PHASE PORTRAITS OF ABEL QUADRATIC DIFFERENTIAL SYSTEMS OF SECOND KIND WITH SYMMETRIES

ANTONI FERRAGUT, JOHANNA D. GARCÍA-SALDAÑA, CLAUDIA VALLS

Abstract. We provide normal forms and the global phase portraits on the Poincaré disk of the Abel quadratic differential equations of the second kind having a symmetry with respect to an axis or to the origin. Moreover, we also provide the bifurcation diagrams for these global phase portraits.

1. Introduction and statements of the main results

Abel differential equations of the second kind are named in honor of Niels Henrik Abel because by a direct substitution they are related with the Abel differential equations of the first kind which were obtained by him in his studies of the theory of elliptic functions (see [11]) and after that, he made a crucial research on them. Abel differential equations of the second kind have various applications as they appear to reduce the order of many higher order nonlinear problems. They are also frequently found in the modeling of real problems such as big picture modeling in oceanic circulation (see [1] and the references therein).

An Abel differential equation of second kind has the form

$$y\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x), \tag{1}$$

with $A(x), B(x), C(x) \in \mathbb{R}(x, y)$. This differential equation can be written equivalently as the polynomial differential system

$$\dot{x} = d(x)y, \quad \dot{y} = a(x)y^2 + b(x)y + c(x), \tag{2}$$

where $a(x)$, $b(x)$, $c(x)$ and $d(x)$ are polynomials such that $A(x) = a(x)/d(x)$, $B(x) = b(x)/d(x)$ and $C(x) = c(x)/d(x)$. In this paper we are interested in studying the *Abel quadratic polynomial differential systems*, i.e. the differential systems (2) of degree two:

$$\dot{x} = (d_0 + d_1x)y, \quad \dot{y} = a_0y^2 + (b_0 + b_1x)y + c_0 + c_1x + c_2x^2. \tag{3}$$

All the parameters in (3) are real. We assume that $\dot{x}$ and $\dot{y}$ do not have a common factor; in particular, we assume that $c_0^2 + c_1^2 + c_2^2 \neq 0$. Moreover, we take $a_0 \neq 0$, otherwise this is not the Abel equation of second kind. We also assume that $b_1^2 + c_1^2 + c_2^2 + d_1^2 \neq 0$, otherwise the system does not depend on x and hence it is not of our interest.

In this work we study the global phase portraits of the Abel differential systems of second kind of degree two given by system (3) with $d_1 \neq 0$. The case $d_1 = 0$ is completely studied in [7]. Since this is a huge challenge, we restrict our study to the differential systems (3) having a symmetry with respect to an axis or with respect to the origin, see (4) below. We shall provide all the possible global phase portraits for these families. For that purpose, we shall use the well-known Poincaré compactification of polynomial vector fields, see section 2.3. Before stating our main theorem, we split the family (3) into five different families.

Proposition 1. *System (3) with $d_1 \neq 0$ leads, after an affine change of the variables and a scaling of the time, to the differential system $\dot{x} = xy$, $\dot{y} = Q(x, y)$, where $Q(x, y)$ is one of the following five quadratic polynomials:*

(K1) $Q(x, y) = S(x - r_1)(x - r_2) + (B_0 + B_1x)y + A_0y^2$;

2010 *Mathematics Subject Classification.* 34A34, 34C20, 34C23, 34C05.

Key words and phrases. Abel differential equation of second kind, quadratic differential system, phase portrait, Poincaré disk.