each respective value of $\delta = -1, 1$. The set $V^{-1}(0)$ is, for $\delta = -1$, the straight line $x = 0$ separating the two elliptic sectors.

(VIII.3) The origin is the only finite singular point, and it is an unstable node. The only singular point at infinity is A, which is a saddle–node. The phase portrait is shown in **(37)**, it is the same for both values of δ. The set $V^{-1}(0)$ is the straight line $x = 0$ which is, for $y \leq 0$, the finite separatrix of A.

(VIII.4) The origin is the only finite singular point. It is a saddle if $b < 0$ and it is an unstable node if $b > 0$. The only singular point at infinity is A, and it is a non–elementary singular point. If $b < 0$, then $i(A) = 2$, with $(\mathbf{e}, \mathbf{h}) = (2, 0)$. If $b > 0$, then $i(A) = 0$, with $(\mathbf{e}, \mathbf{h}) = (0, 2)$. Moreover, if $0 < b \leq 2$, then A is a saddle–node.

The phase portraits are shown in **(36)** (for $b<0$), **(37)** (for $0<b\le 2$) and **(38)** (for $b>2$). If $b<0$, then the set $V^{-1}(0)$ forms the separatrices of the saddle at the origin, separating the four canonical regions, two of them elliptic sectors. If $0<b\le 2$, then $V^{-1}(0)$ contains the separatrix $x=0$, $y\le 0$ of A. If $b>2$, then $V^{-1}(0)$ divides the open disc in two canonical regions.

(VIII.5) The origin is the only finite singular point, and it is a saddle. There are two singular points at infinity: A, which is non–elementary with $i(A)=2$, $(\mathbf{e},\mathbf{h})=(2,0)$; and a saddle–node p_1 in U_1.

The phase portraits are shown in **(59)** if $\delta=0$ and in **(58)** if $\delta=1$. If $\delta=0$, then $V^{-1}(0)$ contains the separatrices of the origin, defining two elliptic sectors in $x>0$. If $\delta=1$, then the set $V^{-1}(0)$ contains the separatrices of the origin.

(VIII.6) The origin is the only finite singular point. It is a saddle if $b<0$ and an unstable node if $b>0$. There are two singular points at infinity: A, which is non–elementary; and a saddle–node p_1 in U_1. The point A has index $i(A)=2$ for $b<0$, with $(\mathbf{e},\mathbf{h})=(2,0)$, and the phase portrait of the system is equivalent to the phase portrait **(59)** of (VIII.5) for $\delta=0$. If $b>0$, then $i(A)=0$, with $(\mathbf{e},\mathbf{h})=(0,2)$. Its phase portrait is shown in **(60)**. The set $V^{-1}(0)$ contains the separatrix $y=0$, $x\ge 0$ of the saddle–node and the straight line $x=0$ which is the separatrix of A, so it divides the phase space into three canonical regions.

(VIII.7) We have two finite singular points, a saddle and an unstable node. The number of infinite singular points depends on δ. In addition to A, which is a node, we have a non–elementary singular point p_1 on U_1 if $\delta=0$. It has index $i(p_1)=0$ (with $(\mathbf{e},\mathbf{h})=(0,2)$). If $\delta=-1$, the point p_1 splits into two saddle–nodes on U_1. No infinite singular points except A appear if $\delta=1$.

The phase portraits are shown in **(101)**, **(71)** and **(20)** if $\delta=-1,0,1$, respectively. The set $V^{-1}(0)$ contains the separatrices of the saddle–nodes for $\delta=-1$ and the straight line $y=1/2$ (which is the separatrix of p_1) for $\delta=0$. The separatrices of the finite saddle are not contained in $V^{-1}(0)$.

(VIII.8) The number of finite singular points depends on the value of b_{00}. There are no finite singular points if $b_{00}>0$, and there is a saddle–node at the origin for $b_{00}=0$ which splits into a saddle and an unstable node for $b_{00}<0$.

At infinity, and in addition to A which is a node, we have a non–elementary singular point p_1 on U_1. It has index $i(p_1)=0$, with $(\mathbf{e},\mathbf{h})=(0,2)$.

The phase portraits are shown in **(71)** if $b_{00}<0$, in **(56)** if $b_{00}=0$ and in **(44)** if $b_{00}>0$. If $b_{00}=0$, then $V^{-1}(0)$ contains the separatrices of the saddle–node and of p_1, so it divides the phase space into three canonical regions. If $b_{00}<0$, then $V^{-1}(0)$ divides the phase space into six canonical regions, and it contains the separatrices of the saddle and of p_1.

3.2.7 Systems (IV)

These systems were studied in Subsection 2.3.3. We use the notation given in that subsection.

(IV.1) There are no finite singular points if $b_{00} < 0$. If $b_{00} = 0$, then a non–elementary singular point of index 0 appears, splitting into a saddle and a center for $b_{00} > 0$. At infinity, A is a non–elementary singular point (a triple solution of equation (3.9)). The phase portraits correspond to **(12)** if $b_{00} < 0$, to **(13)** if $b_{00} = 0$ and to **(16)** if $b_{00} > 0$.

(IV.2) There is only one finite singular point, which is a saddle if $\delta = 1$ and a center if $\delta = -1$. At infinity, A is non–elementary with $i(A) = 1$ and there is a semi–hyperbolic singular point p_1 on U_1, which is a node if $\delta = 1$ and a saddle if $\delta = -1$.

The phase portraits are shown in **(62)** and **(52)**, for the respective values $\delta = -1, 1$. The set $V^{-1}(0)$ is, for $\delta = -1$, the straight line $y = 1$ separating the two canonical regions.

(IV.3a) There are no finite singular points and there is an infinite singular point p_1 on chart U_1 which is non–elementary of index 0. If $\delta = -1$, then $V^{-1}(0)$ is formed by two horizontal straight lines which contain an elliptic sector. The phase portraits are shown in **(49)** and **(44)**, for $\delta = -1, 1$, respectively.

(IV.3b) There is only one finite singular point, which is a saddle if $\delta = 1$ and a center if $\delta = -1$. At infinity, A is a node and there is a non–elementary singular point p_1 on U_1, with $(\mathbf{e}, \mathbf{h}) = (1, 1)$ if $\delta = 1$, and $(\mathbf{e}, \mathbf{h}) = (0, 4)$ if $\delta = -1$.

The phase portraits are shown in **(63)** (for $\delta = -1$) and **(52)** (for $\delta = 1$). In the first case, the set $V^{-1}(0)$ is the parabola separating the two canonical regions.

(IV.3c) Six possible phase portrait appear in this case, three of them for $\delta = 1$ and other three for $\delta = -1$. For $\delta = 1$, the behavior of the system depends on the parameter D. It is the same as in case (IV.1).

If $\delta = -1$, there are no finite singular points if $D < 0$, and a non–elementary finite singular point appears when $D = 0$, splitting into a saddle and a center if $D > 0$. There are two infinite saddle–nodes p_1 and p_2 on U_1. The corresponding separatrices are contained in $V^{-1}(0)$ in all cases. A is always a node.

The phase portraits for $\delta = -1$ are shown in **(83)** if $D < 0$, in **(89)** if $D = 0$) and in **(104)** if $D > 0$. In this last case, $V^{-1}(0)$ is the curve separating the period annulus of the center from the other canonical regions. If $D \leq 0$, then if divides the phase space into three canonical regions.

(IV.3d) There are no finite singular points, and there is a non–elementary singular point p_1 on U_1. If $D > 0$, then the phase portrait is **(44)**, which has been studied above. If $D = 0$, then the point p_1 is a saddle–node, and $V^{-1}(0)$ is its separatrix, the straight line $y = -1/2$. This straight line splits into two straight lines for $D < 0$, and an elliptic sector appears between the two lines.

The phase portraits for $D \leq 0$ are shown in **(49)** if $D < 0$ and in **(46)** if $D = 0$.

(IV.3e) The bifurcation values of D are, in this case, $D = -1/16$ and $D = 0$. There are two finite singular points for all possible values of D. One of them is always a saddle. The behavior of the other one depends on D: if $D < -1/16$, it is a center; if $-1/16 < D \leq 0$, it is a stable node; and it is a stable focus if $D > 0$.

If $D < 0$, then there are two singular points on U_1, say p_1 and p_2. They collapse into a point p_3 for $D = 0$, and disappear for $D > 0$. All these points are saddle–nodes. The point A is a node in all cases.

The phase portraits are shown in **(100)** if $D < -1/16$, in **(101)** if $-1/16 < D < 0$, in **(71)** if $D = 0$ and in **(19)** if $D > 0$. In the first case, $V^{-1}(0)$ is formed by the separatrices of the saddle (two of them are also separatrices of the saddle–nodes at infinity), and it divides the phase space in four canonical regions. In the second and third cases, $V^{-1}(0)$ contains the separatrices of the infinite saddle–nodes. If $D > 0$, then $V^{-1}(0)$ is the focus.

3.2.8 Systems (III)

These systems were studied in Subsection 2.3.4. We use the notation given in that subsection.

(III.1) We start studying the infinity. The point A is a non–elementary singular point, with $(\mathbf{e}, \mathbf{h}) = (1, 1)$. Moreover, there is a node p_1 on U_1.

In order to study the finite region, we must consider the sign of $\Delta = -2(2b_{10}^3 + 27b_{00}^2)$, which is the discriminant (3.7) up to a positive constant. If $\Delta < 0$, then the system has one finite singular point, a saddle. If $\Delta = 0$, then in addition to the saddle a non–elementary singular point appears in the finite region. We must also study the particular case $b_{00} = b_{10} = 0$, for which there is only a finite singular point, which is non–elementary and defines four hyperbolic sectors in a neighborhood. Finally, if $\Delta > 0$ then the non–elementary singular point of the case $\Delta = 0$ splits into a saddle and a center, and the saddle persists.

The phase portraits are shown in **(52)** if $\Delta < 0$, in **(52)** if $b_{00} = b_{10} = 0$, in **(69)** if $\Delta = 0$ and in **(80)** if $\Delta > 0$.

(III.2a) The origin is a saddle if $\delta = 1$ and a center if $\delta = -1$. In this second case, the set $V^{-1}(0)$ separates the period annulus of the center from the rest of the phase space.

The point A is in all cases a non–elementary infinite singular point, with $(\mathbf{e}, \mathbf{h}) = (1,1)$ if $\delta = 1$ and $(\mathbf{e}, \mathbf{h}) = (0,4)$ if $\delta = -1$. Moreover, there is a node on U_1.

We note that the behavior of the system in this case is the same as in (IV.3b), so the phase portraits are the same.

(III.2b) In this case, the infinity is degenerated. At the finite region, the number of singular points is related to the sign of the discriminant $\Delta = 4b_{10}^3 - 27b_{00}^2$. If $\Delta < 0$, then we have an unstable focus. If $\Delta = 0$, then we get a saddle–node and an unstable node. In the particular case $b_{00} = b_{10} = 0$, we have a non–elementary singular point, with $(\mathbf{e}, \mathbf{h}) = (1,1)$. Finally, if $\Delta > 0$ then we have a saddle, a stable node and an unstable node.

The phase portraits are shown in **(3)** if $\Delta < 0$, in **(4)** if $b_{00} = b_{10} = 0$, in **(10)** if $\Delta = 0$ and in **(11)** if $\Delta > 0$. If $\Delta < 0$, then $V^{-1}(0)$ contains the focus. In the case $b_{00} = b_{10} = 0$, the set $V^{-1}(0)$ is the straight line $y = 0$, which is a separatrix of the singular point. There are two more separatrices, which are included in $H^{-1}(0)$ and define the elliptic sector. If $\Delta = 0$, then $V^{-1}(0)$ contains the separatrices of the saddle–node. Finally, if $\Delta > 0$, then $V^{-1}(0)$ contains the separatrices of the saddle.

(III.2c) In the finite region, the number of singular points is related to the sign of b_{10}. If $b_{10} < 0$, then we have a center. If $b_{10} = 0$, then we have a non–elementary singular point, with $(\mathbf{e}, \mathbf{h}) = (1,1)$. Finally, if $b_{10} > 0$ then we have a saddle, a stable node and an unstable node.

At infinity, A is a non–elementary singular point with $(\mathbf{e}, \mathbf{h}) = (1,1)$. There is also a saddle on U_1.

The phase portraits are shown in **(62)** if $b_{10} < 0$, in **(53)** if $b_{10} = 0$ and in **(81)** if $b_{10} > 0$. In all cases, $V^{-1}(0)$ contains the separatrices of the infinite saddle.

(III.2d) In the finite region, the number of singular points is related to the sign of $b_{10}m$. At infinity, in addition to A, there is a singular point p_1 on the chart U_1. A is non–elementary, and the behavior of p_1 depends on the sign of $m - 1$.

We first study the finite points. Assume that $b_{10} < 0$. If $m < 0$, then there are three finite singular points: two saddles and one center. If $m > 0$, then there is one center.

Assume that $b_{10} = 0$. The origin is, in this case, a non–elementary singular point. It has four hyperbolic sectors if $m < 1$ and a hyperbolic and an elliptic sector if $m > 1$.

If $b_{10} > 0$, then we have a saddle if $m < 0$ and there are a saddle, a stable node and an unstable node if $m > 0$.

Next we study the infinite singular points. With respect to A, if $0 < m < 1$ and $b_{10} \neq 0$, then $(\mathbf{e}, \mathbf{h}) = (0, 4)$. Otherwise, $(\mathbf{e}, \mathbf{h}) = (1, 1)$. The point p_1 is a node if $m < 1$ and a saddle if $m > 1$.

We finally give the relation of phase portraits. If $b_{10}, m < 0$, then we have **(77)**. $V^{-1}(0)$ contains the separatrices of the saddles, dividing the phase space into five canonical regions. If $b_{10} < 0$ and $0 < m < 1$, then we have **(63)**. In this case, $V^{-1}(0)$ defines the period annulus of the center. If $b_{10} < 0$ and $m > 1$, then we have **(62)**. The set $V^{-1}(0)$ contains the separatrix of the infinite saddle, defining again the period annulus of the center.

If $b_{10} = 0$ and $m < 1$, then we have **(52)**. If $b_{10} = 0$ and $m > 1$, then we have **(54)**. In both cases, the set $V^{-1}(0)$ contains all the separatrices of the finite singular point.

If $b_{10} > 0$ and $m < 0$, then we have **(52)**. If $b_{10} > 0$ and $0 < m < 1$, then we have **(76)**. In this case, $V^{-1}(0)$ separates two hyperbolic sectors of A. Finally, if $b_{10} > 0$ and $m > 1$, then we have **(81)**. The set $V^{-1}(0)$ contains the separatrices of the infinite saddle. In these three cases, the separatrices of the finite saddle are contained in a level set of H.

(III.3) At the finite region, we have an unstable node. At infinity, A is non–elementary with four hyperbolic sectors, and there is a node on the chart U_1. The phase portrait is shown in **(64)**. The separatrices of A are contained in the level set $H^{-1}(-1)$.

(III.4) In the finite region, the number of singular points is related to $b_{10} - 3$. If $b_{10} < 3$, then there is a focus. If $b_{10} = 3$, then we have a node and a saddle–node. If $b_{10} > 3$, then we have a saddle, a stable node and an unstable node.

At infinity, A is a non–elementary singular point with $(\mathbf{e}, \mathbf{h}) = (1, 1)$. There is also a saddle on U_1.

The phase portraits are shown in **(65)** ($b_{10} < 3$), **(70)** ($b_{10} = 3$) and **(81)** ($b_{10} > 3$). If $b_{10} \geq 3$, then $V^{-1}(0)$ contains the separatrices of the finite singular points. If $b_{10} < 3$ then $V^{-1}(0)$ contains the focus.

(III.5) At the finite region, we have an unstable node. At infinity, A is non–elementary with $(\mathbf{e}, \mathbf{h}) = (1, 1)$, and there is a saddle on the chart U_1. The phase portrait is shown in **(65)**. The separatrices of A are contained in the level set $H^{-1}(4)$.

(III.6) At the finite region, the number of singular points is related to the discriminant $16b_{10}^2(1 - 2b_{10})$. If $b_{10} < 0$, then there are two saddles, an unstable node and a stable node. If $0 < b_{10} < 1/2$, then there are no finite singular points. If $b_{10} = 1/2$, we have a non–elementary singular point, which splits in a saddle and a center for $b_{10} > 1/2$.

At infinity, A is a node and there are a saddle and a node on U_1. The set $V^{-1}(0)$ contains the separatrix of this infinite saddle. For the finite singular points, the separatrices are contained in level set $H^{-1}(0)$.

The phase portraits are shown in **(93)** if $b_{10} < 0$, in **(84)** if $0 < b_{10} < 1/2$, in **(87)** if $b_{10} = 1/2$ and in **(99)** if $b_{10} > 1/2$.

(III.7) There are a saddle and a non–elementary singular point at the finite region. At infinity, we have a saddle and two nodes. The phase portrait is shown in **(94)**. The set $V^{-1}(0)$ is the straight line $y = 0$.

3.2.9 Systems (VII)

These systems were studied in Subsection 2.3.5. We use the notation given in that subsection.

(VII.1) There are no finite singular points. At infinity, A is non–elementary, with two hyperbolic sectors and there is a node on U_1. The phase portrait is shown in **(45)**.

Remark 3.2.6. We note that (**45**) is an example of a quadratic polynomial foliation with three separatrices of hyperbolic sectors (see [29]). □

(VII.2) The behavior of this system is similar as in (VII.1), the only difference is the number of parabolic sectors in a neighborhood of A. The phase portrait is shown in (**46**). The separatrix is contained in $V^{-1}(0)$.

(VII.3) Once again, there are no finite singular points. At infinity, A is a non–elementary singular point. It has two hyperbolic sectors if $m < 1$ and it has two elliptic sectors if $m > 1$. There is also a singular point on U_1. It is a node if $m < 1$ and a saddle if $m > 1$. The set $V^{-1}(0)$ contains the finite separatrices of the infinite singular points.

The phase portraits are shown in **(45)** if $m < -1$, in **(46)** if $-1 < m < 1$ and in **(43)** if $m > 1$.

(VII.4) The study of the phase portraits is the same for both values of δ. Its behavior is the same as in (VII.3) for $-1 < m < 1$, so we get (**46**).

(VII.5) Once again, there are no finite singular points. The infinite singular point A, which is non–elementary, defines two elliptic sectors for $\delta = -1$. If $\delta = 0$, then the infinity is degenerated, and A is a non–elementary singular point. In this case, the ω–limit of the left half–plane is A, and the α–limit of the right half–plane is the infinite singular point A'. Both half planes are separated by $V^{-1}(0)$.

If $\delta = -1$, then we have **(35)**. If $\delta = 0$, then we have **(1)**. If $\delta = 1$, then we have **(12)**.

(VII.6) The origin is a saddle–node. The point A is non–elementary, and there is a singular point p_1 on U_1, which is a node if $m < 1$ and a saddle if $m > 1$. In the case $m = 1$, the infinity is degenerated.

The phase portraits are shown in **(56)** if $m < 1$, in **(5)** if $m = 1$ and in **(55)** if $m > 1$. All the separatrices are contained in $V^{-1}(0)$.

(VII.7) The origin is a non–elementary singular point with two hyperbolic sectors, defined by $V^{-1}(0)$. The point A is also non–elementary, and it has two elliptic sectors. Moreover, there is a saddle on U_1. The phase portrait is shown in **(51)**.

(VII.8) The origin is a non–elementary singular point with two hyperbolic sectors, defined by $H^{-1}(0)$. The point A is a node. Moreover, there are a saddle and a node on U_1. The separatrices of this infinite saddle are contained in $V^{-1}(0)$. The phase portrait is shown in **(88)**.

(VII.9) The behavior of the system for $b_{20} > -1$ is the same as in (VII.8), but in this case all the separatrices are contained in $V^{-1}(0)$. If $b_{20} = -1$, then the origin is a non–elementary singular point and an infinite point p_1 on U_1 is a saddle–node. A is a node. Finally, if $b_{20} < -1$ then A is the only infinite singular point, and the origin is a non–elementary singular point. The separatrices of all the singular points are contained in $V^{-1}(0)$.

The phase portraits are shown in **(13)** if $b_{20} < -1$, in **(66)** if $b_{20} = -1$ and in **(88)** if $b_{20} > -1$.

(VII.10) There are no finite singular points if $b_{00} < 0$ and two finite saddle–nodes if $b_{00} > 0$. The separatrices are contained in $V^{-1}(0)$. At infinity, A is a node and there are two singular points on chart U_1. One of them is a node and the other one is a saddle. The separatrices of the saddle are contained in a level set of H.

The phase portraits are shown in **(85)** if $b_{00} < 0$ and in **(91)** if $b_{00} > 0$.

(VII.11) The behavior of the system is the same as in (VII.10) for $b_{00} > 0$, so its phase portrait is **(91)**.

(VII.12) The behavior of the system is the same as in (VII.10) for $b_{00} < 0$, so its phase portrait is **(85)**.

3.2.10 Systems (VI)

These systems were studied in Subsection 2.3.6. We use the notation given in that subsection.

As $\dot{x} = 1 + x^2 > 0$, there are no finite singular points in the following systems, so we just have to study the infinity.

(VI.1) A is a non–elementary singular point with two hyperbolic sectors. Moreover, there is a node on U_1. The phase portrait is shown in **(44)**.

(VI.2) The behavior of this system is the same as in (VI.1).

(VI.3) The behavior of this system is the same as in (VI.1).

(VI.4) The infinity is degenerated. After removing the line of infinite singular points, if $b \neq 0$ then A is a focus, and if $b = 0$ then it is a center. The phase portrait is shown in **(2)**.

Remark 3.2.7. A perturbation of system **(2)** could give a limit cycle from the graphic formed by the infinite line of singular points and the set $V^{-1}(0)$. □

(VI.5) A is non–elementary; it has two elliptic sectors if $m > 1$ and two hyperbolic sectors if $m < 1$. Moreover, there is an infinite singular point p_1 on U_1. It is a saddle if $m > 1$ and a node if $m < 1$. In the case $m > 1$, the straight line $y = 0$ is contained in $V^{-1}(0)$, separates the plane into two canonical regions, and the phase portrait **(42)** is obtained. In the second case, we have a phase portrait as in (VI.1).

(VI.6) The behavior of this system is the same as in (VI.1).

(VI.7) The point A is non–elementary, and it has two elliptic sectors. Moreover, there is a saddle on chart U_1. The phase portrait is shown in **(43)**. The set $V^{-1}(0)$ is not relevant for this system.

(VI.8) The point A is a node. Moreover, there are a saddle and a node on U_1. The separatrices of this infinite saddle are formed by $V^{-1}(0)$. The phase portrait is shown in **(84)**.

(VI.9) For all possible values of b_{00}, the behavior of the system is the same as in (VII.12), so we have the phase portrait shown in **(85)**.

(VI.10) The behavior of this system is the same as in (VI.9).

3.2.11 Systems (V)

These systems were studied in Subsection 2.3.7. We use the notation given in that subsection.

(V.1) There are two finite saddles. At infinity, A is a non–elementary singular point with two elliptic sectors. Moreover, there is a node on U_1. The phase portraits are shown in **(73)** if $\delta = 0$ and in **(72)** if $\delta = 1$. The difference between the two phase portraits is that in the case $\delta = 0$ the saddles are connected.

(V.2) There are no finite singular points. At infinity, A is a non–elementary singular point with two hyperbolic sectors, and there is a node on U_1. The phase portrait is the same for the two possible systems, and it is shown in **(47)**. $V^{-1}(0)$ contains the two straight lines which separate the phase space in three canonical regions.

(V.3) At the finite region, there are a saddle and an unstable node. At infinity, A is non–elementary, with two hyperbolic sectors. Moreover, there is a node on U_1. The set $V^{-1}(0)$ contains the separatrices of the saddle and the straight line $x = 1$ which defines the hyperbolic sectors of A. The phase portrait is shown in **(71)**.

(V.4) The behavior of the system is the same as in (V.3). In this case, the separatrices of the saddle are contained in a level set of H, and $V^{-1}(0)$ is the straight line $x = 1$.

(V.5) The infinity is degenerated. Removing the line of singularities, A is a node if $|b| > 1$ and a saddle if $|b| < 1$. At the finite region, there are a saddle and an unstable node if $|b| > 1$ and two nodes (of different stability) if $|b| < 1$. If $|b| > 1$, then the separatrices of the saddle are contained in $V^{-1}(0)$. For $|b| < 1$, $V^{-1}(0)$ defines two hyperbolic sectors. The phase portraits are shown in **(9)** if $|b| > 1$, and in **(8)** if $|b| < 1$.

(V.6) If $\delta = 0$, then the phase portrait is the same as in (V.5) for $|b| > 1$. In this case, the separatrices of the saddle are not contained in $V^{-1}(0)$, but we have another inverse integrating factor, which is $W(x, y) = (-1 + x^2)y$, and the separatrices are contained in $W^{-1}(0)$. We have here an example of a system having two different polynomial inverse integrating factors, one of them containing separatrices and the other one not.

Assume $\delta = 1$. We have a saddle and an unstable node at the finite region, and A is the only infinite singular point. It is non–elementary, with $(\mathbf{e}, \mathbf{h}) = (1, 1)$. Its phase portrait is shown in **(18)**. The separatrices of the saddle are contained in a level set of H, and $V^{-1}(0)$ contains the separatrix which defines the hyperbolic sector of A'.

(V.7) An unstable node is the only finite singular point. If $\delta = 0$, then $b_{00} \neq 0$, so by an easy change of variables we transform it into 1. In this case, the infinity is degenerated, and A' is a saddle–node. The set $V^{-1}(0)$ contains the finite separatrix which define it. The phase portrait is shown in **(6)**.

If $\delta = 1$, then A is the only infinite singular point, with $(\mathbf{e}, \mathbf{h}) = (1, 3)$ if $b_{00} < -1$ and $(\mathbf{e}, \mathbf{h}) = (0, 2)$ if $b_{00} > -1$. The set $V^{-1}(0)$ contains the separatrices which define all the sectors. The phase portraits are shown in **(40)** if $b_{00} < -1$ and in **(39)** if $b_{00} > -1$.

(V.8) There are two finite singular points. If $m < -|b|$, then we have two saddles. If either $-|b| < m < |b|$ and $m < 1$, or $1 < m < |b|$, then we have a saddle and an unstable node. If either $m > |b|$ and $m > 1$, or $|b| < m < 1$, then we have two nodes of different stability.

At infinity, the point A is non–elementary, and there is a singular point on U_1, which is a saddle or a node depending on the values of m and b.

In all cases, $V^{-1}(0)$ contains all the finite separatrices of the phase portrait. The phase portraits are **(73)** if $m < -|b|$, in **(71)** if $-|b| < m < |b|, 1$, in **(67)** if $1 < m < |b|$, in **(74)** if $m > |b|, 1$ and in **(75)** if $|b| < m < 1$.

(V.9) There are two finite singular points. If $m < -1$, then we have two saddles. The separatrices of the saddle laying on $x = 1$ are contained in $V^{-1}(0)$. If $m > -1$, then we have a saddle and a node. In this case, the separatrices are not contained in $V^{-1}(0)$.

At infinity, A is non–elementary. If $|m| > 1$, then it has two elliptic sectors. If $|m| < 1$, then it has two hyperbolic sectors. There is a singular point on U_1, which is a saddle if $m > 1$ and a node if $m < 1$. In the saddle case, its separatrices are contained in $V^{-1}(0)$.

The phase portraits are shown in **(72)** if $m < -1$, in **(71)** if $-1 < m < 1$ and in **(68)** if $m > 1$.

(V.10) The behavior of the system if $|b| < k-2$ is the one of **(74)**, explained in (V.8), but in this case $V^{-1}(0)$ contains the separatrices on the straight lines $x = \pm 1$. If $|b| > k-2$, the phase portrait is the one corresponding to **(68)**, explained in (V.9), but in this case the straight line $x = -1$, formed by separatrices, is contained in $V^{-1}(0)$.

(V.11) If $q - r - 1 > 0$, then the behavior of the system is the same as in (V.1) for $\delta = 1$, so its phase portrait is **(72)**. The straight lines $x = \pm 1$ are contained in $V^{-1}(0)$. If $q - r - 1 < 0$, then the phase portrait is equivalent to the one in (V.3), but in this case only the separatrices on $x = \pm 1$ are contained in $V^{-1}(0)$.

