INVARIANT CURVES AND INTEGRABILITY OF PLANAR $\mathcal{C}^r$ DIFFERENTIAL SYSTEMS

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Abstract. We improve the known expressions of the $\mathcal{C}^r$ differential systems in the plane having a given $\mathcal{C}^{r+1}$ invariant curve, or a given $\mathcal{C}^{r+1}$ first integral. Their application to polynomial differential systems having either an invariant algebraic curve, or a first integral also improves the known results on such systems.

1. Introduction and statement of the main results

A $\mathcal{C}^r$ *real planar differential system* is a system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1}$$

where $(x,y) \in D \subseteq \mathbb{R}^2$, D is the domain of definition of the system and $P, Q \in \mathcal{C}^r$, where r is a positive integer, or $r = \infty$, or $r = \omega$ (meaning that the system is analytic). The vector field associated to system (1) is

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

In what follows we shall talk indistinctly of the differential system (1) or of its vector field X.

Let $\mathbb{R}[x,y]$ be the ring of polynomials in the variables x, y with real coefficients. If $P, Q \in \mathbb{R}[x,y]$ then we say that the differential system (1) is *polynomial*. In such a case, we define the *degree* of this polynomial system as $\max\{\deg P, \deg Q\}$, where $\deg P, \deg Q$ are the degree of P and Q, respectively.

Let U be an open subset of $\mathbb{R}^2$. A *first integral* of X in U is a locally non–constant function $H : U \to \mathbb{R}$ which is constant on all the solutions of X contained in U, i.e. $X(H) = 0$ in the points of U. In this case we also say that X is *integrable* on U.

Let $g \in \mathcal{C}^r$. The curve $g = 0$ is *invariant* under the flow of system (1) if

$$X(g)|_{g=0} = \left(P\frac{\partial g}{\partial x} + Q\frac{\partial g}{\partial y} \right)\bigg|_{g=0} = 0.$$

The *Nambu bracket* (or the *Lie bracket*, or the *Jacobian*) of two functions $f, g \in \mathcal{C}^1$ is

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

We start our results by finding a way (inspired in Corollary 1.3.3 of [9]) for writing all systems (1) having an invariant curve $g = 0$.

Theorem 1. *Let $g = 0$ be a $\mathcal{C}^{r+1}$ invariant curve of system* (1)*. Then for any $\mathcal{C}^{r+1}$ function f such that $\{g,f\} \not\equiv 0$ we have*

$$\dot{x} = P = \frac{X(g)f_y - X(f)g_y}{\{g,f\}}, \quad \dot{y} = Q = \frac{-X(g)f_x + X(f)g_x}{\{g,f\}}. \tag{2}$$

The following result is well-known, for a proof see for instance [3].

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Theorem 2. *Assume that a polynomial differential system has an invariant algebraic curve $g(x, y) = 0$ such that there are no points at which g and its first derivatives all vanish. If $(g_x, g_y) = 1$, then the system has the following normal form:*

$$\dot{x} = Ag - Dg_y, \quad \dot{y} = Bg + Dg_x, \tag{3}$$

where A, B, D are suitable polynomials.

The hypotheses in the previous theorem are necessary, as the polynomial differential system given in the next example shows. However applying Theorem 1 we can obtain the polynomial differential system of the example if the functions A, B, D are not necessarily polynomials.

Example 1. Consider the polynomial differential system of degree five appearing in [2]

$$\dot{x} = 2x + y - 3x^4, \quad \dot{y} = 4y + 2x^3 - 9x^3y + 3x^5.$$

The algebraic curve $g(x, y) = (y - x^2)(y - x^3) = 0$ is invariant under the flow of this differential system. Observe that g and its first derivatives vanish at the origin, and also at $(1, 1)$.

Note that we cannot obtain $\dot{y}$ from the expression $Bg + Dg_x$ of Theorem 2 because the lowest degree of g is 2 and $x|g_x$. Hence the hypotheses of Theorem 2 do not hold for this example and it is easy to see that there do not exist polynomials A, B, D such that (3) holds because in $Bg + Dg_x$ the term $4y$ of $\dot{y}$ cannot appear.

