MEROMORPHIC FIRST INTEGRALS OF ANALYTIC DIFFEOMORPHISMS

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ABSTRACT. We give an upper bound for the number of functionally independent meromorphic first integrals that a discrete dynamical system generated by an analytic map f can have in a neighborhood of one of its fixed points. This bound is obtained in terms of the resonances among the eigenvalues of the differential of f at this point. Our approach is inspired on similar Poincaré type results for ordinary differential equations. We also apply our results to several examples, some of them motivated by the study of several difference equations.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

One of the first steps to study the dynamics of a discrete dynamical system (DDS) is to determine the number m of functionally independent first integrals that it has. It is clear that each new first integral reduces the region where any orbit can lie, so the bigger is m, the simpler will be the dynamics. For instance, if this DDS is n-dimensional, this number m is at most n, and in the case $m = n$ the DDS is called integrable and it has extremely simple dynamics: in most cases it is globally periodic, that is, there exists $p \in \mathbb{N}$, such that $f^p = \mathrm{Id}$, where f is the invertible map that generates it, see [4]. Similarly, DDS having $m = n-1$ are such that all their orbits lie in one-dimensional manifolds, see some examples in [5].

The aim of this paper is to give an upper bound of the number of meromorphic first integrals that a DDS generated by an invertible analytic map can have in a neighborhood of a fixed point. We follow the approach of Poincaré for studying the same problem for continuous dynamical systems given by analytic ordinary differential equations. It is based on the study of the resonances among the eigenvalues of the differential of the vector field at one of its critical points, see for instance [9] and their references. We will use similar tools that the ones introduced in that paper.

Consider analytic diffeomorphisms in $(\mathbb{C}^n, 0)$, a neighborhood of the origin,

$$(1) \qquad\qquad y = f(x), \qquad x \in (\mathbb{C}^n, 0),$$

with $f(0) = 0$. A function $R(x) = G(x)/H(x)$ with G and H analytic functions in $(\mathbb{C}^n, 0)$ is a *meromorphic first integral* of the diffeomorphism (1) if

$$G(f(x))H(x) = G(x)H(f(x)), \qquad \text{for all } x \in (\mathbb{C}^n, 0).$$

Notice that the above condition implies that

$$R(f(x)) = R(x),$$

for all $x \in (\mathbb{C}^n, 0)$ where both functions are well defined. Specially if G and H are polynomial functions, then $R(x)$ is a *rational first integral* of (1). If H is a non–zero constant, then $R(x)$ is an *analytic first integral* of (1). So meromorphic first integrals include rational and analytic first integrals as particular cases.

Denote by $A = \mathrm{D}f(0)$ the Jacobian matrix of $f(x)$ at $x = 0$. Let $\mu = (\mu_1, \dots, \mu_n)$ be the n–tuple of eigenvalues of A. Notice that since f is a diffeomorphism at 0, we have $\mu_1 \mu_2 \cdots \mu_n \neq 0$.

We say that the eigenvalues μ satisfy a *resonant condition* if

$$\mu^{\mathbf{k}} = 1, \qquad \text{for some } \mathbf{k} \in \mathbb{Z}^n \quad \text{with} \quad \|\mathbf{k}\| \neq 0,$$

where $\mathbb{Z}$ is the set of integers, and $\|\mathbf{k}\| = |k_1| + \cdots + |k_n|$.

The aim of this paper is to prove the next result and give some applications.

Theorem 1. *Assume that the analytic diffeomorphism* (1) *satisfies* $f(0) = 0$ *and let* $\mu = (\mu_1, \ldots, \mu_n)$ *be the eigenvalues of* $\mathrm{D}f(0)$. *Then the number of functionally independent generalized rational first integrals of the analytic diffeomorphism* (1) *in* $(\mathbb{C}^n, 0)$ *is at most the dimension of the* $\mathbb{Z}$–*linear space generated from* $\{\mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1\}$.

The above theorem also extends some results proved for $n = 2$ in [7].

In section 2 we prove Theorem 1. Section 3 is devoted to applications of Theorem 1.

2. Proof of Theorem 1

For an analytic or a polynomial function $R(x)$ in $(\mathbb{C}^n, 0)$, we denote by $R^0(x)$ its homogeneous term of the lowest degree. For a rational or a meromorphic function $R(x) = G(x)/H(x)$ in $(\mathbb{C}^n, 0)$, we denote by $R^0(x)$ the rational function $G^0(x)/H^0(x)$. We expand the analytic functions $G(x)$ and $H(x)$ as

$$G^0(x) + \sum_{i=1}^{\infty} G^i(x) \quad \text{and} \quad H^0(x) + \sum_{i=1}^{\infty} H^i(x),$$

where $G^i(x)$ and $H^i(x)$ are homogeneous polynomials of degrees $\deg G^0(x) + i$ and $\deg H^0(x) + i$, respectively. Then we have

$$R(x) = \frac{G(x)}{H(x)} = \left(\frac{G^0(x)}{H^0(x)} + \sum_{i=1}^{\infty} \frac{G^i(x)}{H^0(x)} \right) \left(1 + \sum_{i=1}^{\infty} \frac{H^i(x)}{H^0(x)} \right)^{-1}$$

$$(2) \qquad\qquad\quad = \frac{G^0(x)}{H^0(x)} + \sum_{i=1}^{\infty} \frac{A^i(x)}{B^i(x)},$$

where $A^i(x)$ and $B^i(x)$ are homogeneous polynomials. Clearly,

$$\deg G^0(x) - \deg H^0(x) < \deg A^i(x) - \deg B^i(x) \qquad \text{for all } i \geq 1.$$

In what follows we will say that $\deg A^i(x) - \deg B^i(x)$ is the degree of $A^i(x)/B^i(x)$, and $G^0(x)/H^0(x)$ is the lowest degree term of $R(x)$ in the expansion (2). For simplicity we denote

$$d(G) = \deg G^0(x), \qquad d(R) = d(G) - d(H) = \deg G^0(x) - \deg H^0(x),$$

and call $d(R)$ the *lowest degree* of R.

The following known result, first proved by Ziglin [21] in 1983, see also [1, 9, 12, 20], will be used in the proof of Theorem 1.