(V.12) If $r > 0$, then the behavior of the system is the same as in (V.11) for $q - r - 1 > 0$, so its phase portrait is **(72)**. If $r < 0$, then the phase portrait is equivalent to the one in (V.11) for $q - r - 1 < 0$.

(V.13) The behavior of the system is the same as in (V.11) for $q - r - 1 > 0$, so its phase portrait is **(72)**.

(V.14) There are four finite singular points, two saddles and two nodes of different stability. The separatrices of the saddles are not contained in $V^{-1}(0)$.

At infinity, A is a node. There are two singular points on U_1: a node and a saddle. The separatrices of the saddle are contained in $V^{-1}(0)$.

The phase portrait is shown in **(93)**.

(V.15) There are two finite singular points if $b_{20} < -1$ (a node and a saddle), three if $b_{20} = -1$ (a node, a saddle and a saddle–node) and four if $b_{20} > -1$ (two saddles and two nodes of different stability).

At infinity, A is a node. If $b_{20} = -1$, then there is a non–elementary singular point p_1 on U_1, which splits into a saddle and a node if $b_{20} > -1$.

If $b_{20} < -1$, then $V^{-1}(0)$ contains the node, and the phase portrait is **(20)**. If $b_{20} = -1$, then $V^{-1}(0)$ contains the separatrices of the saddle–node and p_1, and the phase portrait is **(78)**. If $b_{20} > -1$, the phase portrait is as in (V.14), but in this case $V^{-1}(0)$ contains the separatrices of the infinite saddle and the separatrices of the finite saddle with $x = -1$.

(V.16) If $b_{00} > 0$, then the phase portrait is the one appearing in (V.14), but now $V^{-1}(0)$ contains only the separatrices of the finite saddles.

If $b_{00} = 0$, then we have two finite saddle–nodes. Their separatrices are contained in $V^{-1}(0)$.

At infinity, A is a node, and there are a saddle and a node on U_1. The separatrices of this saddle are not contained in $V^{-1}(0)$. The phase portrait is **(91)**.

If $b_{00} < 0$, the behavior of the system is the same as in (VI.10).

(V.17) The phase portrait is equivalent to the one of (V.14). The set $V^{-1}(0)$ contains the separatrices of the finite saddles which are on the straight lines $x = \pm 1$.

3.2.12 Systems (II)

These systems were studied in Subsection 2.3.8. We use the notation given in that subsection. In the study of the phase portraits of systems (II) we must take into account, for $n \neq 0, 1$ and $l \neq 0$, three discriminants. The first one is $m^2 - 4l(n-1)$, which corresponds to the infinite singular points. The second one belongs to $\dot{y} = 0$ assuming $x = 0$, and it is $b^2 - 4dn$. And the third one belongs to $\dot{y} = 0$ assuming $y = 0$, and it is $a^2 - 4dl$. If $n \in \{0, 1\}$ and/or $l = 0$, it is not necessary to compute some of these discriminants, so the study is *a priori* easier.

We summarize the study of these systems in tables, following this notation: F.S.P.: finite singular points. I.S.P.: infinite singular points. P.P.: phase portrait. $\emptyset$: no singular points. S: saddle. C: center. N: node. F: focus. SN: saddle–node. $n \in \mathbb{Z}$: non–elementary singular point of index n. D.I.: degenerate infinity.

(II.1) We first consider the systems (II.1a). As $n \neq 0, 1$ and $l \neq 0$, we must study all the discriminants. The first one is a constant, and the others are, up to a non–zero constant, b_{00} and $1 - 4b_{00}\delta$. So, depending on b_{00} and $\delta = \pm 1$, we may have ten different phase portraits.

For the systems (II.1b) we must consider the six cases which appear from the combinations of the values of σ and δ. So six phase portraits appear.

For the systems (II.1c) we must consider the three cases corresponding to the different values of σ.

For the systems (II.1d) two different phase portraits may appear.

In Table 3.1 we show the different systems which arise from system (II.1). We give the corresponding number of their phase portrait figure.

(II.2) In the following systems, the infinite singular point A is semi–hyperbolic.

We first consider the system (II.2a). The discriminants to study are, up to a non–zero constant, $b_{20}, (1 - 4b_{20}\delta)$. So, depending on b_{20} and δ, we may have ten different phase portraits.

For the systems (II.2b) we consider the six systems which appear from the combinations of the values of σ and δ.

Subcase	Range of parameters	F.S.P.	I.S.P.	P.P.
(a)	$\delta = 1$, $b_{00} < 0$	S,S	N,N,N	95
(a)	$\delta = 1$, $b_{00} = 0$	S,-1	N,N,N	95
(a)	$\delta = 1$, $0 < b_{00} < 1/4$	S,S,S,C	N,N,N	115
(a)	$\delta = 1$, $b_{00} = 1/4$	S,S,0	N,N,N	116
(a)	$\delta = 1$, $b_{00} > 1/4$	S,S	N,N,N	105
(a)	$\delta = -1$, $b_{00} < -1/4$	$\emptyset$	N	12
(a)	$\delta = -1$, $b_{00} = -1/4$	0	N	13
(a)	$\delta = -1$, $-1/4 < b_{00} < 0$	S,C	N	16
(a)	$\delta = -1$, $b_{00} = 0$	C,-1	N	16
(a)	$\delta = -1$, $b_{00} > 0$	S,S,C,C	N	22
(b)	$\delta = -1$, $\sigma = -1$	$\emptyset$	N	12
(b)	$\delta = -1$, $\sigma = 0$	0	N	13
(b)	$\delta = -1$,$\sigma = 1$	S,S,C,C	N	22
(b)	$\delta = 1$, $\sigma = -1$	S,S	N,N,N	95
(b)	$\delta = 1$, $\sigma = 0$	-2	N,N,N	90
(b)	$\delta = 1$, $\sigma = 1$	S,S	N,N,N	105
(c)	$\sigma = -1$	S	N,1	52
(c)	$\sigma = 0$	-1	N,1	52
(c)	$\sigma = 1$	S,S,C	N,1	77
(d)	$\delta = -1$	$\emptyset$	N,0	44
(d)	$\delta = 1$	S,S	N,2	73

Table 3.1: Relations between the parameters of cases (II.1) and the phase portraits.

In Table 3.2 we show the different systems which arise from system (II.2). The set $V^{-1}(0)$ is the straight line $x = 0$. It plays an important role in the systems where $\delta = 1$, because it is the separatrix of the infinite saddle. If $\delta = -1$, then the set $V^{-1}(0)$ is not relevant.

(II.3) For systems (II.3a) the discriminants are, up to a non–zero constant, b_{20}, δ and $(1 - 4b_{20}\delta)$. So, depending on b_{20} and δ, we may have ten different phase portraits. The different systems are summarized in Table 3.3. For the rest of the systems, as the parameters take discrete values, we consider all these values. The set $V^{-1}(0)$ is the straight line $x = 0$. It plays an important role in all the systems, because it contains the separatrices of the infinite saddle.

(II.4) The number of finite singular points is $\delta + 1$. Moreover, the kind of singular points depend on the sign of b_{00}. The discriminants depend on $b_{00} \neq 0$.

When saddles or saddle–nodes (finite or infinite) appear, the set $V^{-1}(0)$ con-

Subcase	Range of parameters	F.S.P.	I.S.P.	P.P.
(a)	$\delta = 1,\ b_{20} < 0$	C,C	S	27
(a)	$\delta = 1,\ b_{20} = 0$	C	S,1	62
(a)	$\delta = 1,\ 0 < b_{20} < 1/4$	S,C	S,N,N	99
(a)	$\delta = 1,\ b_{20} = 1/4$	0	S,N,N	87
(a)	$\delta = 1,\ b_{20} > 1/4$	$\emptyset$	S,N,N	84
(a)	$\delta = -1,\ b_{20} < -1/4$	$\emptyset$	N	12
(a)	$\delta = -1,\ b_{20} = -1/4$	0	N	13
(a)	$\delta = -1,\ -1/4 < b_{20} < 0$	S,C	N	16
(a)	$\delta = -1,\ b_{20} = 0$	S	N,1	52
(a)	$\delta = -1,\ b_{20} > 0$	S,S	N,N,N	95
(b)	$\delta = -1,\ \sigma = -1$	$\emptyset$	N	12
(b)	$\delta = -1,\ \sigma = 0$	$\emptyset$	N,0	44
(b)	$\delta = -1, \sigma = 1$	S,S	N,N,N	95
(b)	$\delta = 1,\ \sigma = -1$	C,C	S	27
(b)	$\delta = 1,\ \sigma = 0$	$\emptyset$	S,2	42
(b)	$\delta = 1,\ \sigma = 1$	$\emptyset$	S,N,N	84

Table 3.2: Relations between the parameters of cases (II.2) and the phase portraits.

tains their separatrices, except for the finite saddle when $\delta = 1$ and $b_{00} < 0$. So, as we can see in the phase portraits, $V^{-1}(0)$ plays a very important role in the sense that it defines almost all the canonical regions of the system. The results are shown in Table 3.4.

(II.5) For systems (II.5a) the relevant discriminants are, up to a non–zero constant, $\delta, (1 - 4b_{20}\delta)$. Moreover, if $b_{20} = 0$, then the infinity is degenerated. So, depending on b_{20} and δ, we may have ten different phase portraits. When $b_{20} < 0$, the set $V^{-1}(0)$ contains the separatrices of the infinite saddle. Otherwise it is not relevant.

For systems (II.5c), if $\sigma = 0$ then the infinity is degenerated. After removing the line of singularities, if $\delta = 1$ then the point $p_1 = (0, 0)$ on U_1 is a center, and if $\delta = -1$ then it is a saddle. The set $V^{-1}(0)$ does not play an important role if $\sigma = 0$.

The different systems are summarized in Table 3.5.

Subcase	Range of parameters	F.S.P.	I.S.P.	P.P.
(a)	$\delta = 1,\ b_{20} < 0$	C,C	S	27
(a)	$\delta = 1,\ b_{20} = 0$	C	S,1	62
(a)	$\delta = 1,\ 0 < b_{20} < 1/4$	S,C	S,N,N	99
(a)	$\delta = 1,\ b_{20} = 1/4$	0	S,N,N	87
(a)	$\delta = 1,\ b_{20} > 1/4$	$\emptyset$	S,N,N	84
(a)	$\delta = -1,\ b_{20} < -1/4$	N,N	S	30
(a)	$\delta = -1,\ b_{20} = -1/4$	N,N,0	S	31
(a)	$\delta = -1,\ -1/4 < b_{20} < 0$	S,C,N,N	S	32
(a)	$\delta = -1,\ b_{20} = 0$	S,N,N	S,1	81
(a)	$\delta = -1,\ b_{20} > 0$	S,S,N,N	S,N,N	93
(b)	$\sigma = -1$	C,1	S	25
(b)	$\sigma = 0$	1	S,1	53
(b)	$\sigma = 1$	S,1	S,N,N	94
(c)	$\delta = -1,\ \sigma = -1$	N,N	S	30
(c)	$\delta = -1,\ \sigma = 0$	N,N	S,0	74
(c)	$\delta = -1, \sigma = 1$	S,S,N,N	S,N,N	93
(c)	$\delta = 1,\ \sigma = -1$	C,C	S	27
(c)	$\delta = 1,\ \sigma = 0$	$\emptyset$	S,2	42
(c)	$\delta = 1,\ \sigma = 1$	$\emptyset$	S,N,N	84
(d)	$\delta = -1$	2	S	24
(d)	$\delta = 1$	0	S,N,N	88

Table 3.3: Relations between the parameters of cases (II.3) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 0,\ b_{00} < 0$	N	SN,SN	60
$\delta = 0,\ b_{00} > 0$	S	SN,2	59
$\delta = 1,\ b_{00} < 0$	S,N	N,SN,SN	101
$\delta = 1,\ b_{00} > 0$	S,C	N,SN,SN	100

Table 3.4: Relations between the parameters of cases (II.4) and the phase portraits.

Subcase	Range of parameters	F.S.P.	I.S.P.	P.P.
(a)	$\delta = 1,\ b_{20} < 0$	C,C	S	27
(a)	$\delta = 1,\ b_{20} = 0$	C	D.I.	7
(a)	$\delta = 1,\ 0 < b_{20} < 1/4$	S,C	N	16
(a)	$\delta = 1,\ b_{20} = 1/4$	0	N	13
(a)	$\delta = 1,\ b_{20} > 1/4$	$\emptyset$	N	12
(a)	$\delta = -1,\ b_{20} < -1/4$	N,N	S	30
(a)	$\delta = -1,\ b_{20} = -1/4$	N,N,0	S	31
(a)	$\delta = -1,\ -1/4 < b_{20} < 0$	S,C,N,N	S	32
(a)	$\delta = -1,\ b_{20} = 0$	S,N,N	D.I.	11
(a)	$\delta = -1,\ b_{20} > 0$	S,S,N,N	N	21
(b)	$\sigma = -1$	C,1	S	25
(b)	$\sigma = 0$	1	D.I.	4
(b)	$\sigma = 1$	S,1	N	17
(c)	$\delta = -1,\ \sigma = -1$	N,N	S	30
(c)	$\delta = -1,\ \sigma = 0$	N,N	D.I.,S	8
(c)	$\delta = -1,\sigma = 1$	S,S,N,N	N	21
(c)	$\delta = 1,\ \sigma = -1$	C,C	S	27
(c)	$\delta = 1,\ \sigma = 0$	$\emptyset$	D.I.,C	2
(c)	$\delta = 1,\ \sigma = 1$	$\emptyset$	N	12
(d)	$\delta = -1$	2	S	24
(d)	$\delta = 1$	0	N	13

Table 3.5: Relations between the parameters of cases (II.5) and the phase portraits.

(II.6) The number of finite singular points is $2\delta + 1$. Moreover, the local behavior at the singular points depends on the sign of b_{20}. The discriminants are always constant, so they do not affect in the study of the systems. If $\delta = 1$ and $b_{20} < 0$, then the infinite saddle is semi–hyperbolic.

The set $V^{-1}(0)$ contains all the separatrices appearing in the phase portraits, except the separatrices of the saddle when $\delta = 1$ and $b_{20} > 0$. The results are shown in Table 3.6. For $\delta = 0$, the system is homogeneous of degree 2, so the Jacobian matrix at the origin is the zero matrix.

(II.7) The infinity is degenerated. At the finite region, there are a saddle–node and a stable node. The set $V^{-1}(0)$ contains the separatrices of the saddle–node. The phase portrait is shown in **(10)**.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 0,\, b_{20} < 0$	2	S,SN	50
$\delta = 0,\, b_{20} > 0$	0	N,SN	66
$\delta = 1,\, b_{20} < 0$	C,N,SN	S,SN	79
$\delta = 1,\, b_{20} > 0$	S,N,SN	N,SN	78

Table 3.6: Relations between the parameters of cases (II.6) and the phase portraits.

(II.8) The infinity is degenerated. After removing the line of singularities, and in the case $\delta = 0$, A is a saddle if $b_{00} < 0$, a node if $0 < b_{00} \le 1/4$ and a focus if $b_{00} > 1/4$. The set $V^{-1}(0)$ contains all the separatrices of all the phase portraits. The results are shown in Table 3.7.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 1,\, b_{00} < 0$	S,N,N	-	11
$\delta = 1,\, 0 < b_{00} < 1/4$	S,N,N	-	11
$\delta = 1,\, b_{00} = 1/4$	N,SN	-	10
$\delta = 1,\, b_{00} > 1/4$	F	-	3
$\delta = 0,\, b_{00} < 0$	N,N	S	8
$\delta = 0,\, 0 < b_{00} < 1/4$	S	N	9
$\delta = 0,\, b_{00} = 1/4$	SN	N	5
$\delta = 0,\, b_{00} > 1/4$	$\emptyset$	F	2

Table 3.7: Relations between the parameters of cases (II.8) and the phase portraits. The infinity is degenerated.

(II.9) We distinguish six cases, depending on the values of δ and σ. The first discriminant, affecting the infinite, is always $-4b_{20}(n-1)$. The second one is $-4\sigma n$. The third one is $\delta - 4\sigma b_{20}$. So depending on their values, we have different behaviors. The set $V^{-1}(0)$ contains the straight line. Moreover, there is a conic in $V^{-1}(0)$, which may contain separatrices of the saddles, when they exist. If such conic is an ellipse, then it is formed by more than one orbit.

For each one of these six systems which arise from the values of δ and σ we have done a table specifying the singular points, the conditions on the parameters and the number of the corresponding phase portrait.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < 0$, $n < 0$	S,S,C,C	N	22
$b_{20} = 0$, $n < 0$	S,S,C	N,1	77
$0 < b_{20} < 1/4$, $n < 0$	S,S,S,C	N,N,N	115
$b_{20} = 1/4$, $n < 0$	S,S,0	N,N,N	116
$b_{20} > 1/4$, $n < 0$	S,S	N,N,N	105
$b_{20} < 0$, $0 < n < 1$	C,C	S	27
$b_{20} = 0$, $0 < n < 1/2$	C	S,1	62
$b_{20} = 0$, $1/2 < n < 1$	C	S,1	82
$0 < b_{20} < 1/4$, $0 < n < 1$	S,C	S,N,N	99
$b_{20} = 1/4$, $0 < n < 1$	0	S,N,N	87
$b_{20} > 1/4$, $0 < n < 1$	$\emptyset$	S,N,N	84
$b_{20} < 0$, $n > 1$	C,C	S,S,N	106
$b_{20} = 0$, $n > 1$	C	N,-1	63
$0 < b_{20} < 1/4$, $n > 1$	S,C	N	16
$b_{20} = 1/4$, $n > 1$	0	N	13
$b_{20} > 1/4$, $n > 1$	$\emptyset$	N	12

Table 3.8: Relations between the parameters of cases (II.9) with $\delta = \sigma = 1$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < -1/4$, $n < 0$	$\emptyset$	N	12
$b_{20} = -1/4$, $n < 0$	0	N	13
$-1/4 < b_{20} < 0$, $n < 0$	S,C	N	16
$b_{20} = 0$, $n < 0$	S	N,1	52
$b_{20} > 0$, $n < 0$	S,S	N,N,N	95
$b_{20} < -1/4$, $0 < n < 1$	N,N	S	30
$b_{20} = -1/4$, $0 < n < 1$	N,N,0	S	31
$-1/4 < b_{20} < 0$, $0 < n < 1$	S,C,N,N	S	32
$b_{20} = 0$, $0 < n < 1$	S,N,N	S,1	81
$b_{20} > 0$, $0 < n < 1$	S,S,N,N	S,N,N	93
$b_{20} < -1/4$, $n > 1$	N,N	S,S,N	107
$b_{20} = -1/4$, $n > 1$	N,N,0	S,S,N	111
$-1/4 < b_{20} < 0$, $n > 1$	S,C,N,N	S,S,N	117
$b_{20} = 0$, $n > 1$	S,N,N	N,-1	76
$b_{20} > 0$, $n > 1$	S,S,N,N	N	21

Table 3.9: Relations between the parameters of cases (II.9) with $\delta = 1$, $\sigma = -1$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < 0,\ n < 0$	C,-1	N	16
$b_{20} = 0,\ n < 0$	-1	N,1	52
$b_{20} > 0,\ n < 0$	S,-1	N,N,N	95
$b_{20} < 0,\ 0 < n < 1$	C,1	S	26
$b_{20} = 0,\ 0 < n < 1$	1	S,1	54
$b_{20} > 0,\ 0 < n < 1$	S,1	S,N,N	96
$b_{20} < 0,\ n > 1$	C,1	S,S,N	97
$b_{20} = 0,\ n > 1$	1	N,-1	57
$b_{20} > 0,\ n > 1$	S,1	N	17

Table 3.10: Relations between the parameters of cases (II.9) with $\delta = 1$, $\sigma = 0$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < 0,\ n < 0$	S,S,C,C	N	22
$b_{20} = 0,\ n < 0$	S,S	N,2	73
$b_{20} > 0,\ n < 0$	S,S	N,N,N	105
$b_{20} < 0,\ 0 < n < 1$	C,C	S	27
$b_{20} = 0,\ 0 < n < 1$	$\emptyset$	S,2	42
$b_{20} > 0,\ 0 < n < 1$	$\emptyset$	S,N,N	84
$b_{20} < 0,\ n > 1$	C,C	S,S,N	106
$b_{20} = 0,\ n > 1$	$\emptyset$	N,0	44
$b_{20} > 0,\ n > 1$	$\emptyset$	N	12

Table 3.11: Relations between the parameters of cases (II.9) with $\delta = 0$, $\sigma = 1$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < 0,\ n < 0$	$\emptyset$	N	12
$b_{20} = 0,\ n < 0$	$\emptyset$	N,0	44
$b_{20} > 0,\ n < 0$	S,S	N,N,N	95
$b_{20} < 0,\ 0 < n < 1$	N,N	S	30
$b_{20} = 0,\ 0 < n < 1$	N,N	S,0	74
$b_{20} > 0,\ 0 < n < 1$	S,S,N,N	S,N,N	93
$b_{20} < 0,\ n > 1$	N,N	S,S,N	107
$b_{20} = 0,\ n > 1$	N,N	N,-2	75
$b_{20} > 0,\ n > 1$	S,S,N,N	N	21

Table 3.12: Relations between the parameters of cases (II.9) with $\delta = 0$, $\sigma = -1$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{20} < 0,\ n < 0$	0	N	13
$b_{20} < 0,\ 0 < n < 1$	2	S	24
$b_{20} < 0,\ n > 1$	2	S,S,N	86
$b_{20} > 0,\ n < 0$	-2	N,N,N	90
$b_{20} > 0,\ 0 < n < 1$	0	S,N,N	88
$b_{20} > 0,\ n > 1$	0	N	13

Table 3.13: Relations between the parameters of cases (II.9) with $\delta = \sigma = 0$ and the phase portraits.

(II.10) The number of finite singular points depends on the sign of $\Delta = b^2 - 4dn$. Moreover, the behavior of the singular points depends, when $\Delta > 0$ on the relation between b and $\sqrt{\Delta}$.

At infinity the behavior of A depends on the sign of n and $n-1$. There is another infinite singular point p_1 on U_1 which is non–elementary.

The set $V^{-1}(0)$ contains all the separatrices of all the phase portraits. The results are shown in Table 3.14.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\Delta < 0$, $n < 0$	$\emptyset$	N,0	44
$\Delta < 0$, $0 < n < 1$	$\emptyset$	S,2	42
$\Delta < 0$, $n > 1$	$\emptyset$	N,0	44
$\Delta = 0$, $n < 0$	SN	N,0	56
$\Delta = 0$, $0 < n < 1$	SN	S,2	55
$\Delta = 0$, $n > 1$	SN	N,0	56
$\Delta > 0$, $n < 0$, $b < -\sqrt{\Delta}$	S,N	N,0	71
$\Delta > 0$, $n < 0$, $\lvert b\rvert < \sqrt{\Delta}$	S,S	N,2	73
$\Delta > 0$, $n < 0$, $b > \sqrt{\Delta}$	S,N	N,0	71
$\Delta > 0$, $0 < n < 1$, $b < -\sqrt{\Delta}$	S,N	S,2	67
$\Delta > 0$, $0 < n < 1$, $\lvert b\rvert < \sqrt{\Delta}$	N,N	S,0	74
$\Delta > 0$, $0 < n < 1$, $b > \sqrt{\Delta}$	S,N	S,2	67
$\Delta > 0$, $n > 1$, $b < -\sqrt{\Delta}$	S,N	N,0	71
$\Delta > 0$, $n > 1$, $\lvert b\rvert < \sqrt{\Delta}$	N,N	N,-2	75
$\Delta > 0$, $n > 1$, $b > \sqrt{\Delta}$	S,N	N,0	71

Table 3.14: Relations between the parameters of cases (II.10) and the phase portraits.

(II.11) The number of singular points and their behavior depends on δ, n and $\Delta = 1 - 4b_{00}(n-1)$. The set $V^{-1}(0)$ contains all the separatrices of all the phase portraits with $\delta = 0$. When $\delta = 1$, $V^{-1}(0)$ contains all the separatrices if there is a center. If $\delta = 1$ and the system does not have a center, $V^{-1}(0)$ contains the separatrices of the infinite saddles, saddles or saddle–nodes which are on $x = 0$.

The results are shown in Tables 3.15 and 3.16.

(II.12) The origin is a non–elementary singular point. There is also an unstable node at the finite region. At infinity we have two saddles and one node. The set $V^{-1}(0)$ contains the separatrices of the origin. The phase portrait is shown in **(98)**.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\Delta \notin \mathbb{R}$, $n < 0$	0	N	13
$\Delta \notin \mathbb{R}$, $0 < n < 1$	2	S	24
$\Delta \notin \mathbb{R}$, $n > 1$	0	N	13
$\Delta = 0$, $n < 0$	0	N,SN	66
$\Delta = 0$, $0 < n < 1$	2	S,SN	50
$\Delta = 0$, $n > 1$	0	N,SN	66
$0 < \Delta < 1$, $n < 0$	0	S,N,N	87
$0 < \Delta < 1$, $0 < n < 1$	2	S,S,N	86
$0 < \Delta < 1$, $n > 1$	0	S,N,N	87
$\Delta > 1$, $n < 0$	-2	N,N,N	90
$\Delta > 1$, $0 < n < 1$	0	S,N,N	87
$\Delta > 1$, $n > 1$	2	S,S,N	86

Table 3.15: Relations between the parameters of cases (II.11) with $\delta = 0$ and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\Delta \notin \mathbb{R}$, $n < 0$	S,F	N	19
$\Delta \notin \mathbb{R}$, $0 < n < 1$	C,F	S	28
$\Delta \notin \mathbb{R}$, $n > 1$	S,F	N	19
$\Delta = 0$, $n < 0$	S,N,SN	N,SN	78
$\Delta = 0$, $0 < n < 1$	C,N,SN	S,SN	79
$\Delta = 0$, $n > 1$	S,N,SN	N,SN	78
$0 < \Delta < 1$, $n < 0$	S,S,N,N	S,N,N	93
$0 < \Delta < 1$, $0 < n < 1$	S,C,N,N	S,S,N	118
$0 < \Delta < 1$, $n > 1$	S,S,N,N	S,N,N	93
$\Delta > 1$, $n < 0$	S,S,S,C	N,N,N	119
$\Delta > 1$, $0 < n < 1$	S,S,N,N	S,N,N	93
$\Delta > 1$, $n > 1$	S,C,N,N	S,S,N	118

Table 3.16: Relations between the parameters of cases (II.11) with $\delta = 1$ and the phase portraits.

(II.13) There are four finite singular points: one saddle (whose separatrices are contained in $V^{-1}(0)$) and three nodes, two of them unstable. At infinity, we have two saddles and one node. The phase portrait is shown in **(120)**.

(II.14) If $\delta = 1$ then we have five phase portraits, which are discussed in Table 3.17. The separatrices of the infinite saddle are not contained in $V^{-1}(0)$, but this set contains the separatrices of finite saddles and saddle–nodes. The subcases depend on the value of b_{00}.

If $\delta = 0$ and $\sigma = -1$, then the phase portrait is the same as the one for $\delta = 1$ and $b_{00} < -1/4$. If $\delta = 0$ and $\sigma = 1$, then the phase portrait is the same as the one for $\delta = 1$ and $b_{00} > 0$.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 1$, $b_{00} < -1/4$	$\emptyset$	S,N,N	85
$\delta = 1$, $b_{00} = -1/4$	SN,SN	S,N,N	91
$\delta = 1$, $-1/4 < b_{00} < 0$	S,S,N,N	S,N,N	93
$\delta = 1$, $b_{00} = 0$	S,N,SN	S,N,N	112
$\delta = 1$, $b_{00} > 0$	S,S,N,N	S,N,N	121
$\delta = 0$, $\sigma = -1$	$\emptyset$	S,N,N	85
$\delta = 0$, $\sigma = 1$	S,S,N,N	S,N,N	121

Table 3.17: Relations between the parameters of cases (II.14) and the phase portraits.