Theorem 1 has no hypotheses besides $\{g, f\} \not\equiv 0$. If we apply it with $f = y$, then we have

$$\begin{aligned}\frac{X(g)f_y - X(f)g_y}{\{g, f\}} &= \frac{(2 - 3x)(4 + 5x + 6x^2)}{x(5x^3 - 2y - 3xy)} g - \frac{4y + 2x^3 - 9x^3y + 3x^5}{x(5x^3 - 2y - 3xy)} g_y \\ &= 2x + y - 3x^4 = \dot{x},\end{aligned}$$

$$\begin{aligned}\frac{-X(g)f_x + X(f)g_x}{\{g, f\}} &= \frac{4y + 2x^3 - 9x^3y + 3x^5}{x(5x^3 - 2y - 3xy)} g_x \\ &= 4y + 2x^3 - 9x^3y + 3x^5 = \dot{y}.\end{aligned}$$

Hence we obtain $\dot{x} = Ag - Dg_y, \dot{y} = Bg + Dg_x$ with the rational functions

$$A = \frac{(2 - 3x)(4 + 5x + 6x^2)}{x(5x^3 - 2y - 3xy)}, \quad B = 0, \quad D = \frac{4y + 2x^3 - 9x^3y + 3x^5}{x(5x^3 - 2y - 3xy)}.$$

□

The arguments used in the previous example to construct the vector field can be generalized. The following theorem includes the condition $g_x \not\equiv 0$ on a curve $g = 0$. This condition is not restrictive. If $g_x \equiv 0$ then $g_y \not\equiv 0$ (otherwise g is a constant), and in this case the theorem applies swapping x and y.

Theorem 3. *Assume that a polynomial differential system has an invariant algebraic curve $g = g(x, y) = 0$. If $g_x \not\equiv 0$, then the system has the following normal form:*

$$\dot{x} = Ag - Bg_y, \quad \dot{y} = Bg_x, \tag{4}$$

where A, B are suitable rational functions. Conversely, if a polynomial differential system can be written as in (4), then the curve $g = 0$ is invariant under the flow of this differential system.

Note that Theorem 3 provides all the polynomial differential systems having a given invariant algebraic curve without any assumptions on the curve, while Theorem 2 needs some assumptions.

When system (1) is polynomial and $g = 0$ is an invariant algebraic curve, there exists a polynomial k, called the *cofactor*, that satisfies the equation $X(g) = kg$. The polynomial cofactors play a main role in the Darboux theory of integrability (see [1, 5, 6, 10]). Of course we can write this equation as $k = X(g)/g$, and this is a polynomial, so $g|X(g)$. If g is an invariant curve of system (1) we can define also its *cofactor* as the function $k = X(g)/g$. We know that in the paper [7] the authors improve the Darboux theory of integrability using invariant curves which are not necessarily algebraic but having polynomial cofactors. Note that in our case the cofactors are in general non-polynomial.

It is widely known that if a linear combination of cofactors of invariant algebraic curves of a polynomial differential system is zero, then this system has a Darboux first integral (see Theorem 8.7 of [6]). We state here our third theorem. It generalizes the classical Darboux Theorem for polynomial differential systems to $\mathcal{C}^r$ differential systems with $\mathcal{C}^{r+1}$ invariant curves.

Theorem 4. *Consider the $\mathcal{C}^r$ differential system* (1) *and let $g_i = 0$ be a $\mathcal{C}^{r+1}$ invariant curve of* (1) *with cofactor k_i for $i = 1, \ldots, M$. Then $H = \prod_{i=1}^{M} g_i^{\nu_i}$ is a first integral of* (1) *if and only if $\sum_{i=1}^{M} \nu_i k_i = 0$, for some convenient $\nu_i \in \mathbb{R}$.*

Theorem 4 will be proved in section 4.