Lemma 2. *Let*

$$R_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, R_m(x) = \frac{G_m(x)}{H_m(x)},$$

be functionally independent meromorphic functions in $(\mathbb{C}^n, 0)$. *Then there exist polynomials $P_i(z_1, \ldots, z_i)$ for $i = 2, \ldots, m$ such that $R_1(x)$, $\widetilde{R}_2(x) = P_2(R_1(x), R_2(x))$, ..., $\widetilde{R}_m(x) = P_m(R_1(x), \ldots, R_m(x))$ are functionally independent meromorphic functions, and that $R_1^0(x)$, $\widetilde{R}_2^0(x)$, ..., $\widetilde{R}_m^0(x)$ are functionally independent rational functions.*

Next we give some properties that meromorphic first integrals of diffeomorphism (1) must have. A *rational monomial* is by definition the ratio of two monomials, i.e. of the form $x^{\mathbf{p}}/x^{\mathbf{q}}$ with $\mathbf{p}, \mathbf{q} \in (\mathbb{N}_0)^n$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of positive integers. The rational monomial $x^{\mathbf{p}}/x^{\mathbf{q}}$ is *resonant* if $\mu^{\mathbf{p}-\mathbf{q}} = 1$. A rational function is *homogeneous* if its denominator and numerator are both homogeneous polynomials. A rational homogeneous

function is *resonant* if the ratio of any two elements in the set of all its monomials in both denominator and numerator is a resonant rational monomial.

For the diffeomorphism (1) we assume without loss of generality that $A = \mathrm{D}f(0)$ is in its Jordan normal form and is a lower triangular matrix.

Lemma 3. *Let $R(x) = G(x)/H(x)$ be a meromorphic first integral of the analytic diffeomorphism* (1). *By changing R by $R - a$, for some suitable constant $a \in \mathbb{C}$, if needed, it is not restrictive to assume that $R^0(x) = G^0(x)/H^0(x)$ is non–constant. Moreover R^0 is a resonant rational homogeneous first integral of the linear part of $f(x)$.*

For proving this last lemma we will use the following result, for a proof see any of the references [3, 13, 19, 20].

Lemma 4. *Let $\mathcal{H}_n^p$ be the linear space of complex coefficient homogeneous polynomials of degree p in n variables. For any constant $c \in \mathbb{C}$, define a linear operator on $\mathcal{H}_n^p$ by*
$$\mathcal{L}_c(h)(x) = h(Ax) - c\,h(x), \qquad h(x) \in \mathcal{H}_n^p.$$
Then the spectrum of $\mathcal{L}_c$ is
$$\{\mu^{\mathbf{k}} - c : \ \mathbf{k} \in (\mathbb{N}_0)^n, \ |\mathbf{k}| = k_1 + \ldots + k_n = p\},$$
where μ are the eigenvalues of A.

Proof of Lemma 3. If $R^0(x) = G^0(x)/H^0(x) \equiv a$ is constant, the function $R(x) - a$ is also a meromorphic first integral and $(R(x) - a)^0$ is not constant. Hence, without loss of generality, we can assume that R is a meromorphic first integral and that R^0 is not identically constant.

Let us prove that R^0 is a resonant rational homogeneous first integral. As in (2) we write $R(x)$ as
$$R(x) = R^0(x) + \sum_{i=1}^{\infty} R^i(x),$$
where $R^0(x)$ is the lowest order rational homogeneous function and $R^i(x)$ for $i \in \mathbb{N}$ are rational homogeneous functions of order larger than $R^0(x)$. Since $R(x)$ is a first integral of the diffeomorphism $f(x)$ in a neighborhood of $0 \in \mathbb{C}^n$, we have
$$R(f(x)) = R(x), \qquad x \in (\mathbb{C}^n, 0).$$
Equating the lowest order rational homogeneous functions we get
$$(3) \qquad\qquad R^0(Ax) = R^0(x), \qquad i.e. \quad \frac{G^0(Ax)}{H^0(Ax)} = \frac{G^0(x)}{H^0(x)}.$$

This implies that $R^0(x)$ is a rational homogeneous first integral of the linear part of the analytic diffeomorphism (1).

Next we shall prove that $R^0(x)$ is resonant. From the equality (3) we can assume without loss of generality that $G^0(x)$ and $H^0(x)$ are relative prime. Now equation (3) can be written as
$$H^0(x)G^0(Ax) = G^0(x)H^0(Ax).$$
Since G^0 and H^0 are relatively prime, and $\mathbb{C}[x]$ is a unique factorization domain (see e.g. [11, p. 2]), we get that $G^0(x)$ divides $G^0(Ax)$ and $H^0(x)$ divides $H^0(Ax)$. In addition, $G^0(Ax)$ and $G^0(x)$ have the same degree, so there exists a constant c such that
$$G^0(Ax) - c\,G^0(x) \equiv 0, \quad H^0(Ax) - c\,H^0(x) \equiv 0.$$
We remark that if $G^0(x) \equiv 1$ or $H^0(x) \equiv 1$, we have $c = 1$. Set $\deg G^0(x) = l$, $\deg H^0(x) = m$ and $\mathcal{L}_c$ the linear operator defined in Lemma 4. Recall from Lemma 4 that $\mathcal{L}_c$ has respectively the spectrums on $\mathcal{H}_n^p$
$$\mathcal{S}_p := \{\mu^{\mathbf{p}} - c : \ \mathbf{p} \in (\mathbb{Z}^+)^n, \ |\mathbf{p}| = p\},$$
and on $\mathcal{H}_n^q$
$$\mathcal{S}_q := \{\mu^{\mathbf{q}} - c : \ \mathbf{q} \in (\mathbb{Z}^+)^n, \ |\mathbf{q}| = q\}.$$

Separate $\mathcal{H}_n^p = \mathcal{H}_{n,1}^p + \mathcal{H}_{n,2}^p$ in such a way that for any $p(x) \in \mathcal{H}_{n,1}^l$ its monomial $x^{\mathbf{P}}$ satisfies $\mu^{\mathbf{P}} - c = 0$, and for any $Q(x) \in \mathcal{H}_{n,2}^p$ its monomial $x^{\mathbf{P}}$ satisfies $\mu^{\mathbf{P}} - c \neq 0$. Separate $G^0(x)$ in two parts $G^0(x) = G_1^0(x) + G_2^0(x)$ with $G_1^0 \in \mathcal{H}_{n,1}^p$ and $G_2^0 \in \mathcal{H}_{n,2}^p$. Since A is in its Jordan normal form and is lower triangular, it follows that

$$\mathcal{L}_c \mathcal{H}_{n,1}^p \subset \mathcal{H}_{n,1}^p, \quad \text{and} \quad \mathcal{L}_c \mathcal{H}_{n,2}^p \subset \mathcal{H}_{n,2}^p.$$

Hence $\mathcal{L}_c G^0(x) \equiv 0$ is equivalent to

$$\mathcal{L}_c G_1^0(x) \equiv 0 \quad \text{and} \quad \mathcal{L}_c G_2^0(x) \equiv 0.$$

Since $\mathcal{L}_c$ has the spectrum without zero element on $\mathcal{H}_{n,2}^p$ and so it is invertible on $\mathcal{H}_{n,2}^p$, the equation $\mathcal{L}_c G_2^0(x) \equiv 0$ has only the trivial solution, i.e. $G_2^0(x) \equiv 0$. This proves that $G^0(x) = G_1^0(x)$, i.e. each monomial, say $x^{\mathbf{P}}$, of $G^0(x)$ satisfies $\mu^{\mathbf{P}} - c = 0$.