(II.15) Following the values of the discriminants, the behavior of the singular points depends on the value of δ, b_{00} and $1 - 8b_{00}$.

We remark that one of the nodes in the case $\delta = -1$ and $b_{00} = 0$ is semi–hyperbolic. It is a node for $b_{00} < 0$ and it bifurcates into two nodes and one saddle for $0 < b_{00} < 1/8$. This saddle and the other node become a saddle–node for $b_{00} = 1/8$, disappearing for $b_{00} > 1/8$.

In the case $\delta = 1$, a similar behavior happens. The two saddles and one node in the case $b_{00} < 0$ become a semi–hyperbolic saddle when $b_{00} = 0$.

The set $V^{-1}(0)$ contains the separatrices of the finite saddles and saddle–nodes. The results are shown in Table 3.18.

We recall that we do not have the expression of $V(x, y)$ for the following four families.

(II.16) Following the values of the discriminants, the behavior of the singular points depends on the sign of δ, and $p - 1 \neq 0$. The results are shown in Table 3.19.

(II.17) For system (II.17a), the behavior of the singular points depends on the values of b_{00} and $4b_{00} + q - 2$. For system (II.17b) it depends on δ. The results are shown in Table 3.20.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = -1,\ b_{00} < 0$	N,N	S,S,N	107
$\delta = -1,\ b_{00} = 0$	N,N	S,N,N	107
$\delta = -1,\ 0 < b_{00} < 1/8$	S,N,N,N	S,S,N	120
$\delta = -1,\ b_{00} = 1/8$	N,N,SN	S,S,N	113
$\delta = -1,\ b_{00} > 1/8$	F,F	S,S,N	108
$\delta = 1,\ b_{00} < 0$	S,S,N,N	N	21
$\delta = 1,\ b_{00} = 0$	S,N	N	20
$\delta = 1,\ 0 < b_{00} < 1/8$	S,N	N	20
$\delta = 1,\ b_{00} = 1/8$	SN	N	14
$\delta = 1,\ b_{00} > 1/8$	$\emptyset$	N	12

Table 3.18: Relations between the parameters of cases (II.15) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = -1,\ p = 0$	$\emptyset$	S,N,N	85
$\delta = 1,\ p = 0$	S,S,N,N	S,N,N	121
$\delta = -1,\ p > 1$	S,S	N,N,N	95
$\delta = 1,\ p > 1$	S,S	N,N,N	105

Table 3.19: Relations between the parameters of cases (II.16) and the phase portraits.

Subcase	Range of parameters	F.S.P.	I.S.P.	P.P.
(a)	$b_{00} < 0$	S,S,N,N	S,N,N	121
(a)	$b_{00} = 0$	S,N,SN	S,N,N	112
(a)	$0 < b_{00} < (q-2)/4$	S,S,N,N	S,N,N	93
(a)	$b_{00} = (q-2)/4$	SN,SN	S,N,N	91
(a)	$b_{00} > (q-2)/4$	$\emptyset$	S,N,N	85
(b)	$\delta = -1$	$\emptyset$	S,N,N	85
(b)	$\delta = 1$	S,S,N,N	S,N,N	121

Table 3.20: Relations between the parameters of cases (II.17) and the phase portraits.

(II.18) At infinity, we have a saddle and two nodes. The behavior of the finite singular points depends on the sign of δ. If $\delta = -1$, then there are two finite saddles and two finite nodes. The corresponding phase portrait is **(93)**. If $\delta = 1$, then there are no finite singular points. We have phase portrait **(84)**.

(II.19) We have two not connected saddles and two nodes. At infinity, there are two nodes and one saddle. The corresponding phase portrait is **(93)**.

3.2.13 Systems (I)

These systems were studied in Subsection 2.3.8. We use the notation given in that subsection. In the study of the finite singular points of systems (I), we must take into account the discriminant of the polynomial equation of degree at most 4 $lx^4 + ax^3 + (d-m)x^2 - bx + n = 0$, which is shown in (3.6) for $l \neq 0$. The discriminant $m^2 - 4l(n-1)$ corresponds again to the infinite singular points. If $n = 1$ and/or $l = 0$, the study is *a priori* easier.

We summarize the study of the systems in tables, following the same legend as for systems (II).

(I.1) We first study (I.1a). In order to know the behavior of all the singular points, we must take into account the sign of the discriminants $\Delta_1 = 3a_{20}^2 + 4b_{20}$ and $\Delta_2 = 75a_{20}^6 + 220a_{20}^4 b_{20} - 116a_{20}^3 + 208a_{20}^2 b_{20}^2 - 144a_{20}b_{20} + 64b_{20}^3 + 54$. In Figure 3.1 we show the bifurcation diagram of the system. We denote by γ_1 the blue curve; by γ_2 the black one above γ_1; by γ_3 the green one; and by γ_4 the black and red below γ_1. The curve γ_1 corresponds to the equation $\Delta_1 = 0$; the others to the equation $\Delta_2 = 0$. Table 3.21 shows the different cases which arise from system (II.1a). In this table, R_1 is the region between γ_2 and γ_3. R_2 is the intersection of $\{\Delta_1 < 0\}$ and the region between γ_3 and γ_4. R_3 is the region under γ_4.

Next we consider the system (I.1b). The bifurcation values are $b_{00} = -1, -3/4$ for $\sigma = 1$, $b_{00} = -3/4$ for $\sigma = -1$ and $b_{00} = 0$ for $\sigma = 0$. Table 3.22 shows the different cases which arise from this system.

(I.2) First we study the infinity. If $b_{11} = 0$, then we have a non–elementary singular point $p_1 = (0, 0)$ on U_1. If $b_{11} \neq 0$, then there are two on U_1. One of them is p_1, which is semi–hyperbolic. The point A is always a node.

At the finite region the bifurcation values are $b_{11} = 0$ and, if $\delta = 1$, the value $b_{11} = 1/4$.

The set $V^{-1}(0)$ is the finite separatrix of p_1 for $b_{11} \geq 0$. Table 3.23 shows the different cases which arise from system (I.2).

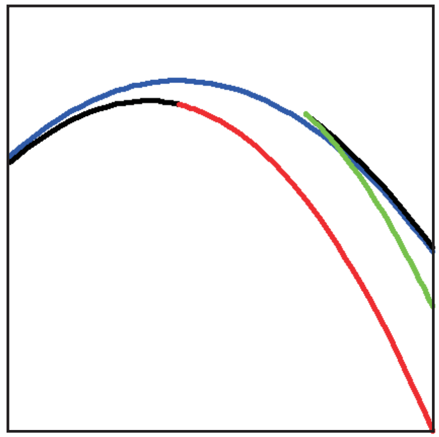


Figure 3.1: Bifurcation diagram of system (I.1a). We have a_{20} in the horizontal axis and b_{20} in the vertical one.

Region	F.S.P.	I.S.P.	P.P.
$\{\Delta_1 > 0\} \setminus R_1$	S,S	N,N,N	95
R_2	S,C	N	16
R_3	$\emptyset$	N	12
$R_1 \cap \{\Delta_1 < 0\}$	S,S,C,C	N	23
$R_1 \cap \{\Delta_1 > 0\}$	S,S,S,C	N,N,N	122
$\{\Delta_1 = 0\} \setminus \{R_1 \cap \gamma_3\}$	S	N,1	52
$\{\Delta_1 = 0\} \cap R_1$	S,S,C	N,1	80
$\gamma_1 \cap \gamma_3$	S,0	N,1	69
$\gamma_2 \cap \gamma_3$	S,-1	N,N,N	95
γ_4	0	N	13
$\gamma_3 \cap \{\Delta_1 > 0\}$	S,S,0	N,N,N	114
γ_2	S,S,0	N,N,N	114
$\gamma_3 \cap \{\Delta_1 < 0\}$	S,S,C,C	N	41

Table 3.21: Relations between the parameters of cases (I.1a) and the phase portraits.

(I.3) The bifurcation values are $b_{11} = -1/4, 0$. The set $V^{-1}(0)$ contains the finite separatrices of the infinite saddle–nodes. Table 3.24 shows the different phase portraits which arise from system (I.3).

Range of parameters	F.S.P.	I.S.P.	P.P.
$\sigma = -1,\, b_{00} < -3/4$	$\emptyset$	N	12
$\sigma = -1,\, b_{00} = -3/4$	$\emptyset$	N,0	44
$\sigma = -1,\, b_{00} > -3/4$	S,S	N,N,N	95
$\sigma = 0,\, b_{00} < 0$	$\emptyset$	N	12
$\sigma = 0,\, b_{00} = 0$	$\emptyset$	N,0	45
$\sigma = 0,\, b_{00} > 0$	S,S	N,N,N	95
$\sigma = 1,\, b_{00} < -1$	$\emptyset$	N	12
$\sigma = 1,\, b_{00} = -1$	0,0	N	15
$\sigma = 1,\, -1 < b_{00} < -3/4$	S,S,C,C	N	23
$\sigma = 1,\, b_{00} = -3/4$	S,S	N,2	72
$\sigma = 1,\, b_{00} > -3/4$	S,S	N,N,N	95

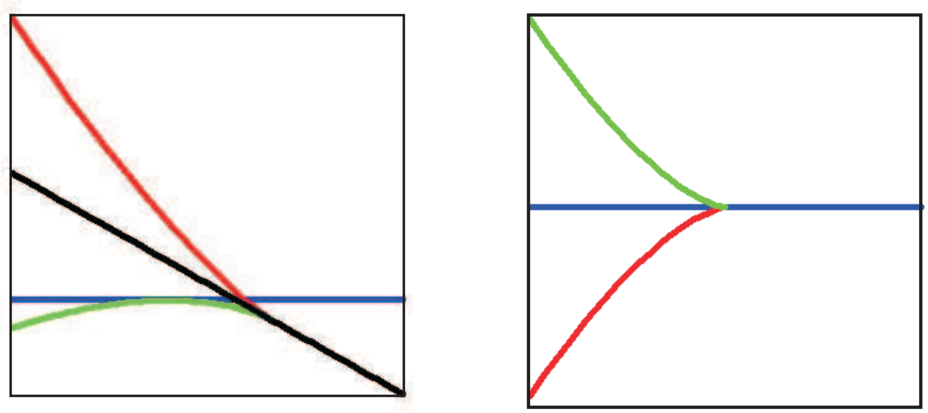
Table 3.22: Relations between the parameters of cases (I.1b) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 0,\, b_{11} < 0$	S,S	N,N,N	95
$\delta = 0,\, b_{11} = 0$	$\emptyset$	N,0	46
$\delta = 0,\, b_{11} > 0$	$\emptyset$	S,N,N	84
$\delta = 1,\, b_{11} < 0$	S,S	N,N,N	95
$\delta = 1,\, b_{11} = 0$	S	N,1	61
$\delta = 1,\, 0 < b_{11} < 1/4$	S,C	S,N,N	99
$\delta = 1,\, b_{11} = 1/4$	0	S,N,N	87
$\delta = 1,\, b_{11} > 1/4$	$\emptyset$	S,N,N	84

Table 3.23: Relations between the parameters of cases (I.2) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{11} < -1/4$	$\emptyset$	N,SN,SN	83
$b_{11} = -1/4$	0	N,SN,SN	89
$-1/4 < b_{11} < 0$	S,C	N,SN,SN	103
$b_{11} = 0$	S	SN,2	58
$b_{11} > 0$	S,C	N,SN,SN	104

Table 3.24: Relations between the parameters of cases (I.3) and the phase portraits.



$(\delta = 1)$ $(\delta = 0)$

Figure 3.2: Bifurcation diagram of system (I.4).

(I.4) The infinity is degenerated. At the finite region the discriminants that must be considered are $\Delta_1 = b_{10}$ and $\Delta_2 = 4b_{00}^3 + 27b_{10}^2 - (b_{00}^2 + 4b_{10} - 18b_{00}b_{10})\delta$. The bifurcation diagrams on the plane (b_{00}, b_{10}) are shown in Figure 3.2. The set $V^{-1}(0)$ contains all the separatrices appearing in the phase portraits.

The different cases are shown in Tables 3.25 and 3.26. In Table 3.25, we denote by γ_1 the curve $\Delta_1 = 0$ (in blue at the picture) and by γ_2 the curve $\Delta_2 = 0$. R_1 and R_2 are the regions outside and inside γ_2, respectively. $A_1 = (a_1, 0)$ is the intersection point of the green and the blue curves. $A_2 = (a_2, 0)$ is the intersection point of the green and the red curves. $A_3 = (a_3, 0)$ is the intersection point of the blue and the red curves. The curve γ_3 is the line $b_{10} = (2 - 9b_{00})/27$, $b_{00} > a_2 = 1/3$, in black in the figure. We denote by $L_1 = (-\infty, a_1) \times \{0\}$, $L_2 = (a_1, a_3) \times \{0\}$ and $L_3 = (a_3, \infty) \times \{0\}$ the three segments, from left to right, in which γ_1 is divided by γ_2.

In Table 3.26, we denote by γ_1 the curve $\Delta_1 = 0$ (in blue at the picture) and by γ_2 the curve $\Delta_2 = 0$. R_1 and R_2 are the regions outside and inside γ_2, respectively. We denote by L_1 and L_2 the two segments, from left to right, in which γ_1 is divided by γ_2.

We note that all from this system we obtain all the phase portrait with degenerated infinity of Section 3.4.

(I.5) If $\delta = 1$ then the bifurcation values of b_{11} are $b_{11} = 0$, $b_{11} = -1/(4(2n+1))$ and $b_{11} = (1-n^2)/(4n^2(2n+1))$. If $\delta = 0$ then we have the bifurcation value $b_{11} = 0$. Moreover in all cases we must take into account the values $n = -1, -1/2, 0, 1$. The bifurcation diagram in the case $\delta = 1$ is shown in Figure 3.3. In this figure, we

Region	F.S.P.	I.S.P.	P.P.
R_1	F	-	3
R_2	S,N,N	-	11
$\gamma_2 \setminus \gamma_1$	N,SN	-	10
γ_3	C	-	7
A_1	N	SN	6
A_2	1	-	4
L_1	N,N	S	8
L_2	$\emptyset$	F	2
L_3	S,N	N	9
A_3	SN	N	5

Table 3.25: Relations between the parameters of cases (I.4) and the phase portraits for $\delta = 1$.

Region	F.S.P.	I.S.P.	P.P.
R_1	F	-	3
R_2	S,N,N	-	11
L_1	N,N	S	8
L_2	$\emptyset$	C	2
$\gamma_1 \cap \gamma_2$	$\emptyset$	1	1
$\gamma_2 \setminus \gamma_1$	N,SN	-	10

Table 3.26: Relations between the parameters of cases (I.4) and the phase portraits for $\delta = 0$.

denote by γ_1 the red curve and by γ_2 the green one. They correspond, respectively, to the bifurcation curves $b_{11} = -1/(4(2n+1))$ and $b_{11} = (1-n^2)/(4n^2(2n+1))$. The blue one is $b_{11} = 0$. The vertical straight lines are, from left to right, $n = -1, -1/2, 0, 1$. For $n < -1$, up to down, the respective regions between the curves are R_1, R_2, R_3 and R_4. For $-1 < n < -1/2$, up to down, we have R_5 to R_8. For $-1/2 < n < 0$, R_9 to R_{12}. For $0 < n < 1$, R_{13} to R_{16}. And for $n > 1$, R_{17} to R_{20}.

The set $V^{-1}(0)$ contains all the separatrices appearing in the phase portraits. The different cases are summarized in Tables 3.27 and 3.28.

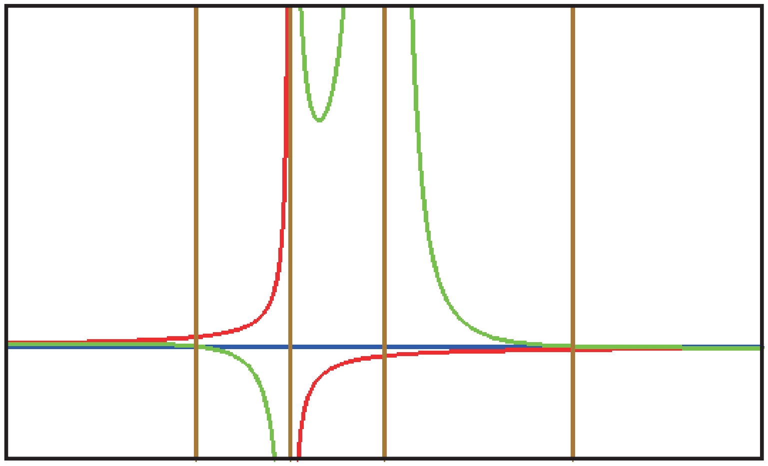


Figure 3.3: Bifurcation diagram of system (I.5) for $\delta = 1$.

Range of parameters	F.S.P.	I.S.P.	P.P.
$n < -1,\ b_{11} > 0$	$\emptyset$	S,N,N	84
$n < -1,\ b_{11} < 0$	S,S,N,N	S,N,N	93
$-1 < n < -1/2,\ b_{11} > 0$	S,S	N,N,N	105
$-1 < n < -1/2,\ b_{11} < 0$	S,S	N,N,N	95
$-1/2 < n < 0,\ b_{11} > 0$	S,S	N,N,N	95
$-1/2 < n < 0,\ b_{11} < 0$	S,S	N,N,N	105
$0 < n < 1,\ b_{11} > 0$	C,C	S,S,N	106
$0 < n < 1,\ b_{11} < 0$	N,N	S,S,N	107
$n > 1,\ b_{11} > 0$	S,S,N,N	S,N,N	93
$n > 1,\ b_{11} < 0$	$\emptyset$	S,N,N	84
$n < -1,\ b_{11} = 0$	$\emptyset$	N,0	46
$-1 < n < 0,\ b_{11} = 0$	$\emptyset$	N,0	45
$0 < n < 1,\ b_{11} = 0$	$\emptyset$	S,2	43
$n > 1,\ b_{11} = 0$	$\emptyset$	N,0	46

Table 3.27: Relations between the parameters of cases (I.5) with $\delta = 0$ and the phase portraits.

Region	F.S.P.	I.S.P.	P.P.
R_1	$\emptyset$	S,N,N	84
R_2	S,C	S,N,N	99
R_3	S,S,N,N	S,N,N	93
R_4	S,S,N,N	S,N,N	93
R_5	S,S	N,N,N	105
R_6	S,S,S,C	N,N,N	109
R_7	S,S,S,C	N,N,N	115
R_8	S,S	N,N,N	95
R_9	S,S	N,N,N	95
R_{10}	S,S,S,C	N,N,N	115
R_{11}	S,S,S,C	N,N,N	109
R_{12}	S,S	N,N,N	105
R_{13}	C,C	S,S,N	106
R_{14}	S,C,N,N	S,S,N	117
R_{15}	S,C,N,N	S,S,N	110
R_{16}	N,N	S,S,N	107
R_{17}	S,S,N,N	S,N,N	93
R_{18}	S,S,N,N	S,N,N	93
R_{19}	S,C	S,N,N	99
R_{20}	$\emptyset$	S,N,N	84
$\{n < -1\} \cap \gamma_1$	0	S,N,N	87
$\{n < -1\} \cap \gamma_2$	S,1	S,N,N	96
$\{-1 < n < -1/2\} \cap \gamma_1$	S,S,0	N,N,N	116
$\{-1 < n < -1/2\} \cap \gamma_2$	S,-1	N,N,N	95
$\{-1/2 < n < 0\} \cap \gamma_2$	S,-1	N,N,N	95
$\{-1/2 < n < 0\} \cap \gamma_1$	S,S,0	N,N,N	116
$\{0 < n < 1\} \cap \gamma_2$	C,1	S,S,N	97
$\{0 < n < 1\} \cap \gamma_1$	N,N,0	S,S,N	111
$\{n > 1\} \cap \gamma_2$	S,1	S,N,N	94
$\{n > 1\} \cap \gamma_1$	0	S,N,N	87
$\{n < -1\} \cap \{b_{11} = 0\}$	S,N	N,0	71
$\{-1 < n < 0\} \cap \{b_{11} = 0\}$	S,S	N,2	72
$\{0 < n < 1\} \cap \{b_{11} = 0\}$	S,N	S,2	68
$\{n > 1\} \cap \{b_{11} = 0\}$	S,N	N,0	71

Table 3.28: Relations between the parameters of cases (I.5) with $\delta = 1$ and the phase portraits. For more information about the definition of the regions, see the legend of Figure 3.3.

(I.6) The bifurcation values are $b_{00} = 0$ and $b_{00} = 1/2$. At infinity, A is always a saddle and there is a singular point p_1 on the U_1. If $b_{00} \leq 0$, then the set $V^{-1}(0)$ contains the finite separatrices of p_1. If $0 < b_{00} \leq 1/2$, the horizontal separatrices of the finite saddle are contained in $V^{-1}(0)$. The results are shown in Table 3.29.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{00} < 0$	N,N	S,0	74
$b_{00} = 0$	N	S,1	48
$0 < b_{00} < 1/2$	S,N	S,2	68
$b_{00} = 1/2$	SN	S,2	51
$b_{00} > 1/2$	$\emptyset$	S,2	43

Table 3.29: Relations between the parameters of cases (I.6) and the phase portraits.

(I.7) The bifurcation values are $b_{00} = 0$ for the infinite singular points and $b_{00} = -\sqrt{2}$ for the finite singular points. The set $V^{-1}(0)$ contains the separatrices of the finite singular points. The results are shown in Table 3.30.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = 1,\ b_{00} < -\sqrt{2}$	S,S,N,N	S,N,N	93
$\delta = 1,\ b_{00} = -\sqrt{2}$	SN,SN	S,N,N	91
$\delta = 1,\ b_{00} > -\sqrt{2}$	$\emptyset$	S,N,N	85
$\delta = -1$	F,F	S	29

Table 3.30: Relations between the parameters of cases (I.7) and the phase portraits.

(I.8) If $\delta = -1$ then there is a singular point p_1 on the U_1; it has two hyperbolic sectors, determined by the set $V^{-1}(0)$. The point A is a saddle, and there are two finite nodes. We are in the phase portrait **(74)**.

If $\delta = 1$ then p_1 has two elliptic sectors, determined by the finite separatrices of A, a saddle. There are no finite singular points. The phase portrait is **(43)**.

(I.9) The study of this system is exactly the same as in (I.6). The only difference is that all the separatrices of finite saddles or saddle-nodes are contained in $V^{-1}(0)$.

(I.10) The bifurcation values are $b_{00} = 0$ for the infinite singular points and $b_{00} = 5$ for the finite singular points. The set $V^{-1}(0)$ is relevant only for $b_{00} = 5$. In this case, it contains the two separatrices of the hyperbolic sector of the non–elementary finite singular point. The results are shown in Table 3.31.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{01} < 0$	S,N	N	20
$b_{00} = 0$	N	N,-1	64
$0 < b_{00} < 5$	C,N	S,S,N	102
$b_{00} = 5$	N,1	S,S,N	98
$b_{00} > 5$	S,N,N,N	S,S,N	120

Table 3.31: Relations between the parameters of cases (I.10) and the phase portraits.

(I.11) If $\delta = -1$ then there are two saddles on U_1. The point A is a node, and there are also two finite nodes. We have phase portrait **(107)**.

If $\delta = 1$ then the only infinite singular point is A, which is a node. There are no finite singular points. The corresponding phase portrait is **(12)**.

In both cases, the set $V^{-1}(0)$ is not relevant in the global phase portrait.

(I.12) The behavior of the system is, in all the cases, the same as in (I.11) with $\delta = -1$.

We recall that we do not have the expression of $V(x, y)$ for the following families, except for (I.17).

(I.13) The bifurcation values are $b_{00} = (Q-1)/(Q(2Q-1)), (Q+1)/Q$ if $\delta = 1$ and $b_{00} = (Q-1)/Q$ if $\delta = -1$. The results are shown in Table 3.32.

(I.14) The bifurcation values are $b_{11} = \delta_1$ and $b_{11} = \delta_2$, where

$$\delta_1 = -\frac{jq(q-2)(q-2-j)}{4(q-2-2j)^2} < 0, \quad \delta_2 = \frac{j^2q(q-2)(q-2-j)^2}{(q-2-2j)^4} > 0.$$

The results are shown in Table 3.33.

(I.15) The bifurcation values are $b_{00} = 1 - 2\delta/q$. The results are shown in Table 3.34.

(I.16) The study is the same as in (I.14).

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = -1,\ b_{00} < (Q-1)/Q$	S,S,N,N	S,N,N	93
$\delta = -1,\ b_{00} = (Q-1)/Q$	SN,SN	S,N,N	91
$\delta = -1,\ b_{00} > (Q-1)/Q$	$\emptyset$	S,N,N	85
$\delta = 1,\ b_{00} < (Q-1)/(Q(2Q-1))$	$\emptyset$	S,N,N	85
$\delta = 1,\ b_{00} = (Q-1)/(Q(2Q-1))$	$\emptyset$	S,N,N	84
$\delta = 1,\ (Q-1)/(Q(2Q-1)) < b_{00} < (Q+1)/Q$	$\emptyset$	S,N,N	85
$\delta = 1,\ b_{00} = (Q+1)/Q$	SN,SN	S,N,N	92
$\delta = 1,\ b_{00} > (Q+1)/Q$	S,S,N,N	S,N,N	93

Table 3.32: Relations between the parameters of cases (I.13) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{11} < \delta_1$	$\emptyset$	S,N,N	85
$b_{11} = \delta_1$	SN,SN	S,N,N	91
$\delta_1 < b_{11} \neq \delta_2$	S,S,N,N	S,N,N	93
$b_{11} = \delta_2$	S,N,SN	S,N,N	112

Table 3.33: Relations between the parameters of cases (I.14) and the phase portraits.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = -1,\ b_{00} < 1 + 2/q$	$\emptyset$	S,N,N	85
$\delta = -1,\ b_{00} = 1 + 2/q$	SN,SN	S,N,N	91
$\delta = -1,\ b_{00} > 1 + 2/q$	S,S,N,N	S,N,N	93
$\delta = 1,\ b_{00} < 1 - 2/q$	S,S,N,N	S,N,N	93
$\delta = 1,\ b_{00} = 1 - 2/q$	SN,SN	S,N,N	92
$\delta = 1,\ b_{00} > 1 - 2/q$	$\emptyset$	S,N,N	85

Table 3.34: Relations between the parameters of cases (I.15) and the phase portraits.

(I.17) The bifurcations values of b_{11} are $-5/12$ and 10. The set $V^{-1}(0)$ contains the separatrices of the connected saddles if $b_{11} < 10$, and the separatrices of the non–elementary singular point if $b_{11} = 10$. Otherwise $V^{-1}(0)$ is not relevant. The results are shown in Table 3.35.

(I.18) First we study the finite region. If $\delta = -1$, then there are no finite singular points. If $\delta = 1$, then we have two saddles and two nodes. At infinity, the point A is a saddle, and there are also two nodes on U_1. The corresponding phase portraits are **(85)** if $\delta = -1$ and **(121)** if $\delta = 1$.

Range of parameters	F.S.P.	I.S.P.	P.P.
$b_{11} < -5/12$	S,S	N,N,N	105
$b_{11} = -5/12$	S,S,0	N,N,N	116
$-5/12 < b_{11} < 10$	S,S,S,C	N,N,N	115
$b_{11} = 10$	S,-1	N,N,N	95
$b_{11} > 10$	S,S	N,N,N	95

Table 3.35: Relations between the parameters of cases (I.17) and the phase portraits.

(I.19) The behavior of the singular points depends on the values of δ, and p. The results are shown in Table 3.36.

Range of parameters	F.S.P.	I.S.P.	P.P.
$\delta = -1,\, p = 0$	S,S,N,N	S,N,N	121
$\delta = 1,\, p = 0$	$\emptyset$	S,N,N	85
$\delta = -1,\, p > 1$	S,S	N,N,N	105
$\delta = 1,\, p > 1$	S,S	N,N,N	95

Table 3.36: Relations between the parameters of cases (I.19) and the phase portraits.

(I.20) The behavior of this system is the same as (I.18), interchanging the values of δ.

3.3 Conclusions

From the study of the polynomial inverse integrating factors and the correspon-ding first integrals and phase portraits, we extract some conclusions.

