SOME NEW RESULTS ON DARBOUX INTEGRABLE DIFFERENTIAL SYSTEMS

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Abstract. We deal with complex planar differential systems having a Darboux first integral H. We present a definition of remarkable values and remarkable curves associated to H and characterize the existence of a polynomial inverse integrating factor for these systems. Furthermore, we study the relation between the characteristic polynomial $\mathcal{F}$ and the inverse integrating factors of the system and show the importance of the numerator of the exponential factor of H in the construction of $\mathcal{F}$.

1. Introduction and preliminary definitions

A *complex planar polynomial differential system* of degree d is

$$\dot{x} = P(x,y), \ \dot{y} = Q(x,y), \tag{1}$$

where $P, Q \in \mathbb{C}[x,y]$ are coprime and $d = \max\{\deg P, \deg Q\}$. We write $X = (P,Q)$ and $Xf = Pf_x + Qf_y$, f a $\mathcal{C}^1$-function.

Let U be an open subset of $\mathbb{C}^2$. A *first integral* of X in U is a non-constant $\mathcal{C}^1$-function $H : U \to \mathbb{C}$, possibly multi-valued, which is constant on all the solutions of X contained in U, i.e. $XH = 0$ on U. We say in this case that X is *integrable* on U.

An *inverse integrating factor* of X in U is a $\mathcal{C}^1$-function $V : U \to \mathbb{C}$ satisfying $XV = \mathrm{div}(X)V$, where $\mathrm{div}(X) = P_x + Q_y$ is the divergence function of X. The function V is a very useful tool to study integrable systems (see [5]). Indeed the set $V^{-1}(0)$ contains a lot of information about the 'skeleton' or the separatrices of the phase portrait of X in U, see [3, 16, 15, 18, 4].

We say that *the inverse integrating factor* V *is associated to the first integral* H of system (1) in U if $(P,Q) = (-H_y, H_x)V$ in $U \setminus \{V = 0\}$.

Let $f \in \mathbb{C}[x,y]$. We say that the algebraic curve $f = 0$ is *invariant* if there exists a polynomial $K \in \mathbb{C}[x,y]$ of degree at most $d-1$, called the *cofactor*, such that $Xf = Kf$.

Let $g, h \in \mathbb{C}[x,y]$ be coprime polynomials. The function $F = e^{g/h}$ is an *exponential factor* of system (1) if there exists a polynomial $L \in \mathbb{C}[x,y]$ of degree at most $d-1$, called the *cofactor*, such that $XF = LF$. In this case, $h = 0$ is an invariant algebraic curve. The notion of exponential factor is due to Christopher [7]. An exponential factor appears when an invariant algebraic curve has multiplicity greater than one. For more details on exponential factors see [10].

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