Similarly we can prove that each monomial, say $x^{\mathbf{q}}$, of $H^0(x)$ satisfies $\mu^{\mathbf{q}} - c = 0$. This implies that $\mu^{\mathbf{P} - \mathbf{q}} = 1$. The above proofs show that $R^0(x) = G^0(x)/H^0(x)$ is a resonant rational homogeneous first integral of the linear part of $f(x)$. $\qquad\square$

Having the above lemmas we can prove Theorem 1.

Proof of Theorem 1. Let

$$R_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, R_m(x) = \frac{G_m(x)}{H_m(x)},$$

be the m functionally independent meromorphic first integrals of the diffeomorphism f. Since the polynomial functions of $R_i(x)$ for $i = 1, \ldots, m$ are also meromorphic first integrals of the diffeomorphsim f, by Lemma 2 we can assume without loss of generality that

$$R_1^0(x) = \frac{G_1^0(x)}{H_1^0(x)}, \ldots, R_m^0(x) = \frac{G_m^0(x)}{H_m^0(x)},$$

are functionally independent.

Lemma 3 shows that $R_1^0(x), \ldots, R_m^0(x)$ are resonant rational homogeneous first integrals of the linear part Ax of $f(x)$. So these first integrals can be written as rational functions in the variables given by resonant rational monomials. Write $A = A_S + A_N$ with $A_S = \mathrm{diag}(\mu_1, \ldots, \mu_n)$ and A_N nilpotent. Then direct calculations show that any resonant rational monomial is a first integral of $A_S x$. For instance, let $x^{\mathbf{k}}$ be a resonant rational monomial, then $\mu^{\mathbf{k}} = 1$. Hence, we have $(A_S x)^{\mathbf{k}} = \mu^{\mathbf{k}} x^{\mathbf{k}} = x^{\mathbf{k}}$. This implies that $R_1^0(x), \ldots, R_m^0(x)$ are also first integrals of $A_S x$. We claim that m is less than or equal to the number of elements in a basis of the $\mathbb{Z}$–linear space generated from $\{\mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1\}$. We denote by γ this last number.

Indeed, since $R_1^0(x), \ldots, R_m^0(x)$ are the first integrals of $A_S x$, we only need to prove that the number of functionally independent resonant rational homogeneous first integrals of $A_S x$ is equal to γ. In fact, from Lemma 3 and its proof it follows that any resonant rational homogeneous first integral consists of resonant rational monomials. This means that the maximum number of functionally independent resonant rational homogeneous first integrals is equal to the maximum number of functionally independent resonant rational monomials. Whereas, by definition a resonant rational monomial, saying $x^{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{Z}^n$, satisfies $\mu^{\mathbf{k}} = 1$. This proves the claim.

After the claim, the proof of the theorem is completed. $\qquad\square$

3. Applications

3.1. **Some simple examples.** We start studying the integrability of two simple examples coming from second order difference equations. Recall that the study of the sequences generated by second order difference equations $x_{n+2} = g(x_n, x_{n+1})$ can be reduced to the study of the DDS generated by the planar map $f(x, y) = (y, g(x, y))$. Moreover, the first integrals H for the DDS are usually called *invariants* for the difference equation. Then $H(x_n, x_{n+1}) = H(x_{n+1}, x_{x+2})$ for all $n \in \mathbb{N}$.

As a first example we study the integrability of the planar map

(4) $$f(x,y) = (y, -bx + c/y), \quad b,c \in \mathbb{C}, b \neq 0.$$

coming from the difference equation $x_{n+2} = -bx_n + c/x_{n+1}$. We prove the following proposition.

Proposition 5. *If the planar map* (4) *has a meromorphic first integral then b is a root of the unity. In particular, if $b,c \in \mathbb{R}$ and the map has a meromorphic first integral then $b \in \{-1, 1\}$.*

Somehow the result for the real case is sharp because when $b = 1$ the function $H(x,y) = x^2y^2 - cxy$ is a first integral of (4). Moreover when $b = -1$ and $c = 0$, clearly the map is linear and integrable. The first integral for the case $b = 1$ (in fact for its inverse) and also for many other rational maps are given in [17]. For the complex case, observe that when $c = 0$ and b is a root of the unity the map is linear and globally periodic, i.e. $f^n = \text{Id}$ for some $n \in \mathbb{N}$. For instance, by using the results of [4] it is easy to construct two functionally independent rational first integrals for each one of them.

Proof of Proposition 5. When $c = 0$ the map has $(0,0)$ as a fixed point and the eigenvalues of $\mathrm{D}f(0,0)$ are $\pm\sqrt{b}$. Then, by Theorem 1, to have a meromorphic first integral there must exist $(m,n) \in \mathbb{Z}^2 \setminus (0,0)$ such that $(\sqrt{b})^n(-\sqrt{b})^m = 1$ and, as a consequence b must be a root of the unity.

From now on we can assume that $c \neq 0$. Then, the fixed points of f are (z,z) where $(b+1)z^2 = c$. If $b = -1$ we are done because $b^2 = 1$. Otherwise, the map has two (real or complex) fixed points. Anyhow, for $c(b+1) \neq 0$,

$$\mathrm{D}f(z,z) = \begin{pmatrix} 0 & 1 \\ -b & -c/z^2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -b & -(b+1) \end{pmatrix}.$$

Hence, its eigenvalues satisfy $p(\mu) = \mu^2 + (b+1)\mu + b = 0$. Since $b \neq 0$ we can write them as μ and b/μ. By Theorem 1, if f is meromorphically integrable, there exists $(m,n) \in \mathbb{Z}^2 \setminus (0,0)$ such that $\mu^n(b/\mu)^m = 1$. If $m = n$, then $b^m = 1$ and we are done, again. Hence, from now on, $m - n \neq 0$. Moreover, $\mu = b^{m/(m-n)}$. Thus

$$p(\mu) = \mu^2 + (b+1)\mu + b = b^{\frac{2m}{m-n}} + (b+1)b^{\frac{m}{m-n}} + b = 0.$$

If we introduce a such that $b = a^{m-n}$, the above equality writes as

$$a^{2m} + (a^{m-n} + 1)a^m + a^{m-n} = a^{m-n}(a^{m+n} + a^m + a^n + 1) = a^{m-n}(a^m + 1)(a^n + 1) = 0.$$

Hence a must be a root of the unity, and as a consequence $b = a^{m-n}$ also must, as we wanted to prove. $\square$

As a second example, next lemma studies conditions for a planar map to have two functionally independent meromorphic first integrals.

