CONVEX CENTRAL CONFIGURATIONS OF THE 4–BODY PROBLEM WITH TWO PAIRS OF EQUAL ADJACENT MASSES

ANTONIO CARLOS FERNANDES[1], JAUME LLIBRE[2] AND LUIS FERNANDO MELLO[1]

Abstract. We study the convex central configurations of the 4–body problem assuming that they have two pairs of equal masses located at two adjacent vertices of a convex quadrilateral. Under these assumptions we prove that the isosceles trapezoid is the unique central configuration.

1. Introduction and statement of the main results

The classical Newtonian n–body problem studies a system formed by n punctual bodies with positives masses $m_1, \ldots, m_n$ and position vectors $r_1, \ldots, r_n$ in $\mathbb{R}^d$, $d = 2, 3$, interacting under the Newton's gravitational law [20]. The equations of motion of this problem are

$$\ddot{r}_i = \frac{d^2 r_i}{dt^2} = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{m_j}{r_{ij}^3} (r_i - r_j), \tag{1}$$

for $i = 1, \ldots, n$, where $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the bodies at r_i and r_j, and t is the independent variable called time. Taking the unit of mass conveniently we can assume that the gravitational constant $G = 1$ in (1).

An interesting class of particular solutions of the n–body problem (1) are the *homographic* solutions in which the shape of the configuration is preserved as time varies. The first homographic solutions were found by Euler [10] and Lagrange [13] in the 3–body problem.

We say that at a given instant $t = t_0$ the n bodies are in a *central configuration* if for all $i = 1, \ldots, n$ there exists a constant $\lambda \neq 0$ such that $\ddot{r}_i = \lambda(r_i - c)$ where c is the center of mass of the n bodies, that is

$$c = \frac{1}{m_1 + \ldots + m_n} \sum_{j=1}^{n} m_j \, r_j.$$

Such configurations are closely related with homographic solutions. In fact, the configuration of bodies at any time in a homographic solution is a central configuration. For more details see for instance [19, 22, 23, 25].

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To find a central configuration is reduced to find a solution of a nonlinear system of equations, because from equations (1) and the definition of a central configuration, we must solve the system of equations

$$\lambda(r_i - c) = -\sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}\,(r_i - r_j), \tag{2}$$

for $i = 1, \ldots, n$. Equations (2) are called the equations of the central configurations.

Two central configurations $(r_1, \ldots, r_n)$ and $(\bar{r}_1, \ldots, \bar{r}_n)$ of the n bodies are *related* if we can pass from one to the other through a dilation and a rotation (centered at the center of mass). So we can study the classes of central configurations defined by the above equivalence relation.

Taking into account this equivalence relation we have exactly five classes of central configurations in the 3–body problem. The finiteness of the number of central configurations performed by n bodies with positive masses is a question posed by Chazy [6], Wintner [25] and reformulated to the planar case by Smale [24]. For $n = 4$ this problem has an affirmative answer given by Hampton and Moeckel [12]. Recently, another proof of this finiteness for $n = 4$ has been given by Albouy and Kaloshin, see [4], where some results on the finiteness for $n = 5$ are also given. But the problem on the finiteness of the classes of central configurations remains open for $n \geq 5$.

In the planar 4–body problem a configuration is *convex* if there is not a body located in the interior of the convex hull of the other three, otherwise the configuration is *concave*, see Figure 1.

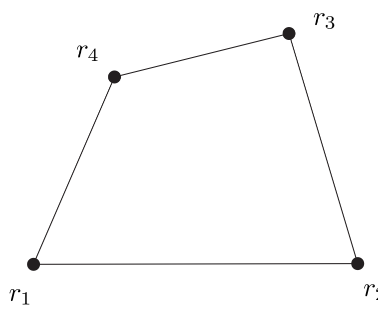

FIGURE 1. A convex 4–body configuration.

In [16], a landmark for the study of convex central configurations in the planar 4–body problem, MacMillan and Bartky proved the following existence theorem.

Theorem 1. *For any positive values of m_1, m_2, m_3 and m_4 there exists a convex planar central configuration of the 4–body problem with these masses.*

MacMillan and Bartky provided information on the admissible shapes of the 4–body convex central configurations.