A function $V(x, y)$ is an *inverse integrating factor* of the differential system (1) if the differential system

$$\dot{x} = \frac{P}{V} = -H_y, \quad \dot{y} = \frac{Q}{V} = H_x,$$

is Hamiltonian, with $H(x, y)$ the Hamiltonian function. For more information about the inverse integrating factor see [8]. Concerning the differential system (1) and the inverse integrating factors, we have the following theorem, inspired in Corollary 1.4.4 of [9].

Theorem 5. *If the differential system* (1) *has a first integral H, then we can write it as*

$$\dot{x} = \frac{X(f)}{\{H, f\}}\{H, x\}, \quad \dot{y} = \frac{X(f)}{\{H, f\}}\{H, y\},$$

where f is an arbitrary function. Moreover,

$$V(x, y) = \frac{X(f)}{\{H, f\}}$$

is an inverse integrating factor of the system.

We finally deal with differential systems having a first integral of the form $ge^{h/g}$. We have the following result.

Proposition 6. *Suppose that the differential system* (1) *has the first integral*

$$H(x, y) = ge^{h/g},$$

where $g, h \in \mathcal{C}^{r+1}$ are such that $\{g, h\} \neq 0$. Then the function $V = -X(g)g/\{g, h\}$ is an inverse integrating factor of the system.

2. Preliminary results

We state here some results of [9] that we shall use later on. The first one (see Corollary 1.3.3 of [9]) provides the planar differential systems having a given invariant curve.

Theorem 7. *Let $g(x, y) = 0$ be a function defined in an open set $D \subseteq \mathbb{R}^2$. Then any differential system defined in D for which $g = 0$ is invariant can be written as*

$$\dot{x} = \phi\frac{\{x, f\}}{\{g, f\}} + \lambda\frac{\{g, x\}}{\{g, f\}},$$
$$\dot{y} = \phi\frac{\{y, f\}}{\{g, f\}} + \lambda\frac{\{g, y\}}{\{g, f\}},$$

where f, ϕ and λ are arbitrary functions such that $\phi|_{g=0} = 0$ and $\{g, f\} \neq 0$ in D.

Next theorem (see Corollary 1.4.4 of [9]) provides the planar differential systems having a given first integral.

Theorem 8. *Let $H(x, y)$ be a function defined in an open set $D \subseteq \mathbb{R}^2$. Then the most general differential systems defined in D which admit the first integral H are*

$$\dot{x} = \lambda \frac{\{H, x\}}{\{H, f\}}, \quad \dot{y} = \lambda \frac{\{H, y\}}{\{H, f\}},$$

where f is an arbitrary function such that $\{H, f\} \neq 0$ in D.

3. Proofs

Proof of Theorem 1. According to Theorem 7, system (1) can be written in D as

$$\begin{aligned} \dot{x} = P(x, y) = \phi \frac{\{x, f\}}{\{g, f\}} + \lambda \frac{\{g, x\}}{\{g, f\}}, \\ \dot{y} = Q(x, y) = \phi \frac{\{y, f\}}{\{g, f\}} + \lambda \frac{\{g, y\}}{\{g, f\}}, \end{aligned} \tag{5}$$

where ϕ is an arbitrary function such that $\phi|_{g=0} = 0$, $\{g, f\} \not\equiv 0$ in D, and λ, f are arbitrary functions.

We note that we can actually compute an expression for ϕ, λ in terms of g, f just solving the linear system (5) that defines P and Q in the unknowns ϕ and λ, which has a unique solution because its determinant is one. This linear system is

$$\begin{aligned} \{x, f\}\phi + \{g, x\}\lambda = P\{g, f\}, \\ \{y, f\}\phi + \{g, y\}\lambda = Q\{g, f\}, \end{aligned}$$

or equivalently

$$\begin{aligned} f_y \phi - g_y \lambda = P\{g, f\}, \\ -f_x \phi + g_x \lambda = Q\{g, f\}, \end{aligned}$$