Lemma 6. *Let f be a real analytic planar map with a fixed point $\mathbf{x} \in \mathbb{R}^2$ such that the characteristic polynomial of $\mathrm{D}f(\mathbf{x})$ is $p(\mu) = \mu^2 + b\mu + c \in \mathbb{R}[\mu]$. Assume that f has two functionally independent meromorphic first integrals. Then, the eigenvalues of $\mathrm{D}f(\mathbf{x})$ are roots of the unity. In particular, if $b^2 - 4c < 0$ then $c = 1$ and otherwise $(b,c) \in \{(\pm 2, 1), (0, -1)\}$.*

Proof. Let $u, v \in \mathbb{C}$ be the two eigenvalues of $\mathrm{D}f(\mathbf{x})$. By Theorem 1, there exist (n,m) and (n', m') in $\mathbb{Z}^2$ such that

$$u^n v^m = 1, \quad u^{n'} v^{m'} = 1, \quad \text{and} \quad mn' - m'n \neq 0.$$

Hence $u^{nm' - n'm} = 1$ and $v^{mn' - m'n} = 1$. Therefore, u and v are roots of the unity. When u and v are complex we are done, because $v = \bar{u}$ and $c = u\bar{u} = |u|^2 = 1$. When $u, v \in \mathbb{R}$ then u and v are either 1 or -1 and the lemma follows. $\square$

We apply the above lemma to study the map

(5) $$f(x, y) = (y, x^p y^q), \quad p, q \in \mathbb{Z}, \ p \neq 0,$$

that describes the difference equation $x_{n+2} = x_n^p x_{n+1}^q$. It has the fixed point $(1, 1)$ for all values of p and q, and provides a good test for our result because it can also be studied by another approach. It can be linearized on $\{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$, because with the new variables $(u, v) = (\ln x, \ln y)$ the DDS generated by f is conjugated to the DDS generated by $g(u, v) = (v, pu + qv)$.

As we have already commented, we will center our attention in finding the values of p and q such that the DDS generated by f can have two functionally independent first integrals. Notice that $f(1, 1) = (1, 1)$ and

$$\mathrm{D}f(1,1) = \begin{pmatrix} 0 & 1 \\ px^{p-1}y^q & qx^p y^{q-1} \end{pmatrix}_{(x,y)=(1,1)} = \begin{pmatrix} 0 & 1 \\ p & q \end{pmatrix}.$$

Hence the eigenvalues μ of $\mathrm{D}f(1, 1)$ satisfy $\mu^2 - q\mu - p = 0$. We can use Lemma 6: since $(p, q) \in \mathbb{Z}^2$, the only cases for which the map (5) can have two functionally independent first integrals are $(p, q) \in \{(-1, -2), (-1, 2), (1, 0)\}$ when $q^2 + 4p \geq 0$, and $p = -1$ when $q^2 + 4p < 0$. Moreover, in this last case all roots of $P(\mu) = \mu^2 - q\mu + 1$ must be roots of the unity. This only happens when $q \in \{-1, 0, 1\}$, because these values are the only ones for which P is a quadratic cyclotomic polynomial. Recall that the p-th cyclotomic polynomial Φ_p is the monic polynomial with integer coefficients, and irreducible in $\mathbb{Q}(x)$, such that its roots are the primitive p-roots of the unity, that is $\mathrm{e}^{2n\pi\mathrm{i}/p}$, for p and n relatively prime. Recall also that only the 3rd, 4th and 6th roots of the unity have quadratic cyclotomic polynomials. Hence, from our tools, we have seen that there are six maps of the form (5) that are candidates to have two functionally independent first integrals:

$$f_1(x, y) = \left(y, \frac{1}{xy^2}\right), \qquad f_2(x, y) = \left(y, \frac{y^2}{x}\right), \qquad f_3(x, y) = (y, x),$$

$$f_4(x, y) = \left(y, \frac{1}{xy}\right), \qquad f_5(x, y) = \left(y, \frac{1}{x}\right), \qquad f_6(x, y) = \left(y, \frac{y}{x}\right).$$

From these candidates it is not difficult to see, by using the tools of [4], that $f_j(x, y)$ $j = 3, 4, 5, 6$ have the desired property. For instance, two functionally independent first integrals for f_6 are

$$H_1(x, y) = x + \frac{1}{x} + y + \frac{1}{y} + \frac{x}{y} + \frac{y}{x}, \quad H_2(x, y) = xy + \frac{1}{xy} + \frac{x^2}{y} + \frac{y}{x^2} + \frac{x}{y^2} + \frac{y^2}{x}.$$

In fact, these four maps are globally periodic, with periods $2, 3, 4$ and 6, respectively.

3.2. **Planar rational maps with rational coefficients.** We consider now rational maps

(6) $$f(x, y) = (R_1(x, y), R_2(x, y)) = \left(\frac{P_1(x, y)}{Q_1(x, y)}, \frac{P_2(x, y)}{Q_2(x, y)}\right),$$

with coefficients in $\mathbb{Q}$, that is, with $P_i, Q_i \in \mathbb{Q}[x, y]$ for $i = 1, 2$.

We introduce some notation. As usual, when two integer numbers p and n are coprime we will write $(p, n) = 1$. Moreover we will denote the degree of the p-th cyclotomic polynomial Φ_p by $\phi(p)$. In fact, $\phi(p)$ is the number of integers between 1 and p that are relatively prime to p.

Given two polynomials $P, Q \in \mathbb{C}[x]$, as usual, we denote by $\mathrm{Res}_x(P(x), Q(x)) \in \mathbb{C}$ the resultant of P and Q with respect to x. Recall that P and Q share some complex root if and only if $\mathrm{Res}_x(P(x), Q(x)) = 0$, see [16].

We will also need the following result, which is essentially contained in ([2, 18]).

Proposition 7. *For each $p \in \mathbb{N}$ there is a polynomial $M_m \in \mathbb{Q}[x]$ of minimal degree $m = \phi(p)/2$ such that $M_m(\cos(2n\pi/p)) = 0$ for all $(p, n) = 1$. Moreover,*

(7) $$\mathrm{Res}_x\left(\Phi_p(x), x^2 - 2xv + 1\right) = M_m^2(v).$$

In particular, $m = 1$ if and only if $p \in \{1, 2, 3, 4, 6\}$; $m = 2$ if and only if $p \in \{5, 8, 10, 12\}$; $m = 3$ if and only if $p \in \{7, 9, 14, 18\}$; and $m = (p-1)/2$ when $p > 2$ is prime.