3.3.1 On the polynomial inverse integrating factors

Phase portraits. From all the families of quadratic systems of the classification of Section 2.3 we have obtained 122 topologically different phase portraits. In our classification there are few quadratic systems having a polynomial first integral, which makes explicit that the inverse integrating factor has, in general, an easier expression (and a bigger domain of definition) than the first integral.

Invariant algebraic curves of arbitrary degree. The classification of the polynomial inverse integrating factors of Chapter 2 provides examples of invariant algebraic curves of arbitrarily high degree, see for example cases (VI.7), (VI.10) or (II.16). There are in the literature some examples of such curves of high degree, but as far as we know all of them have a Darboux first integral, as the ones that we have found.

Some other examples of invariant algebraic curves of arbitrary degree can be extracted from the expressions of the polynomial and rational first integrals. For this purpose, we take an example of (polynomial or rational) first integral H of arbitrary degree and a convenient $h \in \mathbb{R}$ such that the factorized expression of $H(x, y) - h$ contains a polynomial of arbitrary high degree. This can be done with 17 of the 18 polynomial first integrals appearing in Proposition 3.1.3 (the exception is (I.17)) and with 26 of the 49 rational first integrals appearing in Proposition 3.1.4.

The degree of V and the phase portraits. From the study of the phase portraits and the expressions of V, we can state the following result.

Theorem 3.3.1. *For each one of the phase portraits shown in Section 3.4 there is a quadratic system having a polynomial inverse integrating factor of degree at most six.*

We remark that **(92)** is the only phase portrait for which we have not given an explicit expression of a polynomial inverse integrating factor. This phase portrait corresponds to systems (I.13) and (I.15). As the phase portrait does not change if we vary the degree k of V, we have computed a polynomial inverse integrating factor of degree six for system (I.13). So Theorem 3.3.1 does not need to be restricted to $(\star)$ quadratic systems.

3.3.2 On the phase portraits

As we see in the pictures of Section 3.4, the behaviors of the quadratic systems having a polynomial inverse integrating factor can be very different, in the sense that many different topological phase portraits are found in these systems, and then the existence of a polynomial inverse integrating factor is not restricted to a certain kind of quadratic systems, except for the fact that all of them have a Darboux first integral.

We have obtained 122 topologically different phase portraits for the quadratic systems having a polynomial inverse integrating factor. In these phase portraits it has been clearly shown the importance of the inverse integrating factor in the behavior of the orbits of the system. In the most of the cases, the inverse integrating factor contains all or almost all the finite separatrices of the system and many singular points, so the knowledge of such function gives a lot of information about the phase portrait.

The phase portrait **(45)** corresponds to an example given in [29]. It is a foliation (a system without finite singular points) of degree two having three inseparable leaves. With this example, the claim that a quadratic system has at most two inseparable leaves given in [21] becomes false.

3.3.3 On the non–existence of algebraic limit cycles

As we explained in the introduction, one important property of an inverse integrating factor $V(x,y)$ is that all the limit cycles of the system which are in the domain of definition of $V(x,y)$ are contained in the set $V^{-1}(0)$. In our study, the domain of definition of $V(x,y)$ is the whole plane, so $V^{-1}(0)$ contains all the limit cycles of the system whenever they exist. Moreover, as our inverse integrating factors are polynomials, the limit cycles must be algebraic; that is, they are contained in an invariant algebraic curve $f=0$. It is proved that the known quadratic systems having an algebraic limit cycle mentioned in the introduction do not have a Darboux first integral, so they cannot have a polynomial inverse integrating factor.

From the classification of Chapter 2 we stated Theorem 2.4.2: *a $(\star)$ quadratic system having a polynomial inverse integrating factor has no algebraic limit cycles.*

We believe that this theorem is true for all the quadratic system having a polynomial inverse integrating factor.

3.3.4 On the critical remarkable values

In Proposition 3.1.4 of Section 3.1, we have listed the rational first integrals associated to the quadratic systems having a polynomial inverse integrating factor. For some of these first integrals, there exist one or two critical remarkable values. These values are always either $c=-c_2$ or $c=-c_2-c_1^{-1}$ (see Proposition 1.5.3), and then the corresponding critical remarkable invariant algebraic curves are factors of the numerator or the denominator of the first integral. The following theorem gives more information about these curves. Its proof is a consequence of Proposition 3.1.4.

Theorem 3.3.2. *Let $\dot{x}=P(x,y)$, $\dot{y}=Q(x,y)$ be a $(\star)$ quadratic system having a polynomial inverse integrating factor $V(x,y)$ and a rational first integral $H(x,y)$. Assume that $u_1,\ldots,u_r$ are critical remarkable invariant algebraic curves associated to H. Then, the curve $u_i=0$ is completely contained in the curve $V=0$, for all $i\in\{1,\ldots,r\}$.*

3.3.5 Homogeneous quadratic systems

If a quadratic polynomial system has a finite linearly zero singular point, then it is equivalent to the homogeneous quadratic system $\dot{x}=P_2(x,y), \dot{y}=Q_2(x,y)$, where P_2 and Q_2 are quadratic homogeneous polynomials, doing if necessary a translation of the linearly zero singular point to the origin. The global phase portraits of the quadratic homogeneous systems are well known, see [51].

As we know from Example 1.4.4, such homogeneous systems have a polynomial inverse integrating factor of degree 3, which is $yP_2 - xQ_2$. Then these systems appear in our classification. They correspond to the phase portraits **(13)**, **(24)**, **(66)**, **(50)**, **(90)**, **(86)** and **(88)**.

3.3.6 Hamiltonian quadratic systems

The topologically equivalent phase portraits of the Hamiltonian quadratic systems have been classified in [4]. There are 28 topologically equivalent phase portraits of such systems. In two of them the component of $\dot{x}$ is zero and in another one the system has a common factor. The other 25 systems are contained in our study. They correspond to the phase portraits **(16)**, **(22)**, **(23)**, **(41)**, **(77)**, **(80)**, **(115)**, **(109)**, **(119)**, **(122)**, **(12)**, **(44)**, **(45)**, **(13)**, **(36)**, **(52)**, **(72)**, **(73)**, **(69)**, **(90)**, **(116)**, **(95)**, **(105)**, **(114)** and **(15)**.

3.3.7 Quadratic systems having a center

The topologically equivalent phase portraits of the quadratic polynomial systems having a center have been classified in [48]. As we know from Example 1.4.4, such systems have a polynomial inverse integrating factor, so all these systems appear in our classification. There are 32 non–topologically equivalent phase portraits of quadratic systems having a center, but one of them is a linear system and another one has a finite line of singularities, so we do not consider them. The others correspond to the phase portraits **(16)**, **(22)**, **(23)**, **(77)**, **(80)**, **(41)**, **(115)**, **(109)**, **(119)**, **(122)**, **(62)**, **(63)**, **(82)**, **(7)**, **(104)**, **(100)**, **(103)**, **(99)**, **(27)**, **(106)**, **(26)**, **(25)**, **(97)**, **(32)**, **(117)**, **(118)**, **(110)**, **(28)**, **(79)** and **(102)**.

In Example 1.4.4 it is said that these systems have a polynomial inverse integrating factor of degree 3 or 5. With our classification we show that some of the phase portraits may correspond to quadratic systems having a polynomial inverse integrating factor of degree $k < 3$. The only one for which we have a polynomial inverse integrating factor of degree $k > 3$ is **(102)**. In this case, we have a polynomial inverse integrating factor of degree 5.

3.3.8 Quadratic systems having a polynomial first integral

The classification of the quadratic systems having a polynomial first integral is done in [9]. As we mentioned in Section 2.3, we have used some of the techniques of that work.

In Chapter 3 we show the first integrals appearing from our classification, distinguishing their type. That is, either polynomial, or rational, or Darboux but neither polynomial nor rational. In Proposition 3.1.3 we show the quadratic systems having a polynomial inverse integrating factor and a polynomial first integral, giving the expression of such functions. As the systems having a polynomial first integral have a polynomial inverse integrating factor (see [27]), the classification in [9] is strongly related with a part of our classification. Moreover, in [30] it is proved that the set of phase portraits of these systems is included in the set of Hamiltonian quadratic systems.

3.3.9 Quadratic systems having a rational first integral of degree 2

In [7] the global phase portraits of the quadratic systems having a rational first integral of degree 2 are classified. There are 25 non topologically equivalent phase portraits. Six of these phase portraits correspond to quadratic systems with a finite curve of singularities. The others are contained in our classification, and they correspond to the phase portraits **(1)**, **(2)**, **(4)**, **(7)**, **(8)**, **(9)**, **(11)**, **(12)**, **(24)**, **(27)**, **(30)**, **(37)**, **(42)**, **(46)**, **(71)**, **(74)**, **(84)**, **(88)** and **(93)**.

3.3.10 Quadratic systems with degenerated infinity

We know from Subsection 3.2.4 that a quadratic system having degenerated infinity can be written into the form

$$\dot{x} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy, \quad \dot{y} = d + ax + by + a_{20}xy + a_{11}y^2, \tag{3.10}$$

where all the parameters are real. The following result characterizes this family of systems.

Proposition 3.3.3. *Any quadratic system having degenerated infinity has a polynomial inverse integrating factor of degree* 3*.*

Proof: By using Proposition 2.1.2 and after an affine change of variables and a rescaling of the time, we can transform system (3.10) into a family or a subfamily of systems (III.2b), (VII.5), (VII.6), (VI.4), (V.5), (V.6), (V.7), (II.5a), (II.5b), (II.5c), (II.7), (II.8) and (I.4), and as we proved in Chapter 2 we have a polynomial inverse integrating factor of degree 3 for each one of them. ■

The phase portraits obtained from system (3.10) are the phase portraits **(1)**–**(11)** given in Section 3.4. They are also obtained only from system (I.4),

$$\dot{x} = 1 + xy, \quad \dot{y} = b_{00} + b_{10}x + \delta y + y^2,$$

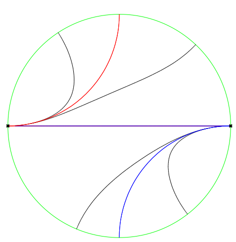
where $\delta \in \{0, 1\}$ and $b_{00}, b_{10} \in \mathbb{R}$.

3.4 List of phase portraits

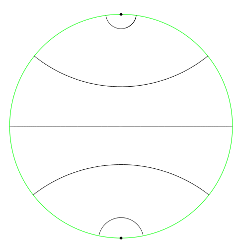
We finally show the phase portraits of the quadratic systems having a polynomial inverse integrating factor. We have used the program P4 for drawing them. In the pictures, a blue (red) curve means stable (unstable) separatrix, green means curve of singularities, black means regular orbit. For the singular points, see Table 3.37.

Symbol	Behavior of the singular point
Green square (rhombus)	Saddle (center)
Green triangle	Semi–hyperbolic saddle
Violet triangle	Saddle–node
Blue (red) square	Stable (unstable) node
Blue (red) triangle	Stable (unstable) semi–hyperbolic node
Blue (red) rhombus	Stable (unstable) strong focus
White cross	Non–elementary singular point

Table 3.37: Legend of the singular points of the phase portraits.



(1)



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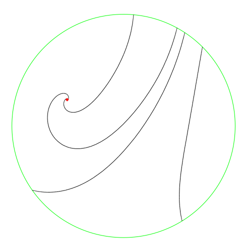
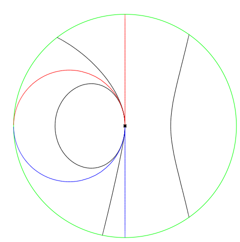
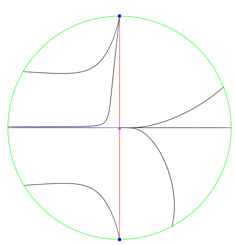


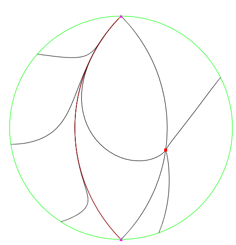
(3)



(4)



(5)



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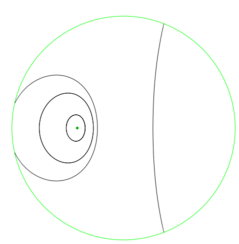
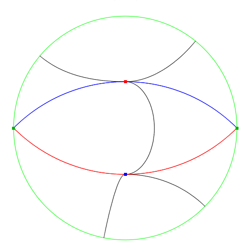
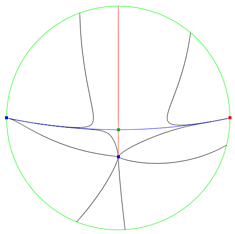


(7)



(8)



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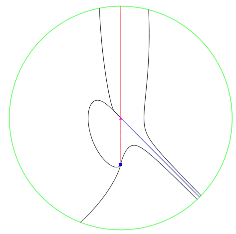

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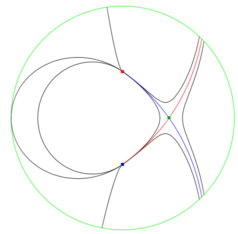

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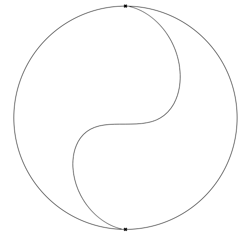

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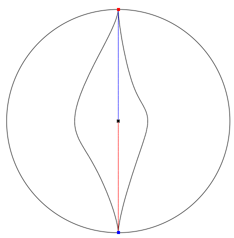

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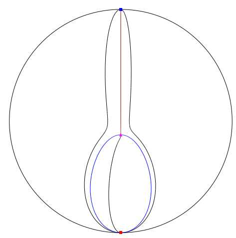

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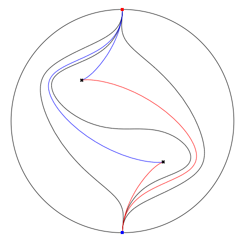

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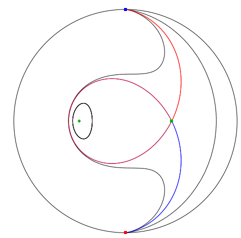

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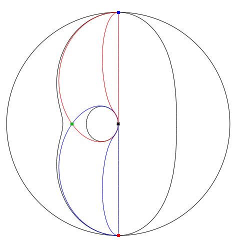

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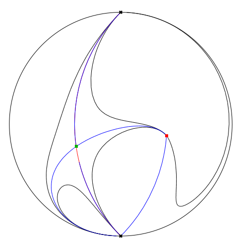

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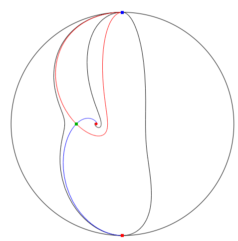

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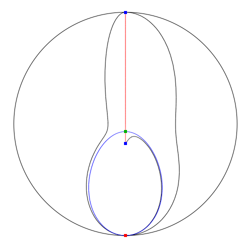

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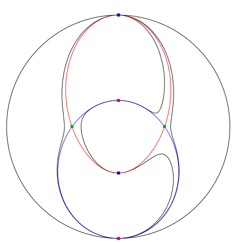

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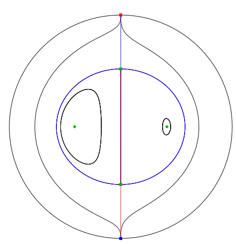

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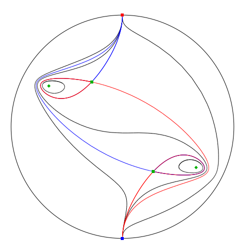

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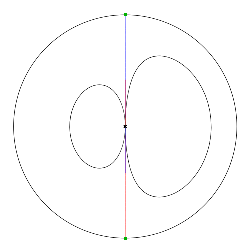

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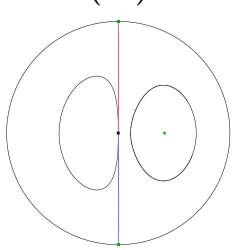

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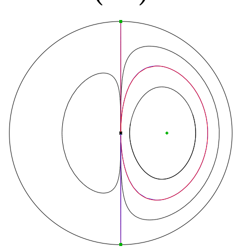

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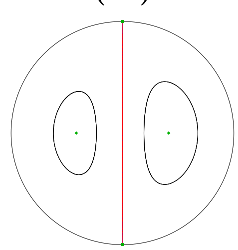

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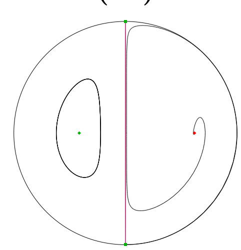

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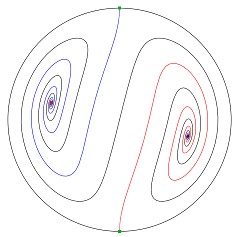
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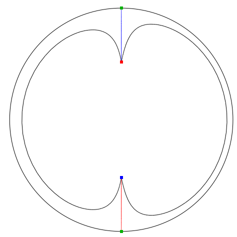
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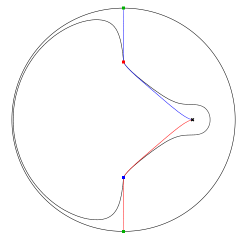
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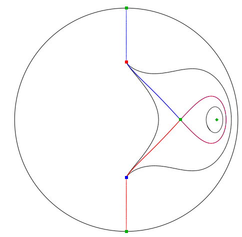
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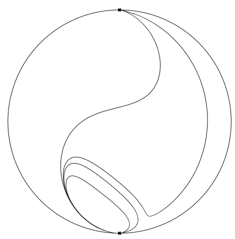
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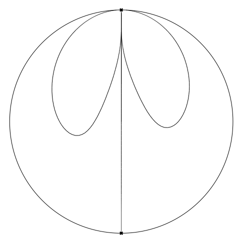
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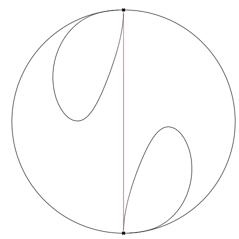
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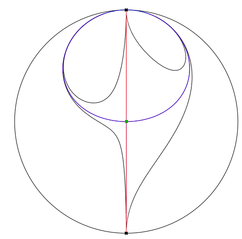
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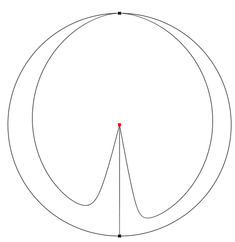
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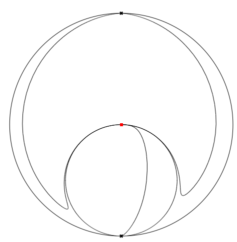
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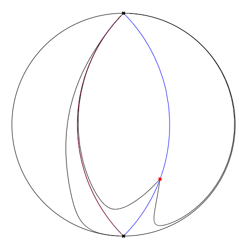
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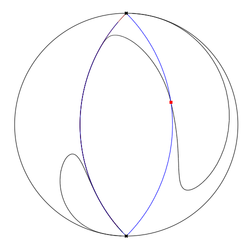
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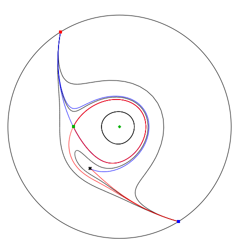
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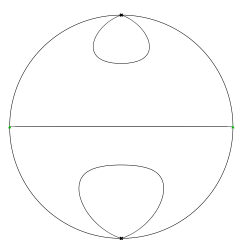
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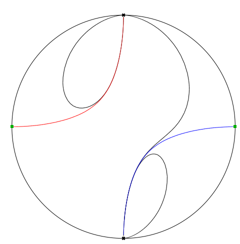
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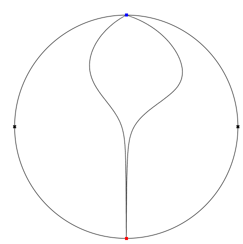
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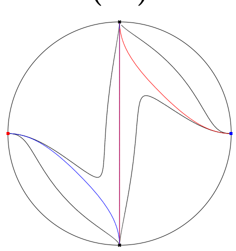
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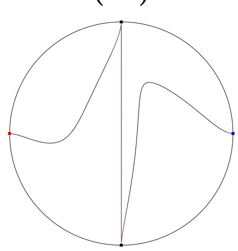
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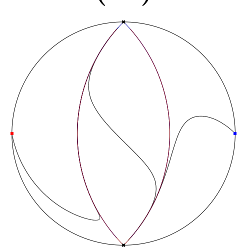
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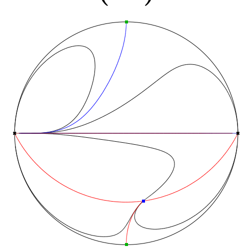
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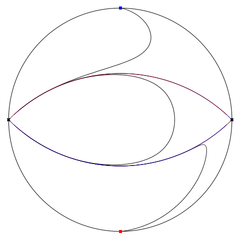
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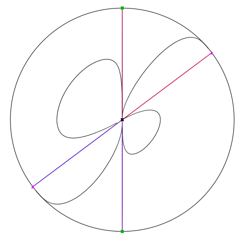
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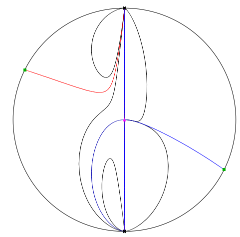
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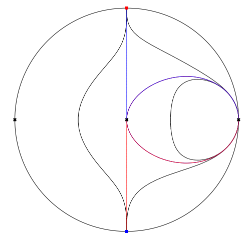
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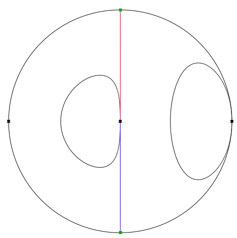
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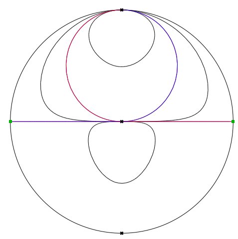
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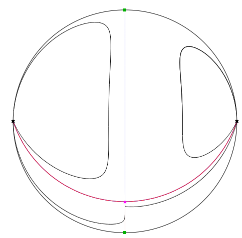
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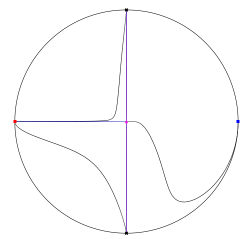
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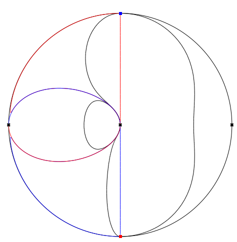
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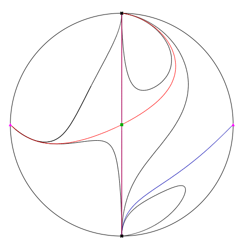
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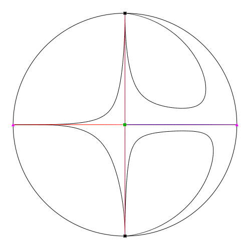
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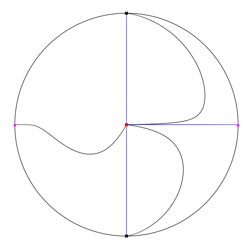
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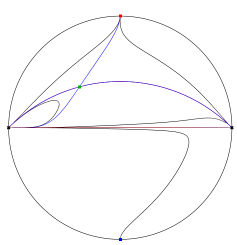
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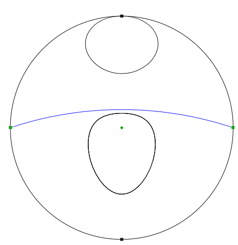
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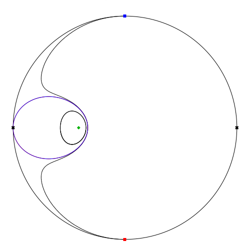
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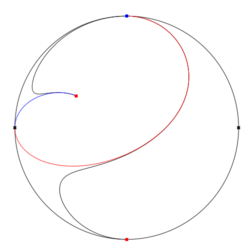
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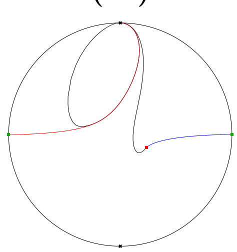
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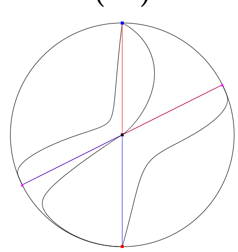
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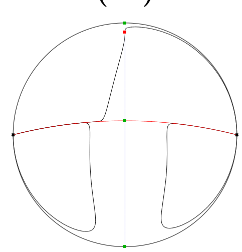
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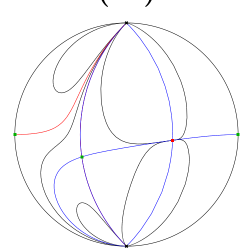
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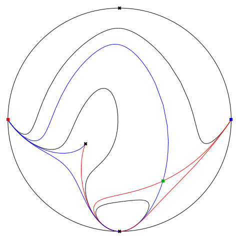

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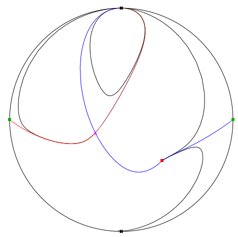

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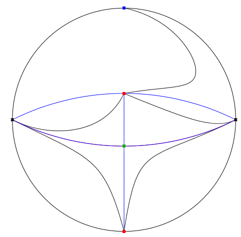

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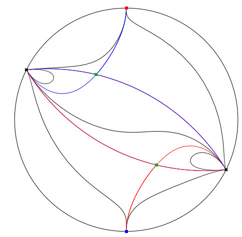

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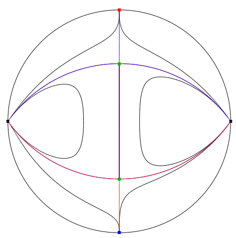

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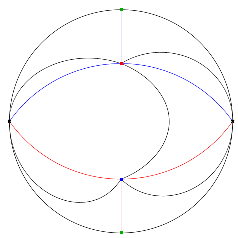

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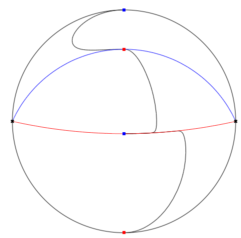

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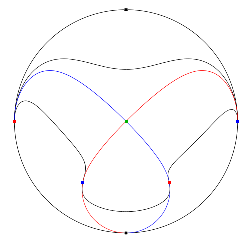

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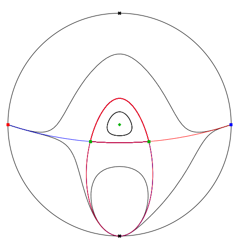

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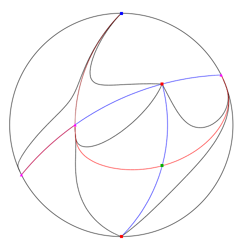

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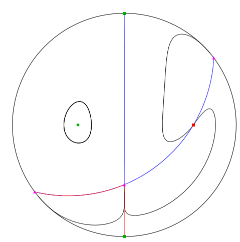

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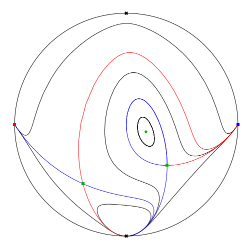

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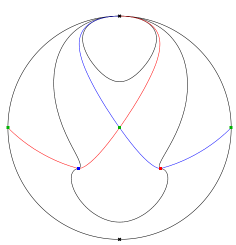

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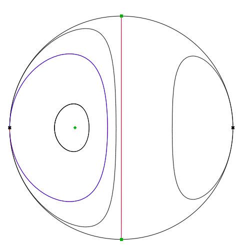

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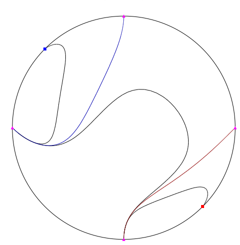

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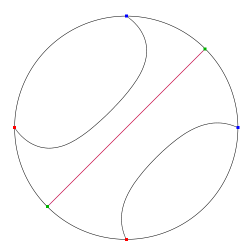

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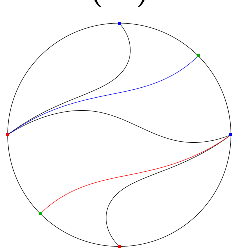

