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INVARIANT CURVES AND INTEGRABILITY OF PLANAR $\mathcal{C}^r$ DIFFERENTIAL SYSTEMS

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ABSTRACT. We improve the known expressions of the $\mathcal{C}^r$ differential systems in the plane having a given $\mathcal{C}^{r+1}$ invariant curve, or a given $\mathcal{C}^{r+1}$ first integral. Their application to polynomial differential systems having either an invariant algebraic curve, or a first integral, also improves the known results on such systems.

1. Introduction and statement of the main results. A $\mathcal{C}^r$ *real planar differential system* is a system of the form

$$\dot{x} = P(x,y), \qquad \dot{y} = Q(x,y), \tag{1.1}$$

where $(x,y) \in D \subseteq \mathbb{R}^2$, D is the domain of definition of the system and $P, Q \in \mathcal{C}^r$, where r is a positive integer, or $r = \infty$, or $r = \omega$ (meaning that the system is analytic). The vector field associated to system (1.1) is

$$X(x,y) = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}. \tag{1.2}$$

In what follows, we shall talk indistinctly of the differential system (1.1) or of its vector field X.

Let $\mathbb{R}[x,y]$ be the ring of polynomials in variables x and y with real coefficients. If $P, Q \in \mathbb{R}[x,y]$, then we say that the differential system (1.1) is *polynomial*. In such a case, we define the *degree* of this

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