We only make some comments about (7). Notice that if $x = \cos(2n\pi/p) + \mathrm{i}\sin(2n\pi/p)$ then $\Phi_p(x) = 0$. Moreover, if we define $v = \cos(2n\pi/p)$ it holds that $v = (x + \bar{x})/2 = (x + 1/x)/2$, or equivalently $(x^2 - 2xv + 1)/x = 0$. Hence the polynomial $\mathrm{Res}_x\left(\Phi_p(x), x^2 - 2xv + 1\right)$ has v as one of its roots. Moreover, if we start with $\bar{x}$ instead of x we obtain again the same value v as a root of this last polynomial, making that all its roots are double.

Notice that, as a consequence of the above result, the only rational values of $\cos(2\pi/p)$ are

$$(8) \quad \cos(2\pi) = 1, \cos(2\pi/2) = -1, \cos(2\pi/3) = -1/2, \cos(2\pi/4) = 0, \cos(2\pi/6) = 1/2,$$

and the only values where $\cos(2\pi/p)$ is in $\mathbb{Q}[\sqrt{q}]$ for $q \in \mathbb{Q}$, but $\sqrt{q} \notin \mathbb{Q}$ are

$$(9) \quad \cos(2\pi/5) = (\sqrt{5} - 1)/4, \quad \cos(2\pi/10) = (\sqrt{5} + 1)/4, \quad \cos(2\pi/12) = \sqrt{3}/2.$$

For instance, the minimal polynomial for $p = 5$ is $M_2(x) = 4x^2 + 2x - 1$; the cases where M_m is cubic are, $8x^3 + 4x^2 - 4x - 1$ when $p = 7$; $8x^3 - 6x + 1$ when $p = 9$; $8x^3 - 4x^2 - 4x + 1$ when $p = 14$; and $8x^3 - 6x - 1$ when $p = 18$.

Next result provides some computable conditions to know whether a planar rational map with rational coefficients has a meromorphic first integral. In particular, notice that for many 1-parametric families of maps it allows to prove that this first integral can only exist for finitely many values of the parameter, see for instance Lemma 11.

Theorem 8. *Consider the rational map (6) with rational coefficients. Assume that f has a real fixed point $(\widehat{x}, \widehat{y}) \in \mathbb{R}^2$ and that $\mathrm{D}f(\widehat{x}, \widehat{y})$ has complex conjugated eigenvalues $\mu, \bar{\mu}$ with modulus different from 1. If the map (6) has a meromorphic first integral, then $\mu/\bar{\mu}$ is a root of the unity and there exists a computable polynomial $U_k \in \mathbb{Q}[x]$, of degree $k \in \mathbb{N}$, such that $U_k\left(\mathrm{Re}(\mu/\bar{\mu})\right) = 0$. Moreover, some of the values*

$$(10) \qquad \mathrm{Res}_x\left(U_k(x), V_p(x)\right), \text{ with } p \text{ such that } \deg(V_p(x)) \le k,$$

must vanish, where V_p is the minimal polynomial of $\mathrm{Re}(\mu/\bar{\mu}) = \cos(2n\pi/p)$, with $(p, n) = 1$, see Proposition 7.

Proof. We introduce the polynomials $S_1, S_2, T_1, T_2, D_1, D_2$ associated to f,

$$S_1(x, y) = P_1(x, y) - xQ_1(x, y), \quad S_2(x, y) = P_2(x, y) - yQ_2(x, y),$$

$$T(x, y) = \frac{T_1(x, y)}{T_2(x, y)} = \frac{\partial R_1(x, y)}{\partial x} + \frac{\partial R_2(x, y)}{\partial y},$$

$$D(x, y) = \frac{D_1(x, y)}{D_2(x, y)} = \frac{\partial R_1(x, y)}{\partial x}\frac{\partial R_2(x, y)}{\partial x} - \frac{\partial R_1(x, y)}{\partial y}\frac{\partial R_2(x, y)}{\partial x}.$$

Notice that if $(\widehat{x}, \widehat{y})$ is a fixed point of f then it is a solution of the system $\{S_1(x, y) = 0, S_2(x, y) = 0\}$, and moreover $Q_1(\widehat{x}, \widehat{y})Q_2(\widehat{x}, \widehat{y}) \neq 0$. Observe also that the eigenvalues $\mu = \mu(\widehat{x}, \widehat{y})$ of $\mathrm{D}f(\widehat{x}, \widehat{y})$ satisfy

$$(11) \qquad P(\mu) = \mu^2 - T(\widehat{x}, \widehat{y})\mu + D(\widehat{x}, \widehat{y}) = \mu^2 - T\mu + D = 0,$$

where, when there is no confusion, we omit the dependence of T, D and μ on the fixed point.

With this notation, the condition of having complex eigenvalues μ and $\bar{\mu}$ with modulus different from one reads simply as $T^2 - 4D < 0$ and $D \neq 1$, because they satisfy (11), $\mu\bar{\mu} = D$, and hence $|\mu|^2 = D^2$.

By Theorem 1, if f has a meromorphic first integral, then there exists $(0, 0) \neq (p, q) \in \mathbb{Z}^2$, such that $\mu^p \bar{\mu}^q = 1$. Taking norms, this means that $|\mu|^{p+q} = 1$. Since by hypothesis $|\mu| \neq 1$, we must have $q = -p$. Hence, $|\mu/\bar{\mu}| = 1$ and $\mu/\bar{\mu}$ is a p-root of the unity. Therefore $\mathrm{Re}(\mu/\bar{\mu}) = \cos(2n\pi/p)$ for some $n \in \{0, 1, \ldots, p-1\}$ with n and p coprime.