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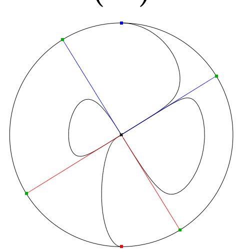

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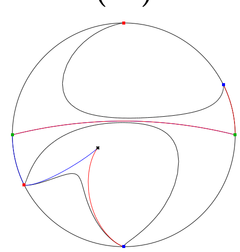

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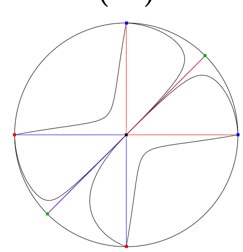

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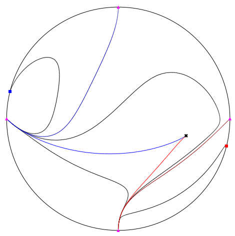

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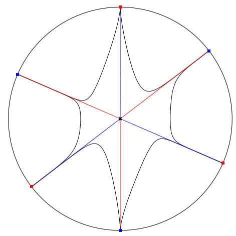

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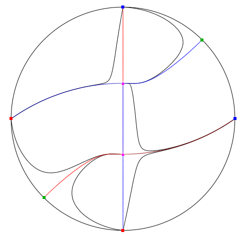

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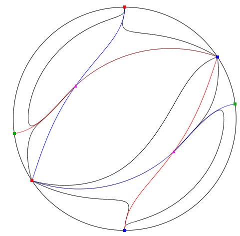

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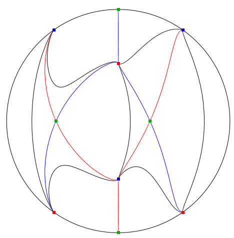

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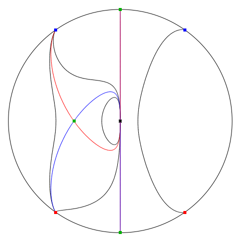

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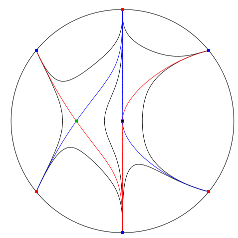

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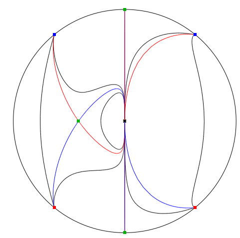

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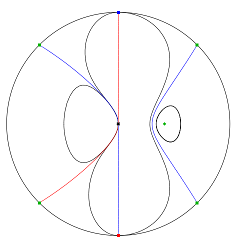

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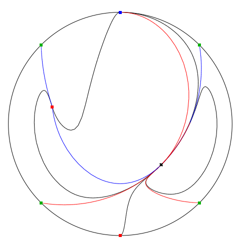

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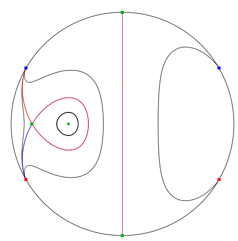

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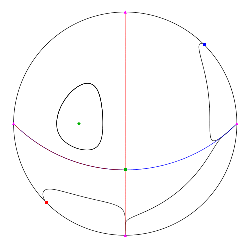

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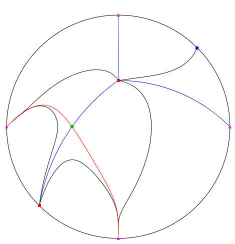

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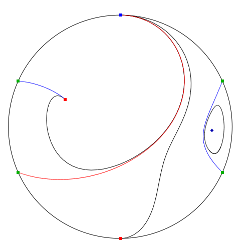

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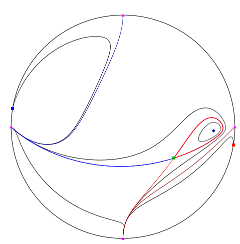

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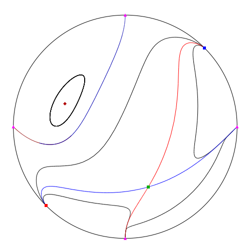

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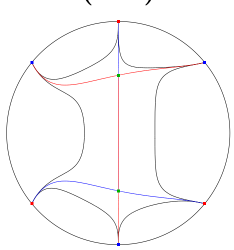

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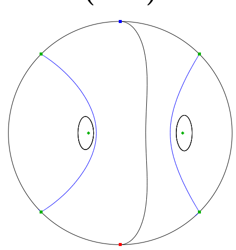

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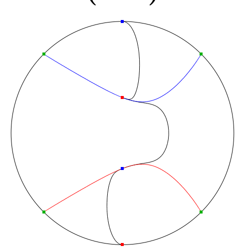

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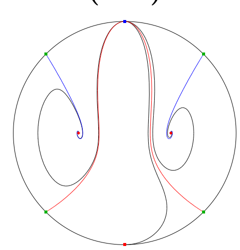

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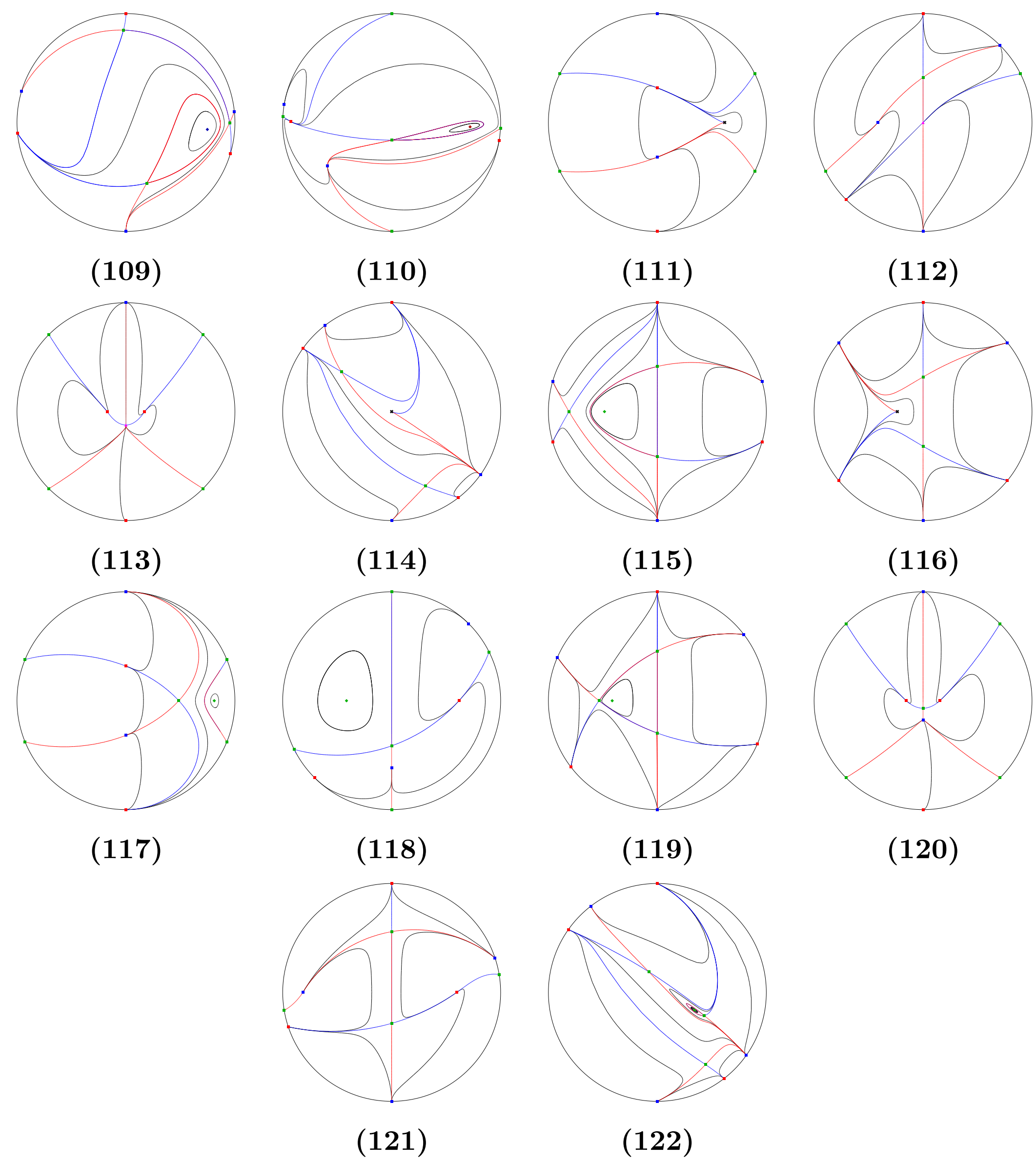

Figure 3.4: Phase portraits corresponding to the quadratic systems having a polynomial inverse integrating factor.

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Part II

Publications

Chapter 4

Polynomial inverse integrating factors for polynomial vector fields

In this first article, we present some results and one open question on the existence of polynomial inverse integrating factors for planar polynomial vector fields. This is a joint work with Jaume Llibre and Adam Mahdi, and it has been accepted for publication in *Discrete and Continuous Dynamical Systems*.

4.1 Introduction

A *polynomial vector field* defined on $\mathbb{C}^2$ (respectively $\mathbb{R}^2$) is a vector field of the form

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{4.1}$$

where P and Q are complex (respectively real) polynomials in the variables x and y. The maximum of the degrees of P and Q is called the *degree* of X. Sometimes to simplify notation we shall write that $X = (P, Q)$.

For simplicity, in the whole paper we will assume that *the polynomials P and Q are coprime* in the ring of all complex polynomials $\mathbb{C}[x,y]$. If they are not coprime, doing easy arguments, we can extend all the results to that case.

We remark that since the real polynomial vector fields are particular cases of the complex ones, the results for the complex are also true for the reals. In what follows we shall give several definitions for polynomial vector fields in $\mathbb{C}^2$, but in a similar way they can be given for polynomial vector fields in $\mathbb{R}^2$.

Let U be an open subset of $\mathbb{C}^2$. If there exists a non–constant C^1 function $H : U \to \mathbb{C}$, eventually multi–valued, which is constant on all the solutions of X contained in U, then we say that H is a *first integral* of X, and that X is integrable on U. Then, we have $XH = 0$ on U.

If $V : U \to \mathbb{C}$ is a function satisfying the linear partial differential equation

$$P\frac{\partial V}{\partial x} + Q\frac{\partial V}{\partial y} = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) V \tag{4.2}$$

on U, then V is called an *inverse integrating factor* of X on U. From its definition, it follows that the curve $V = 0$ is formed by trajectories of the system.

We say that the *inverse integrating factor V is associated to the first integral* H of the vector field X given by (4.1) on U if we have

$$\frac{P}{V} = -\frac{\partial H}{\partial y}, \qquad \frac{Q}{V} = \frac{\partial H}{\partial x}$$

on U.

One of the main open problems in the qualitative theory of planar polynomial vector fields X is to characterize the integrable ones. A good way to study integrable vector fields is through the inverse integrating factor V, for more details see [4]. Moreover, if X is real and $V : U \to \mathbb{R}$ is an inverse integrating factor of X on the open subset U of $\mathbb{R}^2$, then V becomes very important because $\{V = 0\}$ contains the limit cycles of X which are in U, see [9, 10]. Moreover if V is polynomial, then it is defined on the whole $\mathbb{R}^2$ and consequently knowing such kind of V's we can control all limit cycles of X, see for instance [10].

In this paper we are mainly interested in studying the polynomial vector fields having a polynomial inverse integrating factor. But we start presenting the known relationships between the nature of the first integrals and the nature of their associated inverse integrating factors.

Let $f_1, \ldots, f_p, g, h$ be complex polynomials in the variables x and y and let $\lambda_1, \ldots, \lambda_p$ be complex numbers. Then, a function of the form

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} \exp\left(g/h\right)$$

is called *Darboux*.

For a definition of *Liouvillian* function see Singer [14], roughly speaking a Liouvillian function comes from the integral of a Darboux function.

The following theorem summarizes some relations between the first integrals and the inverse integrating factors for a polynomial vector field X in $\mathbb{C}^2$.

Theorem 1. *Let X be a polynomial vector field in $\mathbb{C}^2$.*

(a) If X has a Liouvillian first integral, then it has a Darboux inverse integrating factor.

(b) If X has a Darboux first integral, then it has a rational inverse integrating factor.

(c) If X has a polynomial first integral, then it has a polynomial inverse integrating factor.

Statement (a) of Theorem 1 was proved in [14] and [6]. Statement (b) was proved in [5]. Note that statement (c) corresponds to statement (b) of the Main

Theorem of [5], but there is a misprint in [5], because in the proof of this statement is shown the existence of a polynomial integrating factor, instead of the existence of a polynomial inverse integrating factor. Statement (c) of Theorem 1 is proved in Proposition 8, see Section 4.2.

Note that in statements (a) and (b) of Theorem 1 the expression of the integrating factor is easier than the expression of the first integral. So, in these cases the study of the integrability of such systems is easier looking for their inverse integrating factors instead of looking for their first integrals.

Looking at Theorem 1 a natural question appears: *Suppose that the polynomial vector field X has a rational first integral. When does X have a polynomial inverse integrating factor*? Such X's were characterized in the next result proved in [5]. For the precise definitions of the notions which appear in it, see Section 4.3.

Theorem 2. *Let X be a polynomial vector field. Suppose that $H = f/g$ is a canonical rational first integral and that X has no polynomial first integrals. Then, X has a polynomial inverse integrating factor if and only if H has at most two critical remarkable values.*

We mention here that if X has a rational first integral, then it always has a canonical rational first integral, for a proof see Section 4.3.

As far as we know Theorem 2 was not complete in the sense that there were no examples of polynomial vector fields satisfying its assumptions and without a polynomial inverse integrating factor. In what follows we provide such an example.

Proposition 3. *The polynomial vector field*

$$X = 2x(5 + 30x + 40x^2 + 8y^2)\frac{\partial}{\partial x} + y(5 + 44x + 80x^2 + 16y^2)\frac{\partial}{\partial y}, \tag{4.3}$$

has a rational first integral, and has neither a polynomial first integral, nor a polynomial inverse integrating factor.

In Section 4.3 we present some preliminary results that we need for proving Proposition 3 in Section 4.4. We remark that the polynomial vector field of Proposition 3 also has been studied for other reasons in [5], see there system (23) with $a = 1$.

As we will see in the next result many families of polynomial vector fields in $\mathbb{R}^2$ with a center have a polynomial inverse integrating factor. The definitions of the notions which appear in the statement of these results are given in Section 4.5, where we also give either references for their proofs, or prove them.

Theorem 4. *Assume that X is a polynomial vector field in $\mathbb{R}^2$. Suppose that X satisfies one of the following conditions:*

(i) It is homogeneous.

(ii) It is quasi–homogeneous.

(iii) It has degree 2 *and a center.*

Then, X has a polynomial inverse integrating factor.

All polynomial vector fields of Theorem 4 having a center; of course, also have a polynomial inverse integrating factor. But, as the following result shows, there are polynomial vector fields having a center which do not have polynomial inverse integrating factors.

Proposition 5. *The polynomial vector field*

$$X(x,y) = y^3\frac{\partial}{\partial x} - \frac{1}{2}x^2(2x-y^2)\frac{\partial}{\partial y} \ , \tag{4.4}$$

has a center at the origin and has no polynomial inverse integrating factors.

The polynomial vector field of Proposition 5 was studied by Moussu [11] in order to provide a degenerate center for which does not exist a local analytic first integral. Proposition 5 will be proved in Section 4.6.

We have the following question.

Open Question. *Assume that X is a polynomial vector field having a center. How to characterize the X's having a polynomial inverse integrating factor?*

There are other papers which deal with polynomial inverse integrating factors, but considering distinct aspects of those studied here, see for instance [2, 11, 39].

4.2 X has a polynomial first integral

In the rest of this paper X will always denote a polynomial vector field.

Let $f \in \mathbb{C}[x,y]$. The algebraic curve $f(x,y)=0$ is *invariant* for X if Xf/f is a polynomial of $\mathbb{C}[x,y]$. It is known that if the algebraic curve $f=0$ is invariant, then it is formed by orbits of X, for more details see [7]. We note that the invariant algebraic curves will play a key role in this paper. In fact, they play a main role in the Darboux theory of integrability, and our paper is dedicated to a particular case of this theory, to study the systems which have a polynomial inverse integrating factor.

Proposition 6. *Let g be a polynomial and $g_1^{n_1}\cdots g_r^{n_r}$ its decomposition in irreducible factors in $\mathbb{C}[x,y]$. Then, $g=0$ is an invariant algebraic curve if and only if all the g_j are invariant algebraic curves for $j=1,\dots,r$. Moreover, if K and K_j are the cofactors of f and f_j, then $K=n_1K_1+\cdots+n_rK_r$.*

Proof: See [7]. ∎

We shall need the following result.

Lemma 7. *Let X be a polynomial vector field, H be a first integral of X, and V be an inverse integrating factor of X. Then, VH is another inverse integrating factor of X.*

Proof: It follows easily from the definition (4.2) of inverse integrating factor. ∎

Now we prove statement (c) of Theorem 1.

Proposition 8. *Let X be a polynomial vector field X in $\mathbb{C}^2$. If X has a polynomial first integral, then it has a polynomial inverse integrating factor.*

Proof: Let H be a polynomial first integral of X. We note that a polynomial function is a particular case of a Darboux function. Therefore, by statement (b) of Theorem 1, X has a rational inverse integrating factor $V = f/g$, where f and g are coprime polynomials. It is known that the curves $f = 0$ and $g = 0$ are invariant algebraic curves of X, see for instance [5].

Let $g_1^{n_1} \cdots g_r^{n_r}$ be with $n_i \in \mathbb{N}$ the factorization of g in irreducible factors in $\mathbb{C}[x, y]$. By Proposition 8, $g_j = 0$ is an invariant algebraic curve of X for $j = 1, \ldots, r$. Let h_j be the value of the first integral H on the points of the irreducible invariant algebraic curve $g_j = 0$.

The Hilbert's Nullstellensatz (see for instance, [8]) states: *Set $A, B_i \in \mathbb{C}[x, y]$ for $i = 1, \cdots, s$. If A vanishes in $\mathbb{C}^2$ whenever the polynomials B_i vanish simultaneously, then there exist polynomials $M_i \in \mathbb{C}[x, y]$ and a nonnegative integer n such that $A^n = M_1B_1 + \cdots + M_sB_s$.* Taking $A = H - h_j$, $s = 1$ and $B_1 = g_j$, we get that g_j divides $(H - h_j)^n$ for some nonnegative n. Since g_j is irreducible, g_j divides $H - h_j$. Therefore, there exists a polynomial s_j such that $H - h_j = s_jg_j$.

Since H is a polynomial first integral of X (i.e. H is constant on the solutions of X), it follows that $K = (H - h_1)^{n_1} \cdots (H - h_r)^{n_r}$ is another polynomial first integral of X. By Lemma 7, VK is an inverse integrating factor of X. Since $H - h_j = s_jg_j$, it follows that VK is a polynomial. Hence, the proposition is proved. ∎

4.3 Some preliminary results

Let $H = f/g$ be a rational first integral of the polynomial vector field X. We say that H has *degree* n if n is the maximum of the degrees of f and g. We say that the degree of H is *minimal* between all the degrees of the rational first integrals of X if any other rational first integral of X has degree $\geq n$.

Let $H = f/g$ be a rational first integral of X. According to Poincaré [13] we say that $c \in \mathbb{C} \cup \{\infty\}$ is a *remarkable* value of H if $f + cg$ is a reducible polynomial in $\mathbb{C}[x, y]$. Note that for all $c \in \mathbb{C}$ the curve $f + cg = 0$ is an invariant algebraic curve. Here, if $c = \infty$, then $f + cg$ denotes g. In [5] it is proved that there are finitely many remarkable values for a given rational first integral H.

Now suppose that $c \in \mathbb{C}$ is a remarkable value of a rational first integral H and that $u_1^{\alpha_1} \cdots u_r^{\alpha_r}$ is the factorization of the polynomial $f + cg$ into irreducible factors in $\mathbb{C}[x, y]$. If some of the α_i for $i = 1, \ldots, r$ is larger than 1, then we say (following again Poincaré) that c is a *critical* remarkable value of H, and that $u_i = 0$ having $\alpha_i > 1$ is a *critical remarkable* invariant algebraic curve of X with *exponent* α_i.

Now, if $H = f/g$ is a minimal rational first integral of X of degree n which is not polynomial, then

$$H_1 = \frac{1}{c_1 + f/g} + c_2 = \frac{g + c_2(f + c_1 g)}{f + c_1 g} = \frac{f_1}{g_1}, \tag{4.5}$$

for any $c_1, c_2 \in \mathbb{C}$, is also a rational first integral of X. It is known that there are complex values of c_1 and c_2 for which the numerator and the denominator of H_1 are irreducible polynomials of degree n. One way to see this is the following. We claim that there are finitely many values of c_1 and c_2 such that $g + c_2(f + c_1 g)$ and $f + c_1 g$ are reducible. In order to prove the claim assume that it is not true. Then, in particular there are infinitely many values of c_1 and c_2 for which $g + c_2(f + c_1 g)$ and $f + c_1 g$ factorize in polynomial factors of degree smaller than n. Consequently, the rational first integral H_1 has infinitely many remarkable values, and this is a contradiction.

We say that a rational first integral $H_1 = f_1/g_1$ of a polynomial vector field X is *canonical* if H_1 is minimal and f_1 and g_1 are irreducible polynomials in $\mathbb{C}[x, y]$ having the same degree. Note that the previous arguments show that if a polynomial vector field has a rational first integral, then it has a canonical rational first integral.

Theorem 9. *Let X be a polynomial vector field. Suppose that $H = f/g$ is a canonical rational first integral and that X has no polynomial first integrals. Consider the rational function*

$$V_{f/g} = \frac{g^2}{\prod u_i^{\alpha_i - 1}} \ ,$$

where the product runs over all critical remarkable invariant algebraic curves $u_i = 0$ having exponent α_i of X. Then, $V_{f/g}$ is an inverse integrating factor of X.

For a proof of Theorem 9 see the Main Theorem in [5].

4.4 Proof of Proposition 3

In all this section X is the polynomial vector field of Proposition 3. Then, it is easy to check that X has the rational first integral

$$H(x,y) = \frac{(3\,x+12\,x^2-4\,y^2)^2\,(x+2\,x^2+2\,y^2)^3}{y^{10}};$$

i.e. $XH=0$. From (4.5) and for convenient complex numbers c_1 and c_2, we have that $H_1 = f_1/g_1$ is a canonical rational first integral of X, where

$$\begin{aligned} f_1(x,y) \;=\;& 9\,c_2\,x^5+126\,c_2\,x^6+684\,c_2\,x^7+1800\,c_2\,x^8+2304\,c_2\,x^9+\\ & 1152\,c_2\,x^{10}+30\,c_2\,x^4\,y^2+408\,c_2\,x^5\,y^2+1944\,c_2\,x^6\,y^2+\\ & 3840\,c_2\,x^7\,y^2+2688\,c_2\,x^8\,y^2-20\,c_2\,x^3\,y^4+24\,c_2\,x^4\,y^4+\\ & 768\,c_2\,x^5\,y^4+1280\,c_2\,x^6\,y^4-120\,c_2\,x^2\,y^6-768\,c_2\,x^3\,y^6-\\ & 768\,c_2\,x^4\,y^6-384\,c_2\,x^2\,y^8+(1+128\,c_2+c_1\,c_2)\,y^{10}, \end{aligned}$$

$$\begin{aligned} g_1(x,y) \;=\;& 9\,x^5+126\,x^6+684\,x^7+1800\,x^8+2304\,x^9+1152\,x^{10}+\\ & 30\,x^4\,y^2+408\,x^5\,y^2+1944\,x^6\,y^2+3840\,x^7\,y^2+\\ & 2688\,x^8\,y^2-20\,x^3\,y^4+24\,x^4\,y^4+768\,x^5\,y^4+1280\,x^6\,y^4-\\ & 120\,x^2\,y^6-768\,x^3\,y^6-768\,x^4\,y^6-384\,x^2\,y^8+(c_1+128)\,y^{10}. \end{aligned}$$

Now we show that X has no polynomial first integrals. Assume that F is a polynomial first integral of X of degree m. Let F_m be the homogeneous part of F of degree m. Then, since $XF=0$, taking only the higher order terms of XF, we get that

$$2\,x\,\left(40\,x^2+8\,y^2\right)\frac{\partial F_m}{\partial x}+y\,\left(80\,x^2+16\,y^2\right)\frac{\partial F_m}{\partial y}=0,$$

or equivalently,

$$x\frac{\partial F_m}{\partial x}+y\frac{\partial F_m}{\partial y}=0.$$

Since the general solution of this linear partial differential equation is an arbitrary function in the variable y/x, and F_m must be a homogeneous polynomial of degree m, it follows that X cannot have a polynomial first integral.

We remark that H_1 is a canonical rational first integral of X, and that X has no polynomial first integrals. So, X satisfies the assumptions of Theorem 2. Therefore, if we prove that H_1 has at least three critical remarkable values, by Theorem 2(a), it follows that X has no polynomial inverse integrating factors.

For $c=-c_2$, we have that $f_1+cg_1=y^{10}$. So, $-c_2$ is a critical remarkable value of H_1 and $y=0$ is the corresponding critical remarkable invariant curve of X with exponent 10.

For $c=-(1+128c_2+c_1c_2)/(128+c_1)$, we have that

$$f_1+cg_1=-\frac{x^2}{128+c_1}\,h(x,y),$$

where

$$\begin{aligned} h(x,y) \;=\;& 9\,x^3+126\,x^4+684\,x^5+1800\,x^6+2304\,x^7+1152\,x^8+\\ & 30\,x^2\,y^2+408\,x^3\,y^2+1944\,x^4\,y^2+3840\,x^5\,y^2+2688\,x^6\,y^2-\\ & 20\,x\,y^4+24\,x^2\,y^4+768\,x^3\,y^4+1280\,x^4\,y^4-120\,y^6-\\ & 768\,x\,y^6-768\,x^2\,y^6-384\,y^8. \end{aligned}$$

Hence, $-(1+128c_2+c_1c_2)/(128+c_1)$ is a critical remarkable value of H_1 and $x=0$ is the corresponding critical remarkable invariant curve of X with exponent 2.

For $c=-(1+c_1c_2)/c_1$, we have that

$$f_1+cg_1=-\frac{1}{c_1}\left(3\,x+12\,x^2-4\,y^2\right)^2\left(x+2\,x^2+2\,y^2\right)^3.$$

Therefore, $-(1+c_1c_2)/c_1$ is a critical remarkable value of H_1, and $3\,x+12\,x^2-4\,y^2=0$ and $x+2\,x^2+2\,y^2=0$ are the corresponding critical remarkable invariant curves of X with exponent 2 and 3, respectively.

In short, we have proved that the polynomial vector field X given by (4.3) has a rational first integral and does not have a polynomial inverse integrating factor. Therefore, Proposition 3 is proved.

In fact, we can prove that H_1 has exactly these three critical remarkable values. This follows from statement (c.1) of the Main Theorem of [5], because the rational function

$$V=\frac{g_1^2}{y^9x(3\,x+12\,x^2-4\,y^2)(x+2\,x^2+2\,y^2)^2}$$

is the inverse integrating factor of the polynomial vector field (4.3), having in its denominator all the critical remarkable invariant algebraic curves.

4.5 Proof of Theorem 4

In all this section we assume that X is a polynomial vector field in $\mathbb{R}^2$ having a center.

First we provide all the definitions that appear in the statement of Theorem 4.

Let $p \in \mathbb{R}^2$ be a singular point of X. We say that p is a *center* if there is a neighborhood U of p such that all the orbits of $U \setminus \{p\}$ are periodic.

If X given by (4.1) has the polynomials P and Q homogeneous with the same degree, then we say that the polynomial vector field X is *homogeneous*.

In what follows p and q will denote positive integers. We say that a function $F(x, y)$ is (p, q)*–quasi–homogeneous of weight degree* $m \geq 0$ if $F(\ell^p x, \ell^q y) = \ell^m F(x, y)$ for all $\ell \in \mathbb{R}$.