and by using the Cramer method, we have

$$\phi = \frac{\begin{vmatrix} P\{g, f\} & -g_y \\ Q\{g, f\} & g_x \end{vmatrix}}{\begin{vmatrix} f_y & -g_y \\ -f_x & g_x \end{vmatrix}} = \frac{\{g, f\}(P g_x + Q g_y)}{\{g, f\}} = X(g),$$

and

$$\lambda = \frac{\begin{vmatrix} f_y & P\{g, f\} \\ -f_x & Q\{g, f\} \end{vmatrix}}{\begin{vmatrix} f_y & -g_y \\ -f_x & g_x \end{vmatrix}} = \frac{\{g, f\}(Q f_y + P f_x)}{\{g, f\}} = X(f).$$

Hence the only arbitrary function in (5) is f, and it must satisfy that $\{g, f\} \not\equiv 0$ in D. □

Proof of Theorem 3. We assume in this proof that $g_x \not\equiv 0$. If $g_x \equiv 0$ then we must have $g_y \not\equiv 0$, and we can swap x and y.

Theorem 1 assures that all the differential systems having $g = 0$ as invariant write as system (2). The function f which appears in Theorem 1 is arbitrary, because the simplification of the quotients in (2) provide directly P and Q. In particular, we can fix $f(x, y) = y$, for which $X(f) = Q$ and $\{g, f\} = g_x \not\equiv 0$. Moreover, system (2) writes as

$$\dot{x} = P = \frac{X(g) - Q g_y}{g_x}, \quad \dot{y} = Q = \frac{Q g_x}{g_x}. \tag{6}$$

Since we are assuming that $g = 0$ is invariant, there must exist a polynomial $k(x, y)$ such that $X(g) = kg$. Substituting in (6) we have

$$\dot{x} = P = \frac{kg - Q g_y}{g_x}, \quad \dot{y} = Q. \tag{7}$$

In order to obtain a polynomial differential system we have to impose that $(kg - Qg_y)/g_x$ is a polynomial. That is, we need to fix the coefficients of k and Q in such a way that P is a polynomial. If we succeed, then the differential system (7) is polynomial and moreover $g = 0$ is invariant for system (7).

Clearly there exist two polynomials A and B such that

$$kg - Qg_y = Ag_x + B.$$

Then

$$P = \frac{kg - Qg_y}{g_x} = A + \frac{B}{g_x},$$

with $\deg B < \deg g - 1$. We need to obtain conditions on the coefficients of k and Q such that $B = 0$. The equation $B = 0$ can be written as a linear system of equations, one equation for each monomial of B. The unknowns of this linear system are the coefficients of k and Q. Of course the degrees of k and Q must be large enough to assure that the system is compatible.

We note that we can always find a solution, i.e. the differential system $\dot{x} = g - g_y$, $\dot{y} = g_x$ has the curve $g = 0$ invariant; its cofactor is g_x. So the system coming from $B = 0$ is compatible if we have enough unknowns. More precisely, if there are enough coefficients of k and Q; or equivalently if the degrees of k and Q are large enough.

Since $\deg B \leq \deg g - 2$, the linear system has at most $\binom{\deg g}{2}$ equations. The number of unknowns is $\binom{\deg k + 2}{2} + \binom{\deg Q + 2}{2}$. Hence we must have

$$\binom{\deg k + 2}{2} + \binom{\deg Q + 2}{2} \geq \binom{\deg g}{2}.$$

Once the linear system is solved, some coefficients of k and Q are fixed, and some other may remain free. Therefore $B = 0$ and P is a polynomial. Those free coefficients provide all the polynomial differential systems having the curve g invariant.

We have obtained the most general polynomial differential system $\dot{x} = P$, $\dot{y} = Q$ having the curve $g = 0$ invariant. Moreover, k is the cofactor of $g = 0$. The degree of the system is either $\deg Q$ in the case $\deg k < \deg Q$, or $\deg k + 1$ otherwise. So the theorem follows. $\square$

4. Invariant objects

4.1. **Darboux first integrals.** Next we prove Theorem 4.