Theorem 2. *In a convex 4–body central configuration*

(i) *the diagonals are greater than all exterior sides, and*

(ii) *the biggest side is opposite to the smallest one.*

MacMillan and Bartky also provided information on the isosceles trapezoid central configuration in the 4–body problem assuming the isosceles trapezoid symmetry in the hypotheses.

Theorem 3. *In a convex configuration of* 4 *bodies with position vectors oriented counterclockwise, if* $r_{13} = r_{24}$ *and* $r_{23} = r_{14}$*, then for each pair of positive values* m *and* μ *there exists a unique isosceles trapezoid central configuration such that* $m_1 = m_2 = \mu$ *and* $m_3 = m_4 = m$*.*

In [16] the authors showed that there exists a curve of central configurations connecting the equilateral triangle central configuration and the square central configuration in which the mass ratio m/μ is strictly increasing.

Recently Deng, Li and Zhang in [9] improved Theorem 3 as follows.

Theorem 4. *The thesis of Theorem 3 holds changing the assumption "*$r_{13} = r_{24}$ *and* $r_{23} = r_{14}$*" by "*$r_{13} = r_{24}$ *or* $r_{23} = r_{14}$*".*

In [15] Llibre, assuming that the planar central configurations of the 4–body problem with equal masses have some symmetry, showed numerically that the 4–body problem with equal masses have 50 classes of central configurations. Later on Albouy in [1] and [2] proved that such symmetries always exist and provide an analytical proof of the 50 classes.

Albouy, Fu and Sun [3] studied some symmetric central configurations in the 4–body problem. In particular they showed that in a convex planar central configuration of 4 bodies if two opposite masses are equal then there exists an axis of symmetry passing through the other two masses. The converse of this statement is also true. This kind of central configurations are called *kite* central configurations. Several papers were written studying kite central configurations and their properties, see [5, 14, 17, 18] and references therein. In [21] Perez–Chavela and Santoprete proved that the unique convex planar central configuration with two opposite equal masses is the kite central configuration or the rhombus central configuration when the other two masses are also equal.

Albouy, Fu and Sun [3] stated the following conjecture.

Conjecture 5. *There is a unique convex planar central configuration having two pairs of equal masses located at the adjacent vertices of the configuration and it is an isosceles trapezoid.*

Recently Corbera and Llibre [7] proved this conjecture assuming that two equal masses are sufficiently small.

In this paper we prove Conjecture 5 for all values of the masses. We consider the 4–body problem in the plane with masses $m_1 = m_2$ and $m_3 = m_4$ located at adjacent vertices of a convex quadrilateral as illustrated in Figure 2. Without loss of generality, we can consider $r_1 = (-1, 0)$, $r_2 = (1, 0)$, $r_3 = (x_3, y_3)$, $r_4 = (x_4, y_4)$, $m_1 = m_2 = \mu$ and $m_3 = m_4 = m$. We state the main result of this article.

Theorem 6. *Consider a convex configuration of* 4 *bodies with position vectors* r_1*,* r_2*,* r_3*,* r_4 *and masses* m_1*,* m_2*,* m_3*,* m_4*. Suppose that* $m_1 = m_2 = \mu$*,* $m_3 = m_4 = m$*,*

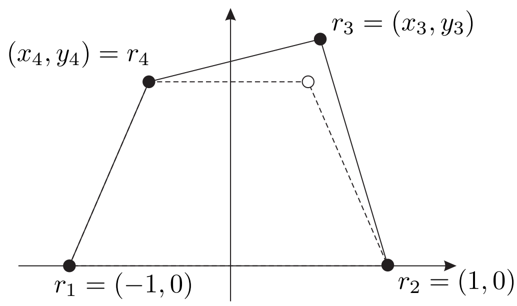

FIGURE 2. Coordinates for the problem. Solid lines indicate our coordinates and dashed lines indicate the isosceles trapezoid configuration.

and r_1, r_2, r_3 and r_4 are disposed counterclockwise at the vertices of a convex quadrilateral. Then the only possible central configuration performed by these bodies is an isosceles trapezoid.

This article is organized as follows. We prove Theorem 6 in Section 3. In Section 2 we prove some preliminary results used in the proof of Theorem 6.

2. Preliminary results

In this section we present a set of equations equivalent to the central configuration equations. The following result is well known, see for instance [11].

Lemma 7. *Consider n bodies with positive masses $m_1, m_2, \ldots, m_n$ and position vectors $r_1, r_2, \ldots, r_n$ in