A polynomial vector field X given by (4.1) is (p, q)*–quasi–homogeneous of weight degree* $m \geq 0$ (or simply *quasi–homogeneous*) if P and Q are (p, q)–quasi–homogeneous functions of weight degrees $p - 1 + m$ and $q - 1 + m$, respectively.

Note that the $(1, 1)$–quasi–homogeneous polynomial vector fields of weight degree m coincide with the homogeneous polynomial vector fields of degree m.

We also note that if X is (p, q)–quasi–homogeneous, then the differential equation $dy/dx = Q/P$ (another way to work with X) is invariant by the change of variables $(x, y) \rightarrow (\ell^p x, \ell^q y)$.

If condition (i) of Theorem 4 holds by X, then it is easy to check that $xQ(x, y) - yP(x, y)$ is a polynomial inverse integrating factor of the homogeneous polynomial vector field X.

If condition (ii) of Theorem 4 holds by X, then it is easy to check that $pxQ(x, y) - qyP(x, y)$ is a polynomial inverse integrating factor of the (p, q)–quasi–homogeneous polynomial vector field X.

If condition (iii) of Theorem 4 is satisfied by X, in [1] and [12] it is proved that X has a polynomial inverse integrating factor of degree 3 or 5 for any quadratic vector field X having a center.

In short, we have proved Theorem 4.

4.6 Proof of Proposition 5

In all this section X will be the polynomial vector field (4.4) of Proposition 5.

The origin is a center of X because it is a monodromic singular point, and X is φ–reversible with respect to the involution $\varphi(x, y) = (x, -y)$; i.e. X satisfies

$$D\varphi(p)X(p) = -X \circ \varphi(p), \qquad p \in \mathbb{R}^2.$$

Suppose that X has a polynomial inverse integrating factor V of degree k. Then, it satisfies equation (4.2). Writing V as a sum of homogeneous polynomials; i.e,

$$V(x, y) = V_0 + \sum_{i=1}^{k} V_i(x, y),$$

where $V_0 \in \mathbb{R}$ and $V_i \in \mathbb{R}[x,y]$ are homogeneous polynomials of degree i for $i = 1, \ldots, k$, we obtain a system of partial differential equations. The partial differential equation of degree $k+3$, using the Euler formula for homogeneous functions, becomes

$$x^2 y \left(V_k(x,y) - \frac{y}{2} \frac{\partial V_k}{\partial y} \right) = 0.$$

From this equation and since V can be determined unless a constant, we get

$$V_k(x,y) = x^{k-2} y^2.$$

Now we can substitute this expression in the partial differential equation of degree $k+2$, and again using the Euler formula for the homogeneous functions, it becomes

$$x^2 y \left(x^{k-5}(-2x^4 + (k-2)y^4) + V_{k-1}(x,y) - \frac{y}{2} \frac{\partial V_{k-1}}{\partial y}(x,y) \right) = 0.$$

¿From this equation, we obtain

$$V_{k-1}(x,y) = x^{k-5}\left(-2x^4 + c_1 x^2 y^2 - (k-2)y^4\right),$$

where $c_1 \in \mathbb{R}$. Finally, substituting the expressions of V_k and V_{k-1} into the partial differential equation of degree $k+1$, we obtain

$$\begin{aligned} x^2 y \Big(&- x^{k-8}(2c_1 x^6 - 2(k-3)x^4 y^2 - 3c_1(k-3)x^2 y^4 + (k-2)(k-5)y^6) \\ &-V_{k-2}(x,y) + \frac{y}{2} \frac{\partial V_{k-2}}{\partial y}(x,y) \Big) = 0. \end{aligned}$$

Therefore, we have

$$\begin{aligned} V_{k-2}(x,y) \; &= \; -2c_1 x^{k-2} + \left(-4(k-3)x^{k-4}\log y + c_2 x^{k-4}\right) y^2 \\ &\quad -c_1(k-3)x^{k-6}y^4 + (k-2)(k-5)x^{k-8}y^6/2, \end{aligned}$$

where $c_2 \in \mathbb{R}$. This function must be a polynomial, and then we must take $k = 3$.

Finally, direct computations show that there is no polynomial inverse integrating factor of degree 3 for the vector field (4.4).

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Periodic orbits for a class of $\mathcal{C}^1$ three–dimensional systems

In this second article, we perturb a reversible polynomial differential system of degree 4 in $\mathbb{R}^3$ by $\mathcal{C}^1$ functions. If the perturbation is strongly reversible, the dynamics of the perturbed system do not change. Otherwise, if the perturbation is non–strongly reversible, we prove the existence of an arbitrary number of symmetric periodic orbits. This is a joint work with Jaume Llibre and Marco António Teixeira, and it has been submitted for publication.

5.1 Introduction

A vector field $X : \mathbb{R}^3 \to \mathbb{R}^3$ of the form $X = (P, Q, R)$ is called a *polynomial vector field of degree* m if P, Q and R are polynomials of degree $\leq m$ and at least one of them has degree m.

A diffeomorphism $\varphi : \mathbb{R}^3 \to \mathbb{R}^3$ is called an *involution* if $\varphi \circ \varphi = Id$. A vector field X is *reversible* if there exists an involution φ such that $\varphi_* X = -X \circ \varphi$; i.e., $d\varphi_p(X(p)) = -X(\varphi(p))$. Let S_φ be the set of fixed points of φ. An orbit γ is *symmetric* if $\varphi(\gamma) = \gamma$. Hence, every singular point of X in S_φ is *symmetric*. Some classical properties of reversible systems are:

(i) The phase portrait of X is symmetric with respect to S_φ.

(ii) A symmetric singular point or a symmetric periodic orbit cannot be an attractor or a repellor.

(iii) If $X(p) = 0$ and $p \notin S_\varphi$, then $X(\varphi(p)) = 0$.

(iv) If an orbit γ intersects S_φ in two distinct points, then γ is a periodic orbit.

(v) If γ is an orbit of X such that $\gamma \not\subset S_\varphi$ and $p \in \gamma \cap S_\varphi$, then $X(p) \notin T_p S_\varphi$.

(vi) Any periodic orbit γ of X not crossing S_φ, has a symmetric one given by $\varphi(\gamma)$.

(vii) Let γ be an orbit having stable manifold $W^s(\gamma)$ and unstable manifold $W^u(\gamma)$. Then,

$$\varphi(W^s(\gamma)) = W^u(\varphi(\gamma)), \quad \varphi(W^u(\gamma)) = W^s(\varphi(\gamma)).$$

In this paper we perturb a polynomial system in $\mathbb{R}^3$ by $\mathcal{C}^1$ functions and we use the symmetry to prove that the perturbed system has an arbitrary number of symmetric periodic orbits.

Let X_0 be the vector field associated to the polynomial differential system of degree 4

$$\dot{x} = (y^2 + z^2)(1 - y^2 - z^2), \quad \dot{y} = -z + xy, \quad \dot{z} = y + xz, \tag{5.1}$$

where $x, y, z \in \mathbb{R}$, or in cylindric coordinates (taking $y = r\cos\theta$ and $z = r\sin\theta$)

$$\dot{x} = r^2(1 - r^2), \quad \dot{r} = xr, \quad \dot{\theta} = 1, \tag{5.2}$$

where $x, r \in \mathbb{R}$, $r \geq 0$, and $\theta \in \Sigma^1$. In both cases, the dot means derivative with respect to the time $t \in \mathbb{R}$.

System (5.2) has a first integral $H : \mathbb{R}^2 \times \Sigma^1 \to \mathbb{R}$ defined by

$$H(x, r, \theta) = -2x^2 + 2r^2 - r^4. \tag{5.3}$$

We denote $H_c = H^{-1}(c)$, for $c \in [0, 1]$. The level H_0 is, topologically, a sphere with two different points identified at the origin. The origin is a singular point, and all the other orbits in H_0 are homoclinic orbits at the origin. The level H_1 is the periodic orbit $x = 0$, $r = 1$, see Figure 5.1.

We denote D_3 the bounded region $H^{-1}((0, 1)) \subset \mathbb{R}^3$. If $0 < c < 1$, then H_c is an invariant torus inside D_3. In the whole paper, all the three–dimensional systems considered are studied only in the bounded region $\overline{D}_3 = H^{-1}([0, 1]) = H_0 \cup D_3 \cup H_1$.

Systems (5.1) and (5.2) are reversible with respect to the linear involution φ defined by

$$\varphi(x, y, z) = (-x, y, -z) \tag{5.4}$$

for system (5.1), and

$$\varphi(x, r, \theta) = (-x, r, -\theta) \tag{5.5}$$

for system (5.2). The set S_φ of fixed points of φ is the segment $x = z = 0$, $y \in [-\sqrt{2}, \sqrt{2}]$. In polar coordinates, S_φ can be written as $x = 0$, $r \in [0, \sqrt{2}]$, $\theta \in \{0, \pi\}$.

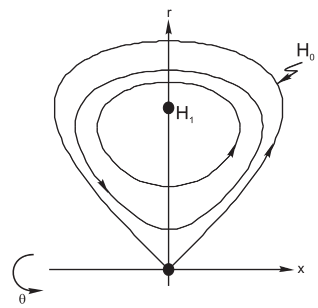

Figure 5.1: Phase portrait of system (5.2) in the region $\overline{D}_3$.

We say that a φ–reversible system of the form

$$\dot{x} = f(x^2, r, \theta), \quad \dot{r} = x\, g(x^2, r, \theta), \quad \dot{\theta} = 1 + h(x^2, r, \theta),$$

is *strongly reversible* if f, g and h do not depend on θ.

For a strongly φ–reversible perturbation Y_ε of the vector field $Y_0 = X_0$, defined by system (5.2), we prove the following theorem.

Theorem 1. *Let Y_ε be the strongly φ–reversible vector field associated to the system*

$$\begin{aligned} \dot{x} &= r^2(1 - r^2) + \varepsilon f(x^2, r, \varepsilon), \\ \dot{r} &= x(r + \varepsilon g(x^2, r, \varepsilon)), \\ \dot{\theta} &= 1 + \varepsilon h(x^2, r, \varepsilon), \end{aligned} \tag{5.6}$$

with $f, g, h \in \mathcal{C}^1$, $f(0, 0, \varepsilon) = 0$. Suppose that H_0 and H_1 are invariant by the flow of Y_ε, and that the system $(\dot{x}, \dot{r})$ restricted to $\overline{D}_3 \cap \{\theta = \text{constant}\}$ has only two singular points, $(x, r) = (0, 0)$ and $(x, r) = (0, 1)$. Then, D_3 is fulfilled of invariant tori.

Let X_ε be the non–strongly φ–reversible perturbation of X_0 associated to the system

$$\begin{aligned} \dot{x} &= r^2(1 - r^2) + \varepsilon f(x, r, \theta, \varepsilon), \\ \dot{r} &= x(r + \varepsilon g(x, r, \theta, \varepsilon)), \\ \dot{\theta} &= 1 + \varepsilon h(x, r, \theta, \varepsilon), \end{aligned} \tag{5.7}$$

with $f, g, h \in \mathcal{C}^1$. The main goal of this paper is to prove that, for $\varepsilon > 0$ sufficiently small and under convenient conditions there exist an arbitrary number of periodic orbits for system X_ε.

Let γ be a periodic orbit of period P such that there exist $m, n \in \mathbb{N}$, $(m, n) = 1$ satisfying the relation

$$2\pi n = mP.$$

We define the *rotation number* of γ as m/n. If a periodic orbit γ of system (5.2) has rotation number m/n, then after a time $2\pi n$ it has made m turns to the periodic orbit H_1.

We prove the following theorem.

Theorem 2. *Let* (5.7) *be a non–strongly φ–reversible $\mathcal{C}^1$–perturbation of system* (5.2) *such that H_0 and H_1 are invariant by the flow of* (5.7) *and $f(0,0,\theta,\varepsilon)=0$. Let $m/n \in (0,\sqrt{2})$, $m,n \in \mathbb{N}$, $(m,n)=1$. Then, there exists $\varepsilon_n > 0$ such that if $\varepsilon \in (0,\varepsilon_n)$, then X_ε has two periodic orbits of period $2\pi n$ and rotation number m/n.*

Let $\widetilde{T}_{n,\varepsilon}$ be the n–th return map of the Poincaré function of system (5.7); that is, the image of the section $\theta=0$ under the flow of system (5.7) after a time $2\pi n$.

Corollary 3. *Under the hypotheses of Theorem 2, all the periodic orbits γ of system* (5.7) *with period $\leq 2\pi n$ and rotation number m/n, satisfy*

$$\widetilde{T}_{n,\varepsilon}(\gamma) \underset{\varepsilon\to 0}{\longrightarrow} \widetilde{T}_n(\gamma),$$

where $\widetilde{T}_n = T_{n,0}$.

The paper is structured as follows. In Section 5.2 we study the dynamics of the vector field X_0. In Section 5.3 and 5.4 we prove Theorem 1 and Theorem 2, respectively. Finally, in Section 5.5 we relate Theorem 2 with the well–known Poincaré–Birkhoff Theorem.

5.2 The dynamics of X_0 on $\overline{D}_3$

As we said in Section 5.1, the level H_0 is formed by homoclinic orbits at the origin. These homoclinic orbits are $\gamma_\mu = \{(x_\mu(t), r_\mu(t), t) : t \in \mathbb{R}\}$, where

$$x_\mu(t) = \frac{2(\mu - t)}{1 + 2(\mu - t)^2}, \quad r_\mu(t) = \frac{\sqrt{2}}{\sqrt{1 + 2(\mu - t)^2}},$$

for all $\mu \in \mathbb{R}$. Then, $H_0 \setminus \{(0,0,0)\} = \{\gamma_\mu : \mu \in \mathbb{R}\}$. The set H_1 can be written as $H_1 = \{(0,1,t) : t \in \mathbb{R} \pmod{2\pi}\}$.

The phase portrait of system (5.2) comes from a rotation of the phase portrait of the system

$$\dot{x} = r^2(1 - r^2), \quad \dot{r} = rx, \tag{5.8}$$

defined in $\mathbb{R}^2_+ = \{(x,r) \in \mathbb{R}^2 : r \geq 0\}$, see Figure 5.1. This planar system has the first integral $K : \mathbb{R}^2_+ \to \mathbb{R}$ given by

$$K(x,r) = -2x^2 + 2r^2 - r^4. \tag{5.9}$$

We note that K is the same function as the first integral H defined in Section 5.1, but we write different names in order to distinguish the space where they are used, $\mathbb{R}^3$ for H and $\mathbb{R}^2_+$ for K.

System (5.8) has a degenerate saddle at the origin, and the curve $K^{-1}(0)$ is a homoclinic loop of this saddle. It also has a center at $x = 0$, $r = 1$, with eigenvalues $\pm i\sqrt{2}$, where $i = \sqrt{-1}$. This center corresponds to $K^{-1}(1)$, and becomes the periodic orbit H_1 when we consider system (5.2).

System (5.8) is reversible with respect to the linear involution

$$\psi(x, r) = (-x, r). \tag{5.10}$$

The set S_ψ of fixed points of ψ in the region $\overline{D}_2 = K^{-1}([0, 1])$ is the segment $x = 0$, $r \in [0, \sqrt{2}]$. The phase portrait of system (5.8) restricted to $\overline{D}_2$ is shown in Figure 5.1.

We can derive the dynamics of system (5.2) restricted to $\overline{D}_3$ from the dynamics of system (5.8) restricted to $\overline{D}_2$ just observing that the angle θ can be taken as the time modulus 2π. The flow of system (5.8) at $t = 2\pi$ gives the first return map of system (5.2) at the transversal section $\theta = 0$.

Lemma 4. *The period function of the periodic orbits $\gamma_c = K^{-1}(c)$, $c \in (0, 1)$, is given by*

$$P(c) = \frac{2\pi}{\sqrt{2c}} \in (\sqrt{2}\pi, +\infty).$$

Proof. Let $P(c)$ be the period of γ_c. The curve γ_c cuts S_ψ at the points $P_0 = (0, \sqrt{1 - \sqrt{1 - c}})$ and $P_1 = (0, \sqrt{1 + \sqrt{1 - c}})$. We will compute the time spent by the flow os system (5.2) for going from P_0 to P_1 through γ_c. Because of the symmetry ψ, this time corresponds to half the period. From $K(\gamma_c) = c$, we have $x = \pm\sqrt{\frac{2r^2 - r^4 - c}{2}}$. We take $x > 0$; if $x < 0$, then we get the other half of the period. Substituting x in the expression $\dot{r} = xr$ and integrating, we obtain

$$\frac{P(c)}{2} = \int_0^{P(c)/2} dt = \int_{\sqrt{1-\sqrt{1-c}}}^{\sqrt{1+\sqrt{1-c}}} \frac{\sqrt{2}}{r\sqrt{2r^2 - r^4 - c}}\, dr = \frac{\pi}{\sqrt{2c}}.$$

Then, the lemma follows. ∎

If $2\pi/P(c) = \sqrt{2c} = m/n \in \mathbb{Q}$, for certain $m, n \in \mathbb{N}$ such that $(m, n) = 1$ and $m/n \in (0, \sqrt{2})$, then the corresponding torus H_c is fulfilled of periodic orbits. This happens if $c = m^2/(2n^2)$. Otherwise, it has a dense orbit, see Appendix 1 of [1].

Proposition 5. *Let $m, n \in \mathbb{N}$, $(m,n) = 1$, such that $m/n \in (0, \sqrt{2})$. Let $c = m^2/(2n^2) \in (0,1)$. Then, there exist two symmetric periodic orbits of system (5.2) in the torus H_c with rotation number m/n. These kind of orbits tend to H_0 as n tends to infinity, for fixed m.*

Proof. Let γ_1, γ_2 be the orbits of H_c such that $(0, \sqrt{1+\sqrt{1-c}}, 0) \in \gamma_1$ and $(0, \sqrt{1-\sqrt{1-c}}, 0) \in \gamma_2$. As $c = m^2/(2n^2)$, we have $2\pi n = mP(c)$, so these two orbits, after a time $t = n\pi$, have done exactly $m/2$ turns to the periodic orbit H_1, and then they cut S_φ twice. So they are closed orbits by the property (iv) of reversible systems stated in Section 5.1.

If $n \to \infty$, then $c \to 0$, so γ_1 and γ_2 get closer to H_0 as n increases, for fixed m. ∎

Consider the image of the section $\theta = n\pi \pmod{2\pi}$ by the flow of system (5.2) at time $n\pi$ and let $S_{n\pi}$ be the image of the curve S_φ under the flow of (5.2) at time $n\pi$. In Figure 5.2 we draw $S_{n\pi}$ for $n = 0, 1, 2, 3$. As H_1 is invariant and the origin is a singular point, $S_{n\pi}$ always cuts the axis $\{x = 0, \theta = \pi\}$ at $r = 0$ and $r = 1$.

At time $t = 0$, we have $S_0 = S_\varphi$. At time $t = \pi$, S_π cuts the axis $\{x = 0, \theta = \pi\}$ four times (two of them at $r = 0, 1$), so there exist two symmetric periodic orbits of period 2π. On these orbits, we have $H = 1/2$ because $m = n = 1$, and the rotation number is 1.

At time $t = 2\pi$, $S_{2\pi}$ cuts the axis $\{x = \theta = 0\}$ six times, two of them at $r = 0, 1$ and two of them corresponding to the curves with rotation number 1. So, there exist two symmetric periodic orbits of period 4π, and on these curves $H = 1/8$ (the rotation number is 1/2).

At time $t = 3\pi$, $S_{3\pi}$ cuts the axis $\{x = 0, \theta = \pi\}$ ten times. Two of them are at $r = 0, 1$ and two of them correspond to the curves of rotation number 1, so there are six symmetric periodic orbits of period 6π, and on these curves either $H = 1/18$ or $H = 2/9$ or $H = 8/9$ (the respective rotation numbers are 1/3, 2/3 and 4/3).

As time tends to infinity increasing by multiples of π, we find more symmetric periodic orbits, corresponding to $H = m^2/(2n^2)$, $(m,n) = 1$, $m/n \in (0, \sqrt{2})$. We can ensure that at least two of such orbits exist, because $(1, n) = 1$, for any $n \in \mathbb{N}$.

In the following proposition we prove that, for any $n \in \mathbb{N}$, $S_{n\pi}$ cuts the y–axis transversally.

Proposition 6. *Let $c \in (0,1)$ and $\gamma \in H_c$. Let $P = (0, r_0, 0) \in \gamma \cap S_\varphi$, $r_0 \in (0, \sqrt{2})$. Let Q be the image of P by the flow of system (5.2) at time $n\pi$. Suppose that $Q =\in S_{n\pi}$ is on the y–axis. Then, $S_{n\pi}$ crosses the y–axis at Q transversally.*

Proof. Let $0 < c_1 < c < c_2 < 1$. For $i = 1, 2$, let γ_i be an orbit of system (5.2) such that $H(\gamma_i) = c_i$ and such that there exists $P_i \in \gamma_i \cap S_\varphi$. Let $P_i^n =$

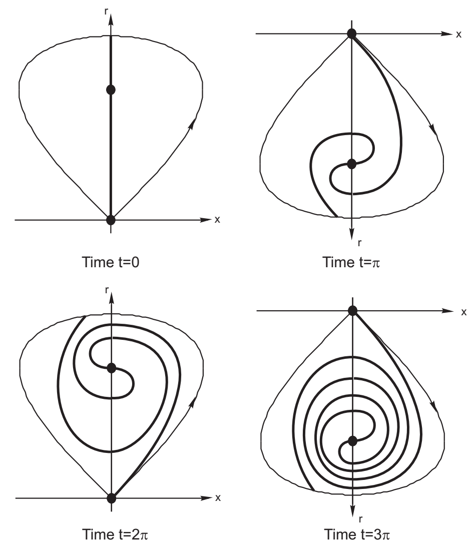

Figure 5.2: The evolution of the set S_φ for some values of the time.

$(x_i^n, r_i^n, n\pi \bmod 2\pi) \in \gamma_i \cap S_{n\pi}$, $i = 1, 2$. The points P_1^n and P_2^n are close to Q if c_1 and c_2 are close to c. Since the period function P is strictly decreasing, $P(c_1) > P(c) > P(c_2)$. Then, as $Q \in S_{n\pi}$, we have $x_1^n \cdot x_2^n < 0$. Taking c_1 and c_2 are as close to c as we want, the proposition follows. ∎

5.3 On Theorem 1

In this section we prove Theorem 1 and we provide a polynomial differential system of degree 4 satisfying all its assumptions.

Proof of Theorem 1: Consider the strongly φ–reversible perturbation (5.6) of system (5.2) defined in Theorem 1. As h is defined in the bounded domain D_3, then it is bounded, so for ε small enough, $\dot\theta = 1 + \varepsilon h > 0$ in D_3. Then, we can take θ as the independent variable to obtain from (5.6) the two–dimensional system

$$\begin{aligned} x' &= F(x^2, r, \varepsilon) = \frac{r^2(1-r^2) + \varepsilon f(x^2, r, \varepsilon)}{1 + \varepsilon h(x^2, r, \varepsilon)}, \\ r' &= x\, G(x^2, r, \varepsilon) = x\, \frac{r + \varepsilon g(x^2, r, \varepsilon)}{1 + \varepsilon h(x^2, r, \varepsilon)}, \end{aligned} \tag{5.11}$$

where $'$ means derivative with respect to θ. This system is reversible with respect to the linear involution ψ defined in (5.10), and it is defined in the bounded domain $\overline{D}_2$ defined in Section 5.2.

By hypotheses, the point $P_1 = (0, 1)$ is a singular point of (5.11). The Jacobian matrix of (5.11) at P_1 is

$$\begin{pmatrix} 0 & \dfrac{-2 + \varepsilon \dfrac{\partial f}{\partial r}(0, 1, \varepsilon)}{1 + \varepsilon h(0, 1, \varepsilon)} \\ \dfrac{1 + \varepsilon g(0, 1, \varepsilon)}{1 + \varepsilon h(0, 1, \varepsilon)} & 0 \end{pmatrix},$$

and it has eigenvalues

$$\pm i \frac{\sqrt{2 - \varepsilon\left(\frac{\partial f}{\partial r}(0, 1, \varepsilon) + 2g(0, 1, \varepsilon)\right) + \varepsilon^2 \frac{\partial f}{\partial r}(0, 1, \varepsilon) g(0, 1, \varepsilon)}}{1 + \varepsilon h(0, 1, \varepsilon)}.$$

So P_1 is either a center or a focus. But, as it belongs to the set S_ψ of fixed points of ψ, it cannot be a focus, so it is a center, and by hypotheses it is the only singular point inside the region $K^{-1}([0, 1]) \setminus \{(0, 0)\}$. So the region $K^{-1}((0, 1))$ is fulfilled of periodic orbits. Going back to (5.6), as $\dot{\theta} > 0$ for ε small enough, the region D_3 is fulfilled of invariant tori, and the theorem follows. ∎

Example 5.3.1. *Consider the differential system of* $\mathbb{R}^3$

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z),$$

where P, Q and R are polynomials of degree 4. *If this system has a singular point at the origin, is strongly φ–reversible with respect to the linear involution* (5.4) *and has H_0 and H_1 invariant, then it becomes, in polar coordinates,*

$$\begin{array}{rcl} \dot{x} & = & a_2 r^2(1 - r^2) + x^2[2a_1 + 2a_3 x^2 - (2a_1 + a_3)r^2], \\ \dot{r} & = & x\, r[2a_1 + a_2 + a_3 x^2 - a_1 r^2], \\ \dot{\theta} & = & c_1 + c_2 x^2 + c_3 r^2. \end{array} \tag{5.12}$$

Observe that system (5.2) *is a particular example of system* (5.12), *just take $a_2 = c_1 = 1$ and $a_1 = a_3 = c_2 = c_3 = 0$. So system* (5.12) *can be written as a perturbation of* (5.2) *(we take $a_2 = c_1 = 1$ for simplicity):*

$$\begin{array}{l} \dot{x} = r^2(1 - r^2) + \varepsilon x^2[2a_1 + 2a_3 x^2 - (2a_1 + a_3)r^2], \\ \dot{r} = x\, r[1 + \varepsilon(2a_1 + a_3 x^2 - a_1 r^2)], \\ \dot{\theta} = 1 + \varepsilon(c_2 x^2 + c_3 r^2). \end{array} \tag{5.13}$$

For this system, $\dot{H} = 4\varepsilon x(a_1(1 - r^2) + a_3 x^2)H$, where H is the function defined in (5.3). *For ε small enough, system* (5.13) *can be written as the two–dimensional system*

$$x' = \frac{r^2(1 - r^2) + \varepsilon x^2[2a_1 + 2a_3 x^2 - (2a_1 + a_3)r^2]}{1 + \varepsilon(c_2 x^2 + c_3 r^2)},$$

$$r' = x\, r\, \frac{1 + \varepsilon(2a_1 + a_3 x^2 - a_1 r^2)}{1 + \varepsilon(c_2 x^2 + c_3 r^2)}.$$

For ε small enough, $(0,0)$ and $(0,1)$ are the only singular points of this system in the region $K^{-1}([0,1])$, where K is the function defined in (5.9). Then, Theorem 1 can be applied. So, for system (5.13), the region D_3 is fulfilled of invariant tori.

5.4 On Theorem 2

In this section we prove Theorem 2 and we provide a polynomial differential system of degree 4 satisfying all its assumptions.

Proof of Theorem 2: According to Proposition 5, there exist two symmetric periodic orbits γ_1 and γ_2 of period $2\pi n$ and rotation number m/n of system (5.2) in the torus H_c, $c = m^2/(2n^2)$. Each one of these symmetric periodic orbits cut transversally the y–axis twice, see Proposition 6. Then, due to the theorem of continuous dependence of the solution of an ODE with respect to initial conditions and parameters, there exist $\varepsilon_n > 0$ such that, for an ε–perturbation (5.7) of system (5.2), with $\varepsilon \in (0, \varepsilon_n)$, two symmetric periodic orbits γ_1^ε, γ_2^ε appear ε–close to γ_1 and γ_2, respectively, and $\gamma_i^\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} \gamma_i$, $i = 1, 2$. So, Theorem 2 is proved. ∎

From the proof of Theorem 2 it follows immediately Corollary 3.

