NON-ALGEBRAIC OSCILLATIONS FOR PREDATOR-PREY MODELS

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Abstract. We prove that the limit cycle oscillations of the celebrated Rosenzweig-MacArthur differential system and other predator-prey models are non-algebraic.

1. Introduction and statement of the main results

It is easy to see that the periodic orbits of the celebrated Lotka-Volterra model

$$\dot{x} = \frac{dx}{dt} = x(\alpha - \beta y), \quad \dot{y} = \frac{dx}{dt} = y(-\delta + \gamma x),$$

where $x, y \geq 0$ and all the parameters are positive, are non-algebraic curves. This holds because it is an integrable system, and their solutions are contained into the level sets

$$H(x, y) = x^{\gamma} y^{\alpha} e^{-\delta x - \beta y} = h \geq 0,$$

which are clearly non-algebraic. The aim of this work is to prove that the attracting periodic orbits (limit cycles) of the Rosenzweig-MacArthur system, as well as the periodic orbits of other non-integrable predator-prey models, are neither given by algebraic curves. Let us introduce with more detail the systems that we will consider.

To study the predator-prey interaction when the prey exhibits group defense, Freedman and Wolkowicz [8], Mischaikow and Wolkowicz [14] and Wolkowicz [19] proposed the following model (see also Lin [13]):

$$\dot{x} = X(x, y) = xg(x, K) - yp(x), \quad \dot{y} = Y(x, y) = y(-D + q(x)). \tag{1}$$

Here, x and y are functions of time representing population densities of prey and predator, respectively, and are assumed to be non-negative; $K > 0$ is the carrying capacity of the prey and $D > 0$ is the death rate of the predator. The function $g(x, K)$ represents the specific growth rate of the prey in the absence of predator and is assumed to satisfy certain conditions. A prototype is the logistic growth

$$g(x, K) = r\left(1 - \frac{x}{K}\right), \tag{2}$$

with $r > 0$, which satisfies all those conditions. The function $p(x)$ denotes the predator response function and is assumed to satisfy $p(0) = 0$ and $p(x) > 0$ for $x > 0$. The rate of conversion of prey to predator is described by $q(x)$. In Gause's model, we have

$$\frac{q(x)}{p(x)} = \gamma \in \mathbb{R}^{+}. \tag{3}$$

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