Let us write

$$\mu = \frac{T}{2} + \mathrm{i}\sqrt{\frac{4D - T^2}{4}} = \alpha + \mathrm{i}\beta, \quad \bar\mu = \frac{T}{2} - \mathrm{i}\sqrt{\frac{4D - T^2}{4}} = \alpha - \mathrm{i}\beta.$$

Then

$$\frac{\mu}{\bar\mu} = \frac{\alpha + i\beta}{\alpha - i\beta} = \frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2} + \mathrm{i}\frac{2\alpha\beta}{\alpha^2 + \beta^2}$$

and, as a consequence,

$$\mathrm{Re}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 + \beta^2}\right) = \frac{T^2}{2D} - 1 = \cos\left(2n\pi/p\right).$$

If we name $v = \cos\left(2n\pi/p\right)$, then the following system of three equations is satisfied:

$$(12) \qquad \begin{cases} S_1(x,y) = 0, \quad S_2(x,y) = 0, \\ W(x,y,v) = T_1^2(x,y)D_2(x,y) - 2(1+v)T_2^2(x,y)D_1(x,y) = 0. \end{cases}$$

As usual, taking successive resultants, as follows

$$T_1(y) = \mathrm{Res}_x\left(S_1(x,y), S_2(x,y)\right), \quad T_2(y,v) = \mathrm{Res}_x\left(S_1(x,y), W(x,y,v)\right),$$
$$U_k(v) = \mathrm{Res}_y\left(T_1(y), T_2(y,v)\right),$$

where k is de degree of U_k, we know that $U_k(v) = 0$ and $U_k \in \mathbb{Q}[x]$, as we wanted to prove, see again [16]. Hence, by Proposition 7, v has to be a root of some V_p, with $\deg(V_p(x)) \le k$. By the properties of the resultant the last statement of the theorem follows. $\square$

Remark 9. *Notice that the conditions in Theorem 8 depend on the value k given in its statement and obtained solving system (12). According to the fixed point $(\widehat{x}, \widehat{y})$ taken into account, some smaller k can be considered. To do this, let us treat this system in another way. We consider first the following two polynomials with rational coefficients,*

$$T_1(y) = \mathrm{Res}_x\left(S_1(x,y), S_2(x,y)\right), \quad T_3(x) = \mathrm{Res}_y\left(S_1(x,y), S_2(x,y)\right).$$

Let V_1 and V_3 be the irreducible factors in $\mathbb{Q}[x]$ of T_1 and T_3, respectively, such that $T_1(\widehat{y}) = 0$ and $T_3(\widehat{x}) = 0$. Then we can follow the same procedure that for system (12), but starting with the system

$$(13) \qquad\qquad V_1(y) = 0, \quad V_3(x) = 0, \quad W(x,y,v) = 0.$$

We arrive at a new $U_{k'} \in \mathbb{Q}[x]$, of degree $k' \le k$ and such that $U_{k'}(v) = 0$. Then the set of conditions (10) has to be satisfied only until k', taking $U_{k'}$ instead of U_k.

Corollary 10. *Consider the rational map (6) with rational coefficients. Assume that f has a fixed point $(\widehat{x}, \widehat{y}) \in \mathbb{Q}^2$ and $\mathrm{D}f(\widehat{x}, \widehat{y})$ has complex conjugated eigenvalues, $\mu, \bar\mu$, solution of*

$$\mu^2 - T(\widehat{x}, \widehat{y})\mu + D(\widehat{x}, \widehat{y}) = 0,$$

and with modulus different of 1. If the map (6) has a meromorphic first integral, then

$$\frac{T^2(\widehat{x}, \widehat{y})}{D(\widehat{x}, \widehat{y})} \in \{0, 1, 2, 3, 4\}.$$

Proof. By using Theorem 8, Remark 9 and that $(\widehat{x}, \widehat{y}) \in \mathbb{Q}^2$, we know that system (13) writes as

$$V_1(y) = y - \widehat{y}, \quad V_3(x) = x - \widehat{x}, \quad W(x,y,v) = 0.$$

Hence,

$$v = \frac{T^2(\widehat{x}, \widehat{y})}{2D(\widehat{x}, \widehat{y})} - 1 \in \mathbb{Q}$$

and $k = 1$ in Theorem 8. Therefore, the only possible values of $v = \cos(2n\pi/p)$ are the ones that are rational. From (8) we get that

$$\frac{T^2(\widehat{x}, \widehat{y})}{2D(\widehat{x}, \widehat{y})} - 1 \in \left\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1\right\}$$

and the corollary follows. $\square$

In the next lemma we apply the above corollary to a simple example.

Lemma 11. *The rational map*

$$f(x,y) = \left(xy, \frac{(a+(2-a)x)y}{1+xy} \right),$$

with $a \in \mathbb{Q}, a > 9/8$, can have meromorhic first integrals only when $a \in \{3/2, 2, 9/4, 9/2\}$.

Proof. It has a fixed point at $(1,1)$; the characteristic polynomial of $\mathrm{D}f(1,1)$ is $p(\mu) = \mu^2 - 3/2\mu + a/2$ and their eigenvalues $(3 \pm \mathrm{i}\sqrt{8a-9})/4$ are complex. Moreover, when $a \neq 2$ they have modulus different from 1. Notice that $T^2(\widehat{x},\widehat{y})/D(\widehat{x},\widehat{y}) = 9/(2a)$. Hence, by Corollary 10, when $a \neq 2$, the conditions for existence of meromorphic first integral are

$$\frac{9}{2a} \in \{0, 1, 2, 3, 4\},$$

giving the result of the statement. $\qquad\square$

Although we have not found any meromorphic first integral for the four possible cases given in the statement it is worth to mention that when $a = 1$ then f has the meromorphic (polynomial) first integral $H(x,y) = y(1+x)$, see also [14]. In fact, let us see which conditions for existence of meromorphic first integral are consequence of Theorem 1 when $a < 9/8$. In this case, let $r \in \mathbb{R}$ be the non-negative solution of $r^2 = 9 - 8a$. Then the eigenvalues at $(\widehat{x},\widehat{y})$ are $\mu_1 = (3-r)/4$ and $\mu_2 = (3+r)/4$. Hence, the condition for the existence of a meromorphic first integral is that $\mu_1^n \mu_2^m = 1$, for some non-zero $(n,m) \in \mathbb{Z}^2$. Equivalently, the only cases that might have a meromorphic first integral are either $r = 1$ or

$$\log\left(\frac{3-r}{4}\right) \Big/ \log\left(\frac{3+r}{4}\right) \in \mathbb{Q}.$$

Notice that $r = 1$ precisely corresponds to the integrable case given above $a = 1$.

Next result is similar to Corollary 10. The only difference is that each one of the coordinates of the fixed point is a zero of a quadratic polynomial with integer coefficients. Its proof is essentially the same, but instead of using (8) we will use (9) and, consequently, that the only values of $\cos(2n\pi/p)$, that are solutions of a quadratic polynomial in $\mathbb{Q}[x]$, irreducible in $\mathbb{Q}[x]$, are

$$\frac{-1 \pm \sqrt{5}}{4}, \pm\frac{\sqrt{2}}{2}, \frac{1 \pm \sqrt{5}}{4}, \pm\frac{\sqrt{3}}{2}.$$

We skip the details of the proof.