Example 5.4.1. *Consider the differential system of $\mathbb{R}^3$*

$$\dot{x} = P(x,y,z), \quad \dot{y} = Q(x,y,z), \quad \dot{z} = R(x,y,z),$$

where P, Q and R are polynomials of degree 4. If this system has a singular point at the origin, is not strongly φ–reversible with respect to the linear involution (5.4) and has H_0 and H_1 invariant, then it becomes, in polar coordinates,

$$\begin{aligned}
\dot{x} &= x^2[2a_1 + 2a_3x^2 - (2a_1 + a_3)r^2] + a_5x\,r(2x^2 - r^2)\sin\theta + \\
&\quad\; r(1 - r^2)[a_2r + a_6\cos\theta + a_7r\cos^2\theta + a_4x\sin\theta], \\
\dot{r} &= x\,r[2a_1 + a_2 + a_3x^2 - a_1r^2] + a_5x^2r^2\sin\theta + \\
&\quad\; x[a_6\cos\theta + a_7r\cos^2\theta + a_4x\sin\theta], \\
\dot{\theta} &= c_1 + c_2x^2 + c_3r^2 + [c_4r + c_5x^2r + c_6r^3]\cos\theta + c_7r^2\cos^2\theta + \\
&\quad\; c_8r^3\cos^3\theta + c_9xr\sin\theta - (a_7 - c_{10}r^2)x\cos\theta\sin\theta,
\end{aligned}$$

with $\dot{\theta} \neq 0$. In this case, $\dot{H} = 4x(a_1(1 - r^2) + a_3x^2 + a_5xr\sin\theta)H$. Observe that if $a_4 = a_5 = a_6 = a_7 = 0$ and $c_4 = \cdots = c_{10} = 0$, we get system (5.12), and we get system (5.2) if we take $a_2 = c_1 = 1$ and the rest of the coefficients zero. So, we obtain an example of (5.7) taking $a_2 = c_1 = 1$ and multiplying the rest of the *coefficients by ε:*

$$\begin{aligned}
\dot{x} &= \varepsilon\big[x^2[2a_1 + 2a_3x^2 - (2a_1 + a_3)r^2] + a_5x\,r(2x^2 - r^2)\sin\theta + \\
&\quad r(1 - r^2)[r + a_6\cos\theta + a_7r\cos^2\theta + a_4x\sin\theta]\big], \\
\dot{r} &= x\,r + \varepsilon\big[x\,r[2a_1 + a_3x^2 - a_1r^2] + a_5x^2r^2\sin\theta + \\
&\quad x[a_6\cos\theta + a_7r\cos^2\theta + a_4x\sin\theta]\big], \\
\dot{\theta} &= 1 + \varepsilon\big[c_2x^2 + c_3r^2 + [c_4r + c_5x^2r + c_6r^3]\cos\theta + c_7r^2\cos^2\theta + \\
&\quad c_8r^3\cos^3\theta + c_9xr\sin\theta - (a_7 - c_{10}r^2)x\cos\theta\sin\theta\big].
\end{aligned}$$

Then, Theorem 2 can be applied to this system. So, there exist an arbitrary number of symmetric periodic orbits for this system in the region D_3.

5.5 The Poincaré map $\widetilde{T}_{n,\varepsilon}$

Let $A = I \times \Sigma^1$ be an annulus, where I is a closed interval. We denote its coordinates by $c \in I$ and $t \pmod{2\pi} \in \Sigma^1$. A *twist map* is a $\mathbb{C}^1$–map $T : A \to A$ such that

$$T(c,t) = \big(c, t + \tau(c) \pmod{2\pi}\big) \tag{5.14}$$

for a certain $\tau(c)$ and such that $\tau'(c)$ is strictly increasing or decreasing.

We extend the notion of twist map as follows. Let $\widetilde{A}$ be an annulus and let $\widetilde{T} : \widetilde{A} \to \widetilde{A}$ be a $\mathbb{C}^1$–map such that the diagram

$$\begin{array}{ccc}
A & \xrightarrow[T]{} & A \\
h \downarrow & & \downarrow h \\
\widetilde{A} & \xrightarrow[\widetilde{T}]{} & \widetilde{A}.
\end{array}$$

commutes, where h is a $\mathbb{C}^1$–diffeomorphism. Then, we also say that $\widetilde{T}$ is a *twist map*.

Let $\widetilde{A}$ be the annulus of $D_3 \cap \{\theta = 0\}$ having boundaries $H_{c_i} \cap \{\theta = 0\}$, where

$$\frac{2\pi}{P(c_i)} = \sqrt{2c_i} \in \mathbb{Q},$$

for $i = 1, 2$, $c_1, c_2 \in (0,1)$ and $c_1 < c_2$. Let $(x(t), r(t))$ be the periodic solution living on $H_c \cap \{\theta = 0\} \subset A$. Suppose that $r(0) = 1$. Then, we define $\widetilde{T}_n : \widetilde{A} \to \widetilde{A}$ by

$$\widetilde{T}_n(x(t), r(t)) = (x(t + 2\pi n), r(t + 2\pi n)).$$

Proposition 7. *The function $\widetilde{T}_n$ is a twist map in $\widetilde{A}$.*

Proof. We define A as the annulus formed by the points $\big(c, t \pmod{2\pi}\big)$, with $c_1 \le c \le c_2$ and $t \in \mathbb{R}$. Let $T_n : A \to A$ be given by

$$T_n(c,t) = (c, t + \tau(c)) = (c, t + 2\pi n \pmod{P(c)}).$$

Since $P(c)$ is strictly decreasing, $\tau(c)$ is strictly increasing. Finally, defining

$$h(c, t \pmod{P(c)}) = (x(t + 2\pi n), r(t + 2\pi n))$$

if $(x(t), r(t))$ is the periodic solution such that $H(x(t), r(t)) = c$, it follows that the diagram

$$\begin{array}{ccc} A & \xrightarrow[T_n]{} & A \\ h\downarrow & & \downarrow h \\ \widetilde{A} & \xrightarrow[\widetilde{T}_n]{} & \widetilde{A}. \end{array}$$

commutes. So, $\widetilde{T}_n$ is a twist map. ■

Let $\widetilde{T}_n$ be the twist map (5.14) and let $\widetilde{T}_{n,\varepsilon} : A \to A$ be defined by

$$\widetilde{T}_{n,\varepsilon}(c,t) = \big(c + \varepsilon f(c,t,\varepsilon), t + \tau(c) + \varepsilon g(c,t,\varepsilon)\big),$$

where f and g are $\mathbb{C}^1$–functions and ε is a small parameter. If $\widetilde{T}_{n,\varepsilon}$ is area–preserving, then the Poincaré–Birkhoff Theorem says that $\widetilde{T}_{n,\varepsilon}$ has two different periodic orbits for each rational number between the rotation numbers of $\widetilde{T}_{n,\varepsilon}$ on the boundary components of the annulus A. The theorem was conjectured by Poincaré (see [8]) and proved by Birkhoff (see [2] and [3]) and Brown and von Newmann (see [4]). Other more recent proofs have weakened the area–preservation hypotheses (see [6]).

We say that a periodic orbit is hyperbolic if the jacobian of the Poincaré map on this orbit has not pure imaginary eigenvalues, and they are different from 1 and -1. If all the eigenvalues are pure imaginary and the matrix diagonalizes, then the periodic orbit is elliptic.

A version of the Poincaré–Birkhoff Theorem is the following one, using the notation introduced in Proposition 7.

Theorem 8 (Poincaré–Birkhoff Theorem). *Let Γ be an invariant curve of $\widetilde{T}_n$, $n \in \mathbb{N}$, formed by fixed points of $(\widetilde{T}_n)^s$, where s is the denominator of $\sqrt{2c} \in \mathbb{Q}$. If $\widetilde{T}_{n,\varepsilon}$ preserves area, then for $\varepsilon > 0$ sufficiently small the map $(\widetilde{T}_{n,\varepsilon})^s$ has $2ks$ fixed points ($k \in \mathbb{N}$) in the neighborhood of the curve Γ, half of them are elliptic and the other half are hyperbolic.*

Note that if our $\widetilde{T}_{n,\varepsilon}$ preserves area, then the two periodic orbits of Theorem 2 correspond to the two periodic orbits of the Poincaré–Birkhoff Theorem. But we do not know in general that our $\widetilde{T}_{n,\varepsilon}$ preserves the area, or that $\widetilde{T}_{n,\varepsilon}$ satisfies other assumptions for which the theses of the Poincaré–Birkhoff Theorem hold.

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Chapter 6

Hyperbolic periodic orbits coming from the bifurcation of a 4–dimensional non–linear center

In this third article, we study the bifurcation of hyperbolic periodic orbits from a 4–dimensional non–linear center in a class of differential systems. The tool for proving these results is the averaging theory. This is a joint work with Jaume Llibre and Marco António Teixeira, and it has been accepted for publication in *International Journal of Bifurcations and Chaos*.

6.1 Introduction

Consider the real differential system

$$\ddot{x} = -x\,(p + q\,x)^{-3}, \quad \ddot{y} = -y\,(p + q\,x)^{-3}, \tag{6.1}$$

where p and q are real parameters. The overdot indicates derivative with respect to time t. This system was studied by Barone–Netto and Cesar in [1], as an example of a non–trivial stable system of the form $\ddot{x} = -x\,f(x), \ddot{y} = -y\,f(x)$. In that work, it was proved that, for $p > 0$, the origin is stable and any trajectory projected into the (x, y)–plane is a conic.

Let $\dot{x} = u$, $\dot{y} = v$. Then, system (6.1) can be transformed into the $\mathbb{R}^4$ differential system

$$\dot{x} = u, \quad \dot{u} = -x\,(p + q\,x)^{-3}, \quad \dot{y} = v, \quad \dot{v} = -y\,(p + q\,x)^{-3}. \tag{6.2}$$

We say that a singular point of system (6.2) is a *center* if it has a neighborhood where all the orbits except the singular point are periodic.

By the change of time $dt = (p + q\,x)^3\,ds$, we obtain the following polynomial system of degree 4, defined in the whole $\mathbb{R}^4$:

$$\dot{x} = u\,(p + q\,x)^3, \quad \dot{u} = -x, \quad \dot{y} = v\,(p + q\,x)^3, \quad \dot{v} = -y. \tag{6.3}$$

Now, the dot denotes derivative with respect to s. This system is equivalent to (6.2) outside the hyperplane $p+q\,x=0$. Since we shall work near the origin and in the case $p\neq 0$, system (6.2) has a center if and only if system (6.3) has a center.

Our first main result is the following theorem, which will be proved in Section 6.2.

Theorem 1. *Differential system* (6.3) *has a center at the origin if and only if* $p>0$.

Since in Mechanics the systems of the form

$$\ddot{x}=-x\,p(x,y),\quad \ddot{y}=-y\,q(x,y)$$

have some relevance, we want to study perturbations of system (6.1) by systems of the form

$$\begin{aligned}\ddot{x}&=-x\,(p+q\,x)^{-3}(1+\varepsilon^2\,g_2(x,\dot{x},y,\dot{y})+\varepsilon\,g_4(x,\dot{x},y,\dot{y})),\\ \ddot{y}&=-y\,(p+q\,x)^{-3}(1+\varepsilon^2\,h_2(x,\dot{x},y,\dot{y})+\varepsilon\,h_4(x,\dot{x},y,\dot{y})),\end{aligned}\tag{6.4}$$

where g_i and h_i are homogeneous polynomials of degree i, for $i=2,4$. We remark that the perturbed system we are presenting is the one of minimum degree from which hyperbolic periodic orbits can be obtained using the first order theory of averaging of Section 6.3, as we will show in Section 6.4.

The same arguments used for passing from system (6.1) to system (6.3) can be applied to system (6.4) to transform it into

$$\begin{aligned}\dot{x}&=u\,(p+q\,x)^3,\\ \dot{u}&=-x(1+\varepsilon^2\,g_2(x,u,y,v)+\varepsilon\,g_4(x,u,y,v)),\\ \dot{y}&=v\,(p+q\,x)^3,\\ \dot{v}&=-y(1+\varepsilon^2\,h_2(x,u,y,v)+\varepsilon\,h_4(x,u,y,v)).\end{aligned}\tag{6.5}$$

A *hyperbolic periodic orbit* of system (6.5) is an isolated periodic orbit in the set of all periodic orbits of (6.5). The *Poincaré map* (or, equivalently, the *displacement map*) is a good tool for studying the hyperbolic periodic orbits of autonomous systems (for more details, see [4, 5] and also the end part of Section 6.3). We recall that a hyperbolic periodic orbit of a system corresponds to an isolated zero of its displacement function.

We study how many hyperbolic periodic orbits can bifurcate from the center of system (6.3) when $p>0$. Our main result is the following.

Theorem 2. *Suppose that* $p>0$. *Using the first order averaging method applied to system* (6.5) *we can obtain at most* 16 *hyperbolic periodic orbits bifurcating from the periodic orbits of the center of system* (6.3). *Moreover, there are systems* (6.5) *having exactly* $0,1,\ldots,16$ *hyperbolic periodic orbits.*

We use the averaging theory for proving Theorem 2, see Section 6.3. For additional results in the use of the averaging theory for computing periodic orbits, see [2, 7]. In general, it is not easy to find a change of variables to pass from a given differential system to its normal form by applying the averaging method for finding periodic orbits. In particular, it is not easy to apply the averaging method for studying the hyperbolic periodic orbits bifurcating from the periodic orbits of a center, mainly if the center is non–linear; for 2–dimensional systems see [7, 11]; for higher dimensional systems see [6, 3, 8]. The general idea is to relate this change of variables to the first integrals of the center.

6.2 Characterization of the center

Let U be an open set of $\mathbb{R}^4$. A $\mathcal{C}^1$ function $H : U \to \mathbb{R}$ is a *first integral* of system (6.3) if it is constant on the solutions of (6.3) contained in U. In other words, if

$$X = u(p+q\,x)^3\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} + v(p+q\,x)^3\frac{\partial}{\partial y} - y\frac{\partial}{\partial v}$$

is the vector field associated to system (6.3), then H is a first integral if and only if $X\,H = 0$ in U.

System (6.3) has three functionally independent first integrals; one of them corresponds to the energy of the mechanical system in the (x, u)–plane:

$$H_1(x, u) = \frac{u^2}{2} + \frac{x^2}{2p\,(p+q\,x)^2}; \tag{6.6}$$

another one corresponds to the angular momentum:

$$H_2(x, u, y, v) = v\,x - u\,y; \tag{6.7}$$

and the last one is

$$H_3(x, u, y, v) = u\,v(p+q\,x) - q\,u^2\,y + \frac{x\,y}{(p+q\,x)^2}. \tag{6.8}$$

Of course, (6.6), (6.7) and (6.8) are also first integrals of system (6.2).

We note that system (6.3) is invariant under the symmetry

$$(x, u, y, v, t) \to (x, -u, -y, v, -t).$$

If $q = 0$, then system (6.3) is a linear center. If $p = 0$, then the plane $x = y = 0$ of $\mathbb{R}^4$ is full of singular points, and then the origin is not an isolated singular point, so it cannot be a center. As we are interested in studying non–linear centers in $\mathbb{R}^4$, we take $p\,q \neq 0$. In this case, the system becomes easier.

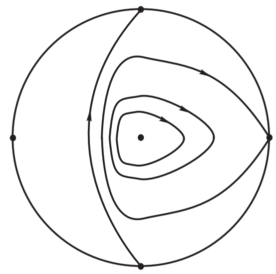

Figure 6.1: Phase portrait of system (6.10) with a plus on the Poincaré disc. The period annulus of the center is the region inside the homoclinic loop of the singular point at infinity. The vertical straight line corresponds to $x = -1$.

Lemma 3. *If* $p\,q \neq 0$, *then system* (6.3) *is topologically equivalent to*

$$\dot{x} = u(x \pm 1)^3, \quad \dot{u} = -x, \quad \dot{y} = v(x \pm 1)^3, \quad \dot{v} = -y, \tag{6.9}$$

where we take the $\pm$ *sign to be* $+$ *if* $p > 0$, *or* $-$ *if* $p < 0$.

Proof. We scale the variables and the time as follows

$$x \to \frac{|p|}{q}x, \quad u \to \frac{u}{q\sqrt{|p|}}, \quad y \to \frac{|p|}{q}y, \quad v \to \frac{v}{q\sqrt{|p|}}, \quad t \to \frac{t}{\sqrt{|p|^3}}.$$

Then, the lemma follows. □

Next we prove that, in order to have a center at the origin, p must be positive.

Proposition 4. *Consider the system*

$$\dot{x} = u(x \pm 1)^3, \quad \dot{u} = -x, \tag{6.10}$$

which corresponds to system (6.9) *restricted to the plane* (x, u), *which is invariant. Then, the following statements hold.*

(a) If in (6.10) *we have a minus, then system* (6.9) *has no center at the origin.*

(b) If in (6.10) *we have a plus, then* (6.10) *has a center at the origin (see Figure 1).*

Proof. The origin is a singular point of system (6.10) with eigenvalues $\pm\sqrt{\mp 1}$. If in (6.10) we have a minus, these eigenvalues are real numbers, and then the origin is a saddle of (6.10). Clearly, in this case, (6.9) cannot have a center. If in (6.10) we have a plus, then the eigenvalues are pure imaginary numbers, and then the origin is either a focus or a center for system (6.10). But the function $H_1(x, u)$ defined in (6.6) with $p = q = 1$ is a first integral of this system defined in a neighborhood of the origin. Therefore, the origin cannot be a focus, so it is a center. □

We note that Proposition 4 proves that if system (6.9) has a center at the origin, then it has the sign plus. That is, using Lemma 3, we have proved the "if" part of Theorem 1.

In order to prove the "only if" part of Theorem 1 the key point will be to show that the projection on the (x, y)–plane of any trajectory of system (6.9) is a conic, and that it is an ellipse if it is sufficiently close to the origin. Before the proof, we give a lemma which will simplify our computations.

Lemma 5. *Let Γ be a trajectory of system* (6.9) *different from the origin for which H_1 is defined. Let γ be its projection in the (x, u)–plane. Assume that γ is in the period annulus of the origin for system* (6.10)*. Then, there exists a point $(x_0, u_0, y_0, v_0) \in \Gamma$ such that $u_0 = 0$.*

Proof. Let $P_1 = (x_1, u_1, y_1, v_1)$ be a point of Γ, and let $h_1 = H_1(x_1, u_1)$. Let x_0 be a solution of the equation

$$\frac{u_1^2}{2} + \frac{x_1^2}{2(1+x_1)^2} = h_1 = \frac{x_0^2}{2(1+x_0)^2}.$$

Then, $(x_0, 0) \in \gamma$ is the projection of a point $P_0 = (x_0, 0, y_0, v_0) \in \Gamma$ in the (x, u)–plane, where y_0 and v_0 can be obtained explicitly from the equations $H_2(P_1) = H_2(P_0)$ and $H_3(P_1) = H_3(P_0)$. $\square$

Proof of Theorem 1. The theorem will be proved if we show that the projection of any trajectory different from the origin and sufficiently close to it in the (x, y)–plane is an ellipse. That is due to the fact that if $x(t)$ and $u(t)$ are periodic functions with the same minimal period T in a neighborhood of the origin (see Proposition 4), then the ellipse $(x(t), y(t))$ implies that $y(t)$ is periodic of period T. Using the last equation of (6.9) it will be proved that $v(t)$ is periodic of period T.

Let Γ be a trajectory of system (6.9) different from the origin such that its projection into the (x, u)–plane, denoted by γ_0, is in the period annulus of the center of system (6.10). Let γ be the projection of Γ into the (x, y)–plane and $P_0 = (x_0, 0, y_0, v_0)$ a point of Γ (see Lemma 5).

If $x_0 = 0$, then the three first integrals vanish on P_0, and then Γ is the origin. So we can assume $x_0 \neq 0$. Moreover, as γ_0 is surrounding the origin for system (6.10), it cuts the straight line $u = 0$ at two points, one of them with positive x–coordinate. Then, we can assume $x_0 > 0$.

Since the hyperplane $1 + x = 0$, where the first integrals H_1 and H_3 are not defined, is far enough from the origin, we can choose the curve Γ contained in

$x > -1$. For the points of Γ we have $H_1(x,u) = H_1(x_0,0)$ and $H_i(x,u,y,v) = H_i(P_0)$, for $i = 2,3$; that is,

$$
\begin{aligned}
u^2 + \frac{x^2}{(1+x)^2} - \frac{x_0^2}{(1+x_0)^2} &= 0, \\
v\,x - u\,y - v_0\,x_0 &= 0, \\
u\,v(1+x) - u^2\,y + \frac{x\,y}{(1+x)^2} - \frac{x_0\,y_0}{(1+x_0)^2} &= 0.
\end{aligned}
\tag{6.11}
$$

Since $x_0 > 0$, it follows that Γ cannot be contained in the hyperplane $x = 0$. For any point $(x,u,y,v) \in \Gamma$, $x \neq 0$, we can isolate u^2 and v in terms of x and y from the first and the second equations of (6.11), respectively:

$$
u^2 = \frac{(x_0 - x)(x - x_0 + 2x_0\,x)}{(1+x)^2(1+x_0)^2}, \quad v = \frac{v_0\,x_0 + u\,y}{x}.
$$

From the third equation of (6.11) we obtain the equation of γ:

$$
(y_0\,x - x_0\,y)^2 + v_0^2(x - x_0)(1+x_0)^2(x - x_0 + 2x_0\,x) = 0. \tag{6.12}
$$

This curve is a conic in the variables x and y. In order to prove that if γ is close enough to the origin, then it is an ellipse, we must prove that the determinant

$$
\begin{vmatrix}
y_0^2 + v_0^2(1+x_0)^2(1+2x_0) & -x_0y_0 & -v_0^2x_0^2(1+x_0)^2 \\
-x_0y_0 & x_0^2 & 0 \\
-v_0^2x_0^2(1+x_0)^2 & 0 & -v_0^2x_0^2(1+x_0)^2
\end{vmatrix}
= -v_0^4x_0^2(1+x_0)^6
$$

is not zero, and that its 2×2 minor

$$
\begin{vmatrix}
y_0^2 + v_0^2(1+x_0)^2(1+2x_0) & -x_0y_0 \\
-x_0y_0 & x_0^2
\end{vmatrix}
= v_0^2x_0^2(1+x_0)^2(1+2x_0)
$$

is positive.

If $v_0 = 0$, then (6.12) is the straight line $y = y_0\,x/x_0$, which is far from the origin. If $v_0 \neq 0$, then we can take x_0 sufficiently small such that both determinants are different from zero and the second one is positive. Then, γ is an ellipse and the theorem follows. □

6.3 First order averaging method for periodic orbits

We consider the differential system

$$
\dot{x}(t) = \varepsilon F(t, x(t)) + \varepsilon^2 R(t, x(t), \varepsilon), \tag{6.13}
$$

with $x \in D \subset \mathbb{R}^n$, D a bounded domain and $t \geq 0$. We assume that $F(t,x)$ and $R(t,x,\varepsilon)$ are T–periodic in t.

The averaged system associated to system (6.13) is defined by

$$\dot{y}(t) = \varepsilon f(y(t)), \tag{6.14}$$

where

$$f(y) = \frac{1}{T}\int_0^T F(s,y)ds. \tag{6.15}$$

The following theorem shows under which sufficient conditions the singular points of the averaged system (6.14) provide T–periodic orbits for system (6.13). For a proof see Theorem 2.6.1 of [10], Theorems 11.5 and 11.6 of [11], and Theorem 4.1.1 of [5].

Theorem 6. *We consider system* (6.13) *and assume that the vector functions F, R, D_xF_1, $D_x^2F_1$ and D_xR are continuous and bounded by a constant M (independent of ε) in $[0,\infty) \times D$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover, we suppose that F and R are T–periodic in t, with T independent of ε.*

(a) If $a \in D$ is a singular point of the averaged system (6.14) *such that the determinant of $D_xf(a)$ is different from zero then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $x_\varepsilon(t)$ of system* (6.13) *such that*

$$x_\varepsilon(t) \underset{\varepsilon\to 0}{\longrightarrow} a.$$

(b) If the singular point $y = a$ of the averaged system (6.14) *is hyperbolic then, for $|\varepsilon| > 0$ sufficiently small, the corresponding periodic solution $x_\varepsilon(t)$ of system* (6.13) *is unique, hyperbolic and of the same stability type as a.*

For every $z \in D$, we denote by $x(\cdot,z,\varepsilon)$ the solution of (6.13) with the initial condition $x(0,z,\varepsilon) = z$. We also consider the function $\zeta : D \times (-\varepsilon_0,\varepsilon_0) \to \mathbb{R}^n$, defined by

$$\zeta(z,\varepsilon) = \int_0^T \left[\varepsilon F(t,x(t,z,\varepsilon)) + \varepsilon^2 R(t,x(t,z,\varepsilon),\varepsilon)\right] dt. \tag{6.16}$$

From (6.13) it follows that, for every $z \in D$,

$$\zeta(z,\varepsilon) = x(T,z,\varepsilon) - x(0,z,\varepsilon). \tag{6.17}$$

The function ζ can be written in the form

$$\zeta(z,\varepsilon) = \varepsilon T f(z) + \varepsilon^2 O(1), \tag{6.18}$$

where f is given by (6.15) and the symbol $O(1)$ denotes a bounded function on every compact subset of $D \times (-\varepsilon_0,\varepsilon_0)$. Moreover, for $|\varepsilon|$ sufficiently small, $z = x_\varepsilon(0)$ is an isolated zero of $\zeta(\cdot,\varepsilon)$. Of course, due to (6.17) the function ζ is a *displacement* function for system (6.13), and its fixed points are the T–periodic solutions of (6.13).

6.4 Proof of Theorem 2

We study in this section hyperbolic periodic orbits bifurcating from the periodic orbits of the 4–dimensional centerr

$$\dot{x} = u(x+1)^3, \quad \dot{u} = -x, \quad \dot{y} = v(x+1)^3, \quad \dot{v} = -y, \tag{6.19}$$

which corresponds to system (6.3) for $p > 0$, see Lemma 3. We perturbe system (6.3) as follows

$$\begin{aligned} \dot{x} &= u(x+1)^3, \quad \dot{u} = -x[1+\varepsilon^2 g_2(x,u,y,v) + \ g_4(x,u,y,v)], \\ \dot{y} &= v(x+1)^3, \quad \dot{v} = -y[1+\varepsilon^2 h_2(x,u,y,v) + \varepsilon h_4(x,u,y,v)], \end{aligned} \tag{6.20}$$

where g_i and h_i are homogeneous polynomials of degree i in the variables x, u, y, v, for $i = 2, 4$. The coefficient of $x^i u^j y^k v^l$ in g (respectively, h) will be denoted by a_{ijkl} (respectively b_{ijkl}).