Proof of Theorem 4. We compute $X(H)$:

$$X(H) = X\left(\prod_{i=1}^{M} g_i^{\nu_i}\right) = \sum_{i=1}^{M} \nu_i X(g_i) g_i^{\nu_i - 1} \prod_{j \neq i} g_j^{\nu_j} = \sum_{i=1}^{M} \nu_i k_i g_i^{\nu_i} \prod_{j \neq i} g_j^{\nu_j} = \left(\sum_{i=1}^{M} \nu_i k_i\right) H.$$

Then the theorem follows. $\square$

In the Darboux theory of integrability for complex polynomial differential systems (and consequently also for real polynomial differential systems) there exists a minimum number of invariant algebraic curves that assures a first integral. This is due to the linear combination of cofactors: these cofactors are polynomials of degree lower than the degree of the polynomial system, say m; that is they belong to the vector space of all complex polynomials in the variables x, y of degree at most $m - 1$. A base of this ring has $\binom{m+1}{2}$ elements. If we have more than $\binom{m+1}{2}$ cofactors then there must exist a linear combination of them being zero.

For the $\mathcal{C}^r$ differential systems in the real plane this kind of facts do not occur in general, so a minimum number of curves that assures a first integral cannot be stablished unless we restrict the class of differential systems.

4.2. **The inverse integrating factor.** We first prove Theorem 5.

Proof of Theorem 5. If the differential system (1) has a first integral H, then from Theorem 8, we know that we can write it as

$$\dot{x} = \lambda \frac{\{H,x\}}{\{H,f\}}, \quad \dot{y} = \lambda \frac{\{H,y\}}{\{H,f\}}, \tag{8}$$

where f is an arbitrary function such that $\{H,f\} \not\equiv 0$. We prove that $X(f) = \lambda$, indeed

$$X(f) = \lambda \frac{\{H,x\}}{\{H,f\}} f_x + \lambda \frac{\{H,y\}}{\{H,f\}} f_y = \lambda \frac{-H_y f_x + H_x f_y}{\{H,f\}} = \lambda.$$

From (8), the function $V = X(f)/\{H,f\}$ is an inverse integrating factor, because we have

$$\dot{x} = P = -V H_y, \quad \dot{y} = Q = V H_x.$$

Notice that $V = 0$ is an invariant curve of system (1). Hence the theorem follows. □

When the differential system (1) has an inverse integrating factor, the associated differential system $\dot{x} = P/V$, $\dot{y} = Q/V$ is Hamiltonian, so the area of any region of the domain of definition of this Hamiltonian system is the same after it is moved forward or backward by the flow of the system. In the set $\{V = 0\}$ this system is not Hamiltonian. The orbits that in their neighborhood do not allow this preservation of the area are contained into $\{V = 0\}$.

In the next example we study a polynomial differential system having a polynomial inverse integrating factor V. The set $\{V = 0\}$ in this example is formed by a focus and a limit cycle. The area is not preserved close to these two orbits, hence the corresponding Hamiltonian system cannot be defined in them.

Example 2. Consider the polynomial differential system

$$\dot{x} = -y + x(x^2 + y^2 - 1), \quad \dot{y} = x + y(x^2 + y^2 - 1),$$

which appears in Example 2 of page 126 of [11]. Using polar coordinates it is easy to see that this system has a focus at the origin and a limit cycle at the circle $x^2 + y^2 = 1$. Moreover it has the polynomial inverse integrating factor $V(x,y) = (x^2 + y^2)(x^2 + y^2 - 1)$. The system has the Darboux first integral

$$H(x,y) = \frac{x^2 + y^2 - 1}{x^2 + y^2}\, e^{2 \arctan \frac{y}{x}}.$$

We can write it in polar coordinates as

$$H(r,\theta) = \frac{r^2 - 1}{r^2}\, e^{2(\theta + k\pi)}, \ k \in \mathbb{Z}.$$

□

4.3. **No exponential factors.** When system (1) is polynomial and we have an invariant algebraic curve $g = 0$ and an additional invariant algebraic curve close to $g = 0$, say $g_\varepsilon = g + \varepsilon f + \mathcal{O}(\varepsilon^2)$, we can define an exponential factor, which is a function $F = e^{f/g}$ such that $X(F)/F$ is a polynomial. So the notion of exponential factor is associated to the notion of multiplicity of an invariant algebraic curve, see [4] for more details.