Corollary 12. *Consider the rational map (6) with rational coefficients. Assume that f has a fixed point $(\widehat{x},\widehat{y})$ with both coordinates in $\mathbb{Q}[\sqrt{s}] \setminus \mathbb{Q}$, $s \in \mathbb{Q}$, and $\mathrm{D}f(\widehat{x},\widehat{y})$ has complex conjugated eigenvalues $\mu, \bar{\mu}$ solution of*

$$\mu^2 - T(\widehat{x},\widehat{y})\mu + D(\widehat{x},\widehat{y}) = 0,$$

and with modulus different from 1. If the map (6) has a meromorphic first integral, then

$$\frac{T^2(\widehat{x},\widehat{y})}{D(\widehat{x},\widehat{y})} \in \left\{ \frac{3 \pm \sqrt{5}}{2}, 2 \pm \sqrt{2}, \frac{5 \pm \sqrt{5}}{2}, 2 \pm \sqrt{3} \right\}.$$

Similar results to Corollaries 10 and 12 could be stated assuming that the rational map f given in (6) has a fixed point $(\widehat{x},\widehat{y})$ with both coordinates being a zero of a polynomial of higher degree and the same conditions for the eigenvalues of $\mathrm{D}f(\widehat{x},\widehat{y})$ hold. As an example we apply our techniques to prove the non-meromorphic integrability of a concrete rational map. Consider

$$(14) \qquad\qquad f(x,y) = \left(x + y^2 - xy, \frac{x^2 + xy + 1}{x^2 - 3y + 1} \right),$$

and its fixed point $(\widehat{x},\widehat{y}) = (s,s)$, where $s \approx 4.836$ is the real root of $P(x) = x^3 - 5x^2 + x - 1$.

It is easy to see that we are under the hypotheses of Theorem 8, because $T^2(s,s) - 4D(s,s) < 0$ and $D(s,s) \neq 1$. For the sake of shortness, we omit the explicit expressions of $T(x,y)$ and $D(x,y)$. Moreover, in the notation of this theorem,

$$S_1(x,y) = y(y-x), \quad S_2(x,y) = -x^2y + x^2 + xy + 3y^2 - y + 1$$

and $W(x,y,v)$ is a polynomial of degree 10 that we do not explicit either. Then,

$$T_1(y) = \mathrm{Res}_x\left(S_1(x,y), S_2(x,y)\right) = -y^2 P(y),$$
$$T_3(x) = \mathrm{Res}_y\left(S_1(x,y), S_2(x,y)\right) = -(x^2+1)P(x).$$

Therefore, in the notation of Remark 9, $V_1(y) = P(y)$ and $V_3(x) = P(x)$. Finally,

$$\mathrm{Res}_x\left(\mathrm{Res}_y(W(x,y,v), V_1(y)), V_3(x)\right) = U_3(v)Z_6(v),$$

where $U_3, Z_6 \in \mathbb{Q}(x)$, $U_3(x) = 5833x^3 + 16607x^2 + 15650x + 4874$ and $U_3(\mathrm{Re}(\mu/\bar{\mu})) = 0$, with μ and $\bar{\mu}$ the eigenvalues of $\mathrm{D}f(s,s)$. Since U_3 has degree 3, to prove that system (14) has no meromorphic first integral it suffices to prove that all the resultants $\mathrm{Res}_x(U_3(x), Q(x)), Q \in \mathcal{Q}$, do not vanish, where

$$\mathcal{Q} = \{x, \, x-1, \, x+1, \, 2x+1, \, 2x-1, \, 2x^2-1, \, 4x^2-3, \, 4x^2+2x-1, \, 4x^2-2x-1,$$
$$8x^3 + 4x^2 - 4x - 1, \, 8x^3 - 6x + 1, \, 8x^3 - 4x^2 - 4x + 1, \, 8x^3 - 6x - 1\}.$$

The above set of polynomials corresponds to the only irreducible ones of degree at most 3 that have a root $\cos(2n\pi/p)$ for some $n, p \in \mathbb{Z}$, see the comments after Proposition 7. These thirteen resultants are all different from zero, and the result follows.

3.3. Higher dimensional examples.
As in the two dimensional case, the study of the sequences generated by n-th order difference equations

$$x_{k+n} = g(x_k, x_{k+1}, \ldots, x_{k+n-2}, x_{k+n-1})$$

can be reduced to the study of the DDS generated by the map

$$f(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_{n-1}, g(x_1, x_2, \ldots, x_n)),$$

from $\mathbb{C}^n$ into itself. Some of the maps that we will consider here have this shape.

We start studying the case of general analytic maps having the maximum possible number of functionally independent meromorphic first integrals: n.

Proposition 13. *Let f be an analytic map from $\mathbb{C}^n$ into itself, with isolated fixed points, and with n functionally independent meromorphic first integrals. If $\mathbf{q}$ is a fixed point of f, then all the eigenvalues μ, of $\mathrm{D}f(\mathbf{q})$ are p-roots of the unity. Moreover, there is a constructive procedure, described in the proof, to find a polynomial $P_k \in \mathbb{C}[x]$ of degree k such that, for all μ, $P_k(\mu) = 0$.*

Additionally, if f is a rational map and all the numerators and denominators of its components are polynomials in $\mathbb{Q}[x_1, \ldots, x_n]$, then $P_k \in \mathbb{Q}[x]$ and the eigenvalues of $Df(q)$ are p-roots of the unity with $p \leq M$, where M is the maximum $m \in \mathbb{N}$ such that the degree of the cyclotomic polynomial Φ_m is k. Furthermore, $\mathrm{Res}_x(\Phi_j(x), P_k(x)) = 0$ for some cyclotomic polynomial Φ_j with $\deg(\Phi_j) \leq k$.

Proof. By Theorem 1, the dimension of the $\mathbb{Z}$–linear space generated from $\{\mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1\}$ is n. Let $\mathbf{k}_i, i = 1, 2, \ldots, n$ be a basis of this space. Then, $\det(K) \neq 0$, where K is the $n \times n$ matrix $K = (\mathbf{k}_{i,j})$ and, moreover it can be seen that $\mu_\ell^{|\det(K)|} = 1$, for all $\ell = 1, 2, \ldots, n$, proving that $p = |\det(K)| \in \mathbb{N}$.

Set $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. Then, the eigenvalues of $\mathrm{D}f$ at a fixed point are the solutions $\mathbf{x}, \mu$ of the system of $n+1$ equations, $f(\mathbf{x}) = \mathbf{x}$, $P(\mathbf{x}, \mu) = 0$, where $P(\mathbf{x}, \mu)$ is the characteristic polynomial of $\mathrm{D}f(\mathbf{x})$ at an arbitrary point $\mathbf{x}$, not necessarily fixed.