Let Γ be a closed trajectory of (6.19). Assume that the three first integrals of (6.19),

$$\begin{aligned} h_1(x,u) &= u^2 + \frac{x^2}{(x+1)^2}, \\ h_2(x,u,y,v) &= v\,x - u\,y, \\ h_3(x,u,y,v) &= u\,v(x+1) - u^2\,y + \frac{x\,y}{(x+1)^2}, \end{aligned} \tag{6.21}$$

take the values $h_1 > 0$, $h_2 \neq 0$ and $h_3 \in \mathbb{R}$ on Γ. Then, we can write x, u, y, v in terms of h_1, h_2, h_3 and a new variable θ as

$$x = \rho\cos\theta, \quad u = \frac{\rho(h_3\cos\theta - 2h_1\sin\theta)}{h_2(1+\rho\cos\theta)},$$

$$y = \rho\sin\theta, \quad v = \frac{(h_2^2(1-2h_1)+h_3^2)\rho\cos\theta - 2h_1(h_2^2 + h_3\rho\sin\theta)}{2h_1h_2(1+\rho\cos\theta)},$$

where

$$\rho = \frac{\sqrt{2h_1}h_2}{\sqrt{h_2^2\cos^2\theta + (h_3\cos\theta - 2h_1\sin\theta)^2} - \sqrt{2h_1}h_2\cos\theta}.$$

In the new variables, system (6.20) writes

$$\dot{h}_s = \frac{\partial h_s}{\partial t}, \ s = 1,2,3, \quad \dot{\theta} = \Omega(h_1, h_2, h_3, \theta, \varepsilon),$$

for a certain function Ω; or, equivalently,

$$h_s' = \frac{\partial h_s}{\partial \theta} = \frac{\partial h_s}{\partial t}\frac{\partial t}{\partial \theta} = \frac{r^2 \dot{h}_s}{\dot{y}x - \dot{x}y},$$

Number of singular points	rank(A)	Number of independent solutions
$n \geq 21$	58	32
$17 \leq n \leq 20$	$2n+17$	$73-2n$
$1 \leq n \leq 16$	$3n$	$3(30-n)$

Table 6.1: Relation between the number of singular points of (6.23) and the number of independent solutions of the linear system $A\mathbf{x}=0$.

for $s=1,2,3$. Taking $h_s=\varepsilon k_s$, this system is transformed into

$$\begin{aligned} k_1' &= \varepsilon^4 F_1(\theta,k_1,k_2,k_3)+\varepsilon^{9/2}R_1(\theta,k_1,k_2,k_3,\varepsilon),\\ k_2' &= \varepsilon^4 F_2(\theta,k_1,k_2,k_3)+\varepsilon^{9/2}R_2(\theta,k_1,k_2,k_3,\varepsilon),\\ k_3' &= \varepsilon^4 F_3(\theta,k_1,k_2,k_3)+\varepsilon^{9/2}R_3(\theta,k_1,k_2,k_3,\varepsilon), \end{aligned} \tag{6.22}$$

for some functions F_s, R_s, with $s=1,2,3$. Applying the averaging theory described in Section 6.3, we obtain the system

$$y_1' = \varepsilon^4 f_1(y_1,y_2,y_3),\ y_2' = \varepsilon^4 f_2(y_1,y_2,y_3),\ y_3' = \varepsilon^4 f_3(y_1,y_2,y_3), \tag{6.23}$$

where the functions f_s are given in Section 6.5.

Monomials of odd degree in the functions g_i and h_i would vanish after applying the averaging theory, so we do not consider them in the perturbed system. If we consider the perturbed system

$$\begin{aligned} \dot{x} &= u(x+1)^3, \quad \dot{u} = -x[1+\varepsilon g(x,u,y,v)],\\ \dot{y} &= v(x+1)^3, \quad \dot{v} = -y[1+\varepsilon h(x,u,y,v)], \end{aligned}$$

where g and h are homogeneous polynomials of the same degree, then the corresponding f_s, for $s=1,2,3$, of the averaged system are also homogeneous polynomials. By the Euler Theorem, we have

$$\sum_{j=1}^{3} y_s \frac{\partial f_s}{\partial y_j} = \deg{(f_s)} f_s,$$

for $s=1,2,3$. So, if $f_s(y_1^0,y_2^0,y_3^0)=0$, $s=1,2,3$, there exists a non–zero linear combination of the columns of the Jacobian matrix at (y_1^0,y_2^0,y_3^0) which vanishes. Then, its determinant is zero and we cannot apply the averaging theory. So, the easiest perturbation (i.e., with the lower degree) that can be considered is the one that we are using.

We find the isolated singular points of system (6.23) in the following way. Let (y_1^r,y_2^r,y_3^r), for $r=1,\dots,n$, be n singular points of the averaged system

(6.23). Substituting these points in $f_s(y_1, y_2, y_3)$ for $s = 1, 2, 3$, we get a linear homogeneous system with a matrix A of dimension $3n \times 90$. Of course, the 90 unknowns are the coefficients a_{ijkl} and b_{ijkl} of g_2, g_4, h_2 and h_4. Using the algebraic manipulator *Mathematica* (see [9]), we can compute the rank of the matrix A as a function of n. The results are given in Table 6.1.

We note that when $n \geq 21$, substituting the 58 depending unknowns, determined as a function of the 32 independent unknowns, in the functions $f_s(y_1, y_2, y_3)$, then the three functions become identically zero. So, we cannot apply the averaging method to determine hyperbolic periodic orbits because the Jacobian matrix is identically zero at the singular points.

When $n \in \{17, 18, 19, 20\}$, substituting the $2n + 17$ depending unknowns, determined as a function of the $73 - 2n$ independent unknowns, in the function $f_1(y_1, y_2, y_3)$, then it becomes identically zero. Hence, we cannot apply the averaging method to determine hyperbolic periodic orbits because the Jacobian matrix has a row identically zero at the singular points.

An example of system (6.23) having 16 singular points with non–zero Jacobian is given in Section 6.6. Therefore, system (6.20) for the values of Section 6.6 has 16 hyperbolic periodic orbits, for $\varepsilon > 0$ sufficiently small, bifurcating from the center of system (6.20) with $\varepsilon = 0$. In a similar way, we can obtain systems (6.20) with $0, 1, \ldots, 14$ or 15 hyperbolic periodic orbits bifurcating from the periodic orbits of the center of system (6.20) with $\varepsilon = 0$.

In short, Theorem 2 is proved. $\square$

6.5 The functions f_1, f_2 and f_3

We give in this section the expression of the functions f_1, f_2 and f_3 of system (6.23):

$$
\begin{aligned}
f_1(y_1, y_2, y_3) = -\frac{1}{64y_1}(&16a_{3100}y_1^4 + 16a_{1300}y_1^4 + 32a_{1100}y_1^3 - 8a_{3010}y_1^3y_2 + 8a_{2101}y_1^3y_2 - \\
&8a_{1210}y_1^3y_2 + 8a_{0301}y_1^3y_2 + 8a_{3001}y_1^3y_3 + 8a_{2110}y_1^3y_3 + 8a_{1201}y_1^3y_3 + 8a_{0310}y_1^3y_3 - \\
&4a_{2011}y_1^2y_2^2 + 4a_{1120}y_1^2y_2^2 + 4a_{1102}y_2^2y_1^2 - 4a_{0211}y_2^2y_1^2 + 4a_{2011}y_3^2y_1^2 + 4a_{1120}y_3^2y_1^2 + \\
&4a_{1102}y_3^2y_1^2 + 4a_{0211}y_3^2y_1^2 + 16a_{0101}y_2y_1^2 - 16a_{1010}y_2y_1^2 + 16a_{0110}y_3y_1^2 + \\
&16a_{1001}y_3y_1^2 - 8a_{2020}y_2y_3y_1^2 + 8a_{2002}y_2y_3y_1^2 - 8a_{0220}y_2y_3y_1^2 + 8a_{0202}y_2y_3y_1^2 - \\
&2a_{1030}y_2^3y_1 - 2a_{1012}y_2^3y_1 + 2a_{0103}y_2^3y_1 + 2a_{0121}y_2^3y_1 + 2a_{1003}y_3^3y_1 + 2a_{1021}y_3^3y_1 + \\
&2a_{0130}y_3^3y_1 + 2a_{0112}y_3^3y_1 - 8a_{0011}y_2^2y_1 + 8a_{0011}y_3^2y_1 - 6a_{1030}y_2y_3^2y_1 + 2a_{1012}y_2y_3^2y_1 + \\
&6a_{0103}y_2y_3^2y_1 - 2a_{0121}y_2y_3^2y_1 + 6a_{1003}y_2^2y_3y_1 - 2a_{1021}y_2^2y_3y_1 + 6a_{0130}y_2^2y_3y_1 - \\
&2a_{0112}y_2^2y_3y_1 - 16a_{0020}y_2y_3y_1 + 16a_{0002}y_2y_3y_1 - a_{0031}y_2^4 - a_{0013}y_2^4 + a_{0031}y_3^4 + \\
&a_{0013}y_3^4 - 4a_{0040}y_2y_3^3 + 4a_{0004}y_2y_3^3 - 4a_{0040}y_2^3y_3 + 4a_{0004}y_2^3y_3),
\end{aligned}
$$

$$
\begin{aligned}
&f_2(y_1,y_2,y_3) = \frac{1}{256y_1^2}(a_{0031}y_2^5 + a_{0013}y_2^5 - b_{0031}y_2^5 - b_{0013}y_2^5 + 2a_{1030}y_1y_2^4 + 2a_{1012}y_1y_2^4 - \\
&2a_{0103}y_1y_2^4 - 2a_{0121}y_1y_2^4 - 2b_{1030}y_1y_2^4 - 2b_{1012}y_1y_2^4 + 2b_{0103}y_1y_2^4 + 2b_{0121}y_1y_2^4 + \\
&5a_{0040}y_3y_2^4 + a_{0004}y_3y_2^4 + a_{0022}y_3y_2^4 - 5b_{0040}y_3y_2^4 - b_{0004}y_3y_2^4 - b_{0022}y_3y_2^4 + 32a_{0200}y_1^3y_3 + \\
&4a_{2011}y_1^2y_2^3 - 4a_{1120}y_1^2y_2^3 - 4a_{1102}y_1^2y_2^3 + 4a_{0211}y_1^2y_2^3 - 4b_{2011}y_1^2y_2^3 + 4b_{1120}y_1^2y_2^3 + \\
&4b_{1102}y_1^2y_2^3 - 4b_{0211}y_1^2y_2^3 + 2a_{0031}y_3^2y_2^3 + 2a_{0013}y_3^2y_2^3 - 2b_{0031}y_3^2y_2^3 - 2b_{0013}y_3^2y_2^3 + \\
&8a_{0011}y_1y_2^3 - 8b_{0011}y_1y_2^3 + 4a_{1003}y_1y_3y_2^3 + 4a_{1021}y_1y_3y_2^3 - 8a_{0130}y_1y_3y_2^3 - 32b_{0200}y_1^3y_3 - \\
&4b_{1003}y_1y_3y_2^3 - 4b_{1021}y_1y_3y_2^3 + 8b_{0130}y_1y_3y_2^3 + 8a_{3010}y_1^3y_2^2 - 8a_{2101}y_1^3y_2^2 - 96b_{2000}y_1^3y_3 + \\
&8a_{1210}y_1^3y_2^2 - 8a_{0301}y_1^3y_2^2 - 8b_{3010}y_1^3y_2^2 + 8b_{2101}y_1^3y_2^2 - 8b_{1210}y_1^3y_2^2 + 8b_{0301}y_1^3y_2^2 + \\
&10a_{0040}y_3^3y_2^2 + 2a_{0004}y_3^3y_2^2 + 2a_{0022}y_3^3y_2^2 - 10b_{0040}y_3^3y_2^2 - 2b_{0004}y_3^3y_2^2 - 2b_{0022}y_3^3y_2^2 - \\
&16a_{0101}y_1^2y_2^2 + 16a_{1010}y_1^2y_2^2 + 16b_{0101}y_1^2y_2^2 - 16b_{1010}y_1^2y_2^2 + 12a_{1030}y_1y_3^2y_2^2 + \\
&4a_{1012}y_1y_3^2y_2^2 - 12b_{1030}y_1y_3^2y_2^2 - 4b_{1012}y_1y_3^2y_2^2 + 12a_{2020}y_1^2y_3y_2^2 + 12a_{2002}y_1^2y_3y_2^2 - \\
&4a_{1111}y_1^2y_3y_2^2 + 12a_{0220}y_1^2y_3y_2^2 - 4a_{0202}y_1^2y_3y_2^2 - 12b_{2020}y_1^2y_3y_2^2 - 12b_{2002}y_1^2y_3y_2^2 + \\
&4b_{1111}y_1^2y_3y_2^2 - 12b_{0220}y_1^2y_3y_2^2 + 4b_{0202}y_1^2y_3y_2^2 + 24a_{0020}y_1y_3y_2^2 + 8a_{0002}y_1y_3y_2^2 - \\
&24b_{0020}y_1y_3y_2^2 - 8b_{0002}y_1y_3y_2^2 - 16a_{3100}y_1^4y_2 - 16a_{1300}y_1^4y_2 + 16b_{3100}y_1^4y_2 + \\
&16b_{1300}y_1^4y_2 + a_{0031}y_3^4y_2 + a_{0013}y_3^4y_2 - b_{0031}y_3^4y_2 - b_{0013}y_3^4y_2 - 32a_{1100}y_1^3y_2 + \\
&32b_{1100}y_1^3y_2 + 4a_{1003}y_1y_3^3y_2 + 4a_{1021}y_1y_3^3y_2 - 8a_{0130}y_1y_3^3y_2 - 4b_{1003}y_1y_3^3y_2 - \\
&4b_{1021}y_1y_3^3y_2 + 8b_{0130}y_1y_3^3y_2 + 12a_{2011}y_1^2y_3^2y_2 - 12a_{1120}y_1^2y_3^2y_2 + 4a_{1102}y_1^2y_3^2y_2 - \\
&4a_{0211}y_1^2y_3^2y_2 - 12b_{2011}y_1^2y_3^2y_2 + 12b_{1120}y_1^2y_3^2y_2 - 4b_{1102}y_1^2y_3^2y_2 + 4b_{0211}y_1^2y_3^2y_2 + \\
&8a_{0011}y_1y_3^2y_2 - 8b_{0011}y_1y_3^2y_2 + 32a_{3001}y_1^3y_3y_2 - 16a_{2110}y_1^3y_3y_2 - 16a_{0310}y_1^3y_3y_2 - \\
&32b_{3001}y_1^3y_3y_2 + 16b_{2110}y_1^3y_3y_2 + 16b_{0310}y_1^3y_3y_2 - 32a_{0110}y_1^2y_3y_2 + 32a_{1001}y_1^2y_3y_2 + \\
&32b_{0110}y_1^2y_3y_2 - 32b_{1001}y_1^2y_3y_2 + 5a_{0040}y_3^5 + a_{0004}y_3^5 + a_{0022}y_3^5 - 5b_{0040}y_3^5 - b_{0004}y_3^5 - \\
&b_{0022}y_3^5 + 10a_{1030}y_1y_3^4 + 2a_{1012}y_1y_3^4 + 2a_{0103}y_1y_3^4 + 2a_{0121}y_1y_3^4 - 10b_{1030}y_1y_3^4 - \\
&2b_{1012}y_1y_3^4 - 2b_{0103}y_1y_3^4 - 2b_{0121}y_1y_3^4 + 20a_{2020}y_1^2y_3^3 + 4a_{2002}y_1^2y_3^3 + 4a_{1111}y_1^2y_3^3 + \\
&4a_{0220}y_1^2y_3^3 + 4a_{0202}y_1^2y_3^3 - 20b_{2020}y_1^2y_3^3 - 4b_{2002}y_1^2y_3^3 - 4b_{1111}y_1^2y_3^3 - 4b_{0220}y_1^2y_3^3 - \\
&4b_{0202}y_1^2y_3^3 + 24a_{0020}y_1y_3^3 + 8a_{0002}y_1y_3^3 - 24b_{0020}y_1y_3^3 - 8b_{0002}y_1y_3^3 + 40a_{3010}y_1^3y_3^2 + \\
&8a_{2101}y_1^3y_3^2 + 8a_{1210}y_1^3y_3^2 + 8a_{0301}y_1^3y_3^2 - 40b_{3010}y_1^3y_3^2 - 8b_{2101}y_1^3y_3^2 - 8b_{1210}y_1^3y_3^2 - \\
&8b_{0301}y_1^3y_3^2 + 16a_{0101}y_1^2y_3^2 + 48a_{1010}y_1^2y_3^2 - 16b_{0101}y_1^2y_3^2 - 48b_{1010}y_1^2y_3^2 + 80a_{4000}y_1^4y_3 + \\
&16a_{2200}y_1^4y_3 + 16a_{0400}y_1^4y_3 - 80b_{4000}y_1^4y_3 - 16b_{2200}y_1^4y_3 - 16b_{0400}y_1^4y_3 + 96a_{2000}y_1^3y_3),
\end{aligned}
$$

$$f_3(y_1, y_2, y_3) = -\frac{1}{64y_1^2(2y_1+3)}(32a_{0400}y_2y_1^5 - 16a_{0310}y_2^2y_1^4 + 16a_{0310}y_3^2y_1^4 + 80a_{4000}y_2y_1^4 +$$
$$16a_{2200}y_2y_1^4 - 16b_{4000}y_2y_1^4 - 16b_{2200}y_2y_1^4 - 80b_{0400}y_2y_1^4 - 16b_{2000}y_2y_1^4 - 48b_{0200}y_2y_1^4 +$$
$$16a_{3100}y_3y_1^4 + 16a_{1300}y_3y_1^4 + 16a_{1100}y_3y_1^4 + 16b_{3100}y_3y_1^4 + 16b_{1300}y_3y_1^4 + 16b_{1100}y_3y_1^4 +$$
$$32a_{0301}y_2y_3y_1^4 - 8a_{1111}y_2^3y_1^3 + 8a_{0220}y_2^3y_1^3 + 8a_{0202}y_2^3y_1^3 + 8a_{1102}y_3^3y_1^3 + 8a_{0211}y_3^3y_1^3 -$$
$$8a_{0110}y_2^2y_1^3 + 40a_{3001}y_2^2y_1^3 - 8a_{2110}y_2^2y_1^3 + 8a_{1201}y_2^2y_1^3 + 24a_{1001}y_2^2y_1^3 + 24b_{0110}y_2^2y_1^3 -$$
$$8b_{3001}y_2^2y_1^3 + 8b_{2110}y_2^2y_1^3 - 8b_{1201}y_2^2y_1^3 + 40b_{0310}y_2^2y_1^3 - 8b_{1001}y_2^2y_1^3 + 8a_{0110}y_3^2y_1^3 +$$
$$8a_{3001}y_3^2y_1^3 + 8a_{2110}y_3^2y_1^3 + 8a_{1201}y_3^2y_1^3 + 8a_{1001}y_3^2y_1^3 + 8b_{0110}y_3^2y_1^3 + 8b_{3001}y_3^2y_1^3 +$$
$$8b_{2110}y_3^2y_1^3 + 8b_{1201}y_3^2y_1^3 + 8b_{0310}y_3^2y_1^3 + 8b_{1001}y_3^2y_1^3 + 8a_{1111}y_2y_3^2y_1^3 - 8a_{0220}y_2y_3^2y_1^3 +$$
$$24a_{0202}y_2y_3^2y_1^3 - 24b_{2000}y_2y_1^3 - 72b_{0200}y_2y_1^3 + 24a_{2000}(2y_1+3)y_2y_1^3 +$$
$$8a_{0200}(2y_1+3)y_2y_1^3 + 24a_{1102}y_2^2y_3y_1^3 - 8a_{0211}y_2^2y_3y_1^3 + 24a_{1100}y_3y_1^3 + 24b_{1100}y_3y_1^3 +$$
$$16a_{0101}y_2y_3y_1^3 + 32a_{3010}y_2y_3y_1^3 + 16a_{2101}y_2y_3y_1^3 + 16a_{1010}y_2y_3y_1^3 - 16b_{0101}y_2y_3y_1^3 -$$
$$16b_{3010}y_2y_3y_1^3 - 16b_{1210}y_2y_3y_1^3 - 32b_{0301}y_2y_3y_1^3 - 16b_{1010}y_2y_3y_1^3 + 4a_{1021}y_2^4y_1^2 -$$
$$4a_{0130}y_2^4y_1^2 - 4a_{0112}y_2^4y_1^2 + 4a_{1021}y_3^4y_1^2 + 4a_{0130}y_3^4y_1^2 + 4a_{0112}y_3^4y_1^2 + 4a_{0020}y_2^3y_1^2 +$$
$$12a_{0002}y_2^3y_1^2 + 4a_{2020}y_2^3y_1^2 + 20a_{2002}y_2^3y_1^2 - 12b_{0020}y_2^3y_1^2 - 4b_{0002}y_2^3y_1^2 - 4b_{2020}y_2^3y_1^2 -$$
$$4b_{2002}y_2^3y_1^2 + 4b_{1111}y_2^3y_1^2 - 20b_{0220}y_2^3y_1^2 - 4b_{0202}y_2^3y_1^2 + 4a_{0011}y_3^3y_1^2 + 4a_{2011}y_3^3y_1^2 +$$
$$4a_{1120}y_3^3y_1^2 + 4b_{0011}y_3^3y_1^2 + 4b_{2011}y_3^3y_1^2 + 4b_{1120}y_3^3y_1^2 + 4b_{1102}y_3^3y_1^2 + 4b_{0211}y_3^3y_1^2 +$$
$$8a_{1012}y_2y_3^3y_1^2 + 16a_{0103}y_2y_3^3y_1^2 - 12a_{0110}y_2^2y_1^2 + 36a_{1001}y_2^2y_1^2 + 36b_{0110}y_2^2y_1^2 -$$
$$12b_{1001}y_2^2y_1^2 + 8a_{1021}y_2^2y_3^2y_1^2 + 12a_{0110}y_3^2y_1^2 + 12a_{1001}y_3^2y_1^2 + 12b_{0110}y_3^2y_1^2 +$$
$$12b_{1001}y_3^2y_1^2 + 4a_{0020}y_2y_3^2y_1^2 + 12a_{0002}y_2y_3^2y_1^2 + 12a_{2020}y_2y_3^2y_1^2 + 12a_{2002}y_2y_3^2y_1^2 -$$
$$12b_{0020}y_2y_3^2y_1^2 - 4b_{0002}y_2y_3^2y_1^2 - 12b_{2020}y_2y_3^2y_1^2 + 4b_{2002}y_2y_3^2y_1^2 - 4b_{1111}y_2y_3^2y_1^2 -$$
$$12b_{0220}y_2y_3^2y_1^2 - 12b_{0202}y_2y_3^2y_1^2 + 8a_{1012}y_2^3y_3y_1^2 + 16a_{0103}y_2^3y_3y_1^2 + 4a_{0011}y_2^2y_3y_1^2 +$$
$$12a_{2011}y_2^2y_3y_1^2 - 4a_{1120}y_2^2y_3y_1^2 + 4b_{0011}y_2^2y_3y_1^2 - 4b_{2011}y_2^2y_3y_1^2 + 12b_{1120}y_2^2y_3y_1^2 -$$
$$4b_{1102}y_2^2y_3y_1^2 + 12b_{0211}y_2^2y_3y_1^2 + 24a_{0101}y_2y_3y_1^2 + 24a_{1010}y_2y_3y_1^2 - 24b_{0101}y_2y_3y_1^2 -$$
$$24b_{1010}y_2y_3y_1^2 + 2a_{0040}y_2^5y_1 + 10a_{0004}y_2^5y_1 + 2a_{0022}y_2^5y_1 + 2a_{0031}y_3^5y_1 + 2a_{0013}y_3^5y_1 +$$
$$10a_{1003}y_2^4y_1 - 2b_{1003}y_2^4y_1 - 2b_{1021}y_2^4y_1 + 10b_{0130}y_2^4y_1 + 2b_{0112}y_2^4y_1 + 2a_{1003}y_3^4y_1 +$$
$$2b_{1003}y_3^4y_1 + 2b_{1021}y_3^4y_1 + 2b_{0130}y_3^4y_1 + 2b_{0112}y_3^4y_1 + 2a_{0040}y_2y_3^4y_1 + 10a_{0004}y_2y_3^4y_1 +$$
$$2a_{0022}y_2y_3^4y_1 + 6a_{0020}y_2^3y_1 + 18a_{0002}y_2^3y_1 - 18b_{0020}y_2^3y_1 - 6b_{0002}y_2^3y_1 +$$
$$4a_{0031}y_2^2y_3^3y_1 + 4a_{0013}y_2^2y_3^3y_1 + 6a_{0011}y_3^3y_1 + 6b_{0011}y_3^3y_1 + 4a_{1030}y_2y_3^3y_1 -$$
$$8b_{1030}y_2y_3^3y_1 - 4b_{0103}y_2y_3^3y_1 - 4b_{0121}y_2y_3^3y_1 + 4a_{0040}y_2^3y_3^2y_1 + 20a_{0004}y_2^3y_3^2y_1 +$$
$$4a_{0022}y_2^3y_3^2y_1 + 12a_{1003}y_2^2y_3^2y_1 + 12b_{0130}y_2^2y_3^2y_1 + 4b_{0112}y_2^2y_3^2y_1 + 6a_{0020}y_2y_3^2y_1 +$$
$$18a_{0002}y_2y_3^2y_1 - 18b_{0020}y_2y_3^2y_1 - 6b_{0002}y_2y_3^2y_1 + 2a_{0031}y_2^4y_3y_1 + 2a_{0013}y_2^4y_3y_1 +$$
$$4a_{1030}y_2^3y_3y_1 - 8b_{1030}y_2^3y_3y_1 - 4b_{0103}y_2^3y_3y_1 - 4b_{0121}y_2^3y_3y_1 + 6a_{0011}y_2^2y_3y_1 +$$
$$6b_{0011}y_2^2y_3y_1 - 5b_{0040}y_2^5 - b_{0004}y_2^5 - b_{0022}y_2^5 + b_{0031}y_3^5 + b_{0013}y_3^5 - 5b_{0040}y_2y_3^4 -$$
$$b_{0004}y_2y_3^4 - b_{0022}y_2y_3^4 + 2b_{0031}y_2^2y_3^3 + 2b_{0013}y_2^2y_3^3 - 10b_{0040}y_2^3y_3^2 - 2b_{0004}y_2^3y_3^2 -$$
$$2b_{0022}y_2^3y_3^2 + b_{0031}y_2^4y_3 + b_{0013}y_2^4y_3).$$

6.6 An example with 16 hyperbolic periodic orbits

We take the coefficients

$$
\begin{array}{ll}
a_{0110} = -0.44273799230932176, & a_{0101} = -0.0016215945505887238, \\
a_{0020} = -0.00097478324838714, & a_{0011} = -0.003918882434202215, \\
a_{0002} = 0.00007081854058168514, & a_{4000} = 0.5989505745384144, \\
a_{3100} = 0.000002966367759708, & a_{3010} = -0.1687679624360404, \\
a_{3001} = 0.18506308130804394, & a_{2020} = -0.012049647635249473, \\
a_{2002} = -0.0036240063728033025, & a_{2110} = -0.18620230772935153, \\
a_{2101} = -0.17263861380369933, & a_{2011} = 0.00047403650848489584, \\
a_{1030} = -0.0018042655006471255, & a_{1003} = -0.0007465749618595678, \\
a_{1120} = 0.00009368584384093783, & a_{1021} = 0.0020131588337237534, \\
a_{1012} = 0.005506754204431025, & a_{1102} = -0.00011351408717201784, \\
a_{1111} = 0.007471300514491378 5, & a_{0400} = -0.004655785701991735, \\
a_{0040} = 0.00001968558790431901, & a_{0004} = -0.0000032848225314553, \\
a_{0310} = 0.0012461786194435333, & a_{0301} = 0.003941103289490351, \\
a_{0130} = 0.0007943513793541091, & a_{0031} = 0.00005280878950993970 4, \\
a_{0103} = -0.001807654766145478, & a_{0220} = 0.00940846361405681, \\
a_{0202} = 0.0009673162067761705, & a_{0211} = -0.0003272826514772228, \\
a_{0121} = 0.005645104995454165, & a_{0112} = -0.002154963920384549, \\
a_{2000} = 0.3346899322743752, & a_{1100} = 0.00002777934549378803 8, \\
a_{1010} = -0.0005686779752235013, & a_{1001} = 0.44083258814665977, \\
b_{0110} = -0.8793311376819993, & b_{0101} = -0.009884290550174158, \\
b_{0011} = 0.0026112577489967063, & b_{4000} = 0.5980524777204888, \\
b_{3100} = -0.000037200708610095267, & b_{3010} = -0.16878494791125745, \\
b_{3001} = 0.44488101166264415, & b_{2020} = -0.009125904399372506, \\
b_{2110} = 0.3344329805234952, & b_{2101} = -0.16839063503055093, \\
b_{0200} = 1, &
\end{array}
$$

and the rest of the coefficients zero. The set of solutions (y_1, y_2, y_3) is

$$
\begin{array}{l}
\{(41,35,-19),(18,-13,9),(37,-18,-35),(31,-47,14),(22,3,-15),(33,5,-16), \\
(35,-28,2),(37,-31,28),(16,9,5),(30,-28,21),(27,-33,33),(17,-32,-50), \\
(22,27,28),(5,26,-25),(43,-7,29),(7,-31,-20)\}.
\end{array}
$$

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