When dealing with $\mathcal{C}^{r+1}$ invariant curves of a $\mathcal{C}^r$ differential system the notion of exponential factor has no sense, because close to a $\mathcal{C}^{r+1}$ invariant curve there are infinitely many other $\mathcal{C}^{r+1}$ invariant curves.

5. A special Darboux function

We consider in this section differential systems (1) having a first integral of the form $H(x,y) = ge^{h/g}$, where $h, g \in \mathcal{C}^{r+1}$. We prove Proposition 6.

Proof of Proposition 6. We have

$$\frac{\partial \log H}{\partial x} = \frac{gg_x + h_x g - hg_x}{g^2} = \frac{Q}{V}, \quad \frac{\partial \log H}{\partial y} = \frac{gg_y + h_y g - hg_y}{g^2} = -\frac{P}{V},$$

where V is the inverse integrating factor associated to the first integral $\log H$. Note that $V = -P/(\log H)_y = Q/(\log H)_x$. Since $g = 0$ is invariant under the flow of system (1), from the expressions of P and Q given in (2) we have

$$V = -\frac{(X(g)f_y - X(f)g_y)g^2}{\{g,f\}(g_y g + h_y g - hg_y)} = \frac{(-X(g)f_x + X(f)g_x)g^2}{\{g,f\}(g_x g + h_x g - hg_x)},$$

where f is an arbitrary function such that $\{g,f\} \neq 0$. This last equality can be written as

$$gX(f)\{h,g\} + (h-g)X(g)\{g,f\} - gX(g)\{h,f\} = 0. \tag{9}$$

Since f is arbitrary and $\{g,h\} \neq 0$, we can set $f = h$. Then

$$\big(X(g)(h-g) - gX(h)\big)\{g,h\} = 0,$$

or equivalently

$$X(h) = \frac{X(g)}{g}(h-g). \tag{10}$$

When the differential system (1) is polynomial, equation (10) is equivalent to say that $-X(g)/g$ is the cofactor of the exponential factor $e^{h/g}$.

Back to the expression of V we have

$$V = \frac{Q}{(\log H)_x} = \frac{\dfrac{-X(g)h_x + X(h)g_x}{\{g,h\}}}{\dfrac{g_x g + h_x g - hg_x}{g^2}} = \frac{X(g)g(-gh_x + (h-g)g_x)}{\{g,h\}(g_x g + h_x g - hg_x)} = -\frac{X(g)g}{\{g,h\}}.$$

Then the proposition follows. □

Note that if the differential system of Proposition 6 is polynomial, then $e^{h/g}$ is an exponential factor with cofactor $-X(g)/g$.

Remark 1. *If $\{g,h\} = 0$ in the proof of Proposition 6, then, from (9), we can obtain*

$$(h-g)\{g,f\} - g\{h,f\} = 0,$$

for all f such that $\{g,f\} \neq 0$. In particular, for $f = \log H$, we have

$$(h-g)\{g,\log H\} - g\{h,\log H\} = (h-g)\left(-g_x\frac{P}{V} - g_y\frac{Q}{V}\right) - g\left(-h_x\frac{P}{V} - h_y\frac{Q}{V}\right)$$

$$= -(h-g)\frac{X(g)}{V} + g\frac{X(h)}{V} = -\frac{X(g)(h-g) - gX(h)}{V} = 0.$$

So we also obtain (10) in the case $\{g,h\} = 0$.

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