Instead of considering them, we take their numerators as a polynomial system of $n+1$ equations and $n+1$ unknowns,

$$(15) \qquad g(\mathbf{x}) = \mathrm{Num}(f(\mathbf{x}) - \mathbf{x}) = 0, \quad R(\mathbf{x}, \mu) = \mathrm{Num}(P(\mathbf{x}, \mu)) = 0,$$

where Num denotes the numerator of a quotient of polynomials. Doing successive resultants following [16], as in the proof of Theorem 8, and because f has no continuum of fixed points, we arrive at a non-zero polynomial $P_k \in \mathbb{C}[x]$, such that $P_k(\mu) = 0$ for all eigenvalues μ of $Df(\mathbf{x})$ at any $\mathbf{x}$, fixed point of f, as we wanted to prove. Moreover, if f is rational, with numerator and denominator in $\mathbb{Q}[\mathbf{x}]$, then $P_k \in \mathbb{Q}[x]$.

Finally, it is known that given a primitive p-th root of the unity, μ, the minimal degree of a polynomial $S \in \mathbb{Q}[x]$ such that $S(\mu) = 0$ is $\phi(p) = \deg(\Phi_p(x))$. Hence, $p \le M$, as we wanted to prove. Moreover P_k must share some root with one of the polynomials Φ_j with $\phi(j) \le k$, and as a consequence, for this value of j, $\mathrm{Res}_x(\Phi_j(x), P_k(x)) = 0$. $\square$

As an example of application we prove the following lemma.

Lemma 14. *Consider the map*

$$(16) \qquad f(x, y, z) = \left(y, z, \frac{a + y + z}{x} \right), \quad a \in \mathbb{Q}.$$

If it has 3 functionally independent meromorphic first integrals, then $a \in \{-1, 1\}$.

Proof. We will apply Proposition 13. We start constructing a reduced, but equivalent, version of system (15). Notice that the fixed points of f are (x, x, x) such that $a + 2x = x^2$ and

$$\mathrm{D}f(x, x, x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{a+y+z}{x^2} & \frac{1}{x} & \frac{1}{x} \end{pmatrix}_{(x,y,z)=(x,x,x)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & \frac{1}{x} & \frac{1}{x} \end{pmatrix}.$$

Hence, the characteristic polynomial at any fixed point is $(\mu + 1)(\mu^2 - (1 + 1/x)\mu + 1)$. Therefore, since $\mu = -1$ is a root of the unity, system (15) with 4 unknowns can be simply reduced to a system of 2 equations and 2 unknowns, x and μ, and a parameter a:

$$-x^2 + 2x + a = 0, \quad (\mu^2 - \mu + 1)x - \mu = 0.$$

Hence

$$P_4(\mu) = \mathrm{Res}_x(-x^2 + 2x + a, (\mu^2 - \mu + 1)x - \mu) = a\mu^4 - 2(a-1)\mu^3 + 3(a-1)\mu^2 - 2(a-1)\mu + a = 0.$$

Therefore, we only have to consider the values of p for which the cyclotomic polynomial Φ_p has degree at most 4. They are $p \in \mathcal{P} = \{1, 2, 3, 4, 5, 6, 8, 10, 12\}$. By doing all the resultants $\mathrm{Res}_x(P_4, \Phi_p(x))$, with $p \in \mathcal{P}$, we can discard all the values $a \in \mathbb{Q}$ but $a = -1, 7/9, 5/4, 3, 1$, that correspond to $p = 1, 2, 3, 4, 8$, respectively. For instance, we get that $\mathrm{Res}_x(P_4, \Phi_3(x)) = (4a - 5)^2$ and $\mathrm{Res}_x(P_4, \Phi_8(x)) = (a - 1)^2$, but $\mathrm{Res}_x(P_4, \Phi_{10}(x)) = (a^2 - a - 1)^2$. Finally, the values $a \in \{7/9, 5/4, 3\}$ are also discarded because, for them, some of the roots of P_4 are not roots of the unity. For instance, $P_4(\mu)|_{a=3} = (\mu^2 + \mu + 1)(5\mu^2 - 7\mu + 5)/4$. On the other hand,

$$P_4(\mu)|_{a=-1} = -(\mu - 1)^4 \quad \text{and} \quad P_4(\mu)|_{a=1} = \mu^4 + 1,$$

and all the roots of both polynomials are roots of the unity. $\square$

It is worth to comment that for $a = 1$ it is well known that the map (16) has effectively 3 functionally independent rational first integrals. Two of them exist for any $a \in \mathbb{C}$. They are

$$H_1(x, y, z) = \frac{(x + 1)(y + 1)(z + 1)(a + x + y + z)}{xyz},$$

$$H_2(x, y, z) = \frac{(1 + x + y)(1 + y + z)(a + x + y + z + xz)}{xyz},$$

see for instance [6] and its references, and a third one can be seen in [4] and it is found by using the tools introduced in that paper. It exists because this map, for $a = 1$, corresponds to the celebrated 3$^{\mathrm{rd}}$ order Todd's difference equation $x_{n+3} = (1 + x_{n+2} + x_n)/x_n$ which is globally 8-periodic, that is $x_{n+8} = x_n$ for all n, whenever x_k is well defined. We believe that when $a = -1$ the map has only the above 2 functionally independent meromorphic first integrals, but from our approach we cannot discard the existence of a third one. In

fact, it is known, even for $a \in \mathbb{C}$, that the only globally periodic map corresponds to $a = 1$, see [8, 10].

There is also a simple case for which the non-existence of meromorphic first integral can be easily established.

Corollary 15. *Let f be an analytic diffeomorphism with a fixed point $\mathbf{q} \in \mathbb{C}^n$ and assume that the eigenvalues of $\mathrm{D}f(\mathbf{q})$ are n different prime numbers. Then f has no meromorphic first integral.*

Proof. By Theorem 1 we have to calculate the dimension of the $\mathbb{Z}$–linear space generated from $\{\mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1\}$ where here $\mu = (p_1, p_2, \ldots, p_n)$ and the p_j are the n different prime numbers. Hence, from the condition $p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n} = 1$ we get $k_1 = k_2 = \cdots = k_n = 0$ and this dimension is 0. As a consequence, the map has no meromorphic first integral. $\square$

There are many other rational maps in dimension $n > 2$ that admit meromorphic (indeed rational) first integrals. As a final example we show one of them, obtained from the paper [15], dedicated to study systems of difference equations with invariants.

The map

$$f(x, y, z, t) = \left(z, t, \frac{az + bt + c}{x}, \frac{az + bt + c}{y} \right)$$

has the first integral

$$H(x, y, z, t) = \frac{(xy + ay + bx)(zt + at + bz)(ax + az + bt + by + c)}{xyzt}.$$

By using our result we could embed the above map into a large family, with more parameters, and find necessary conditions among them for the cases with one or more meromorphic first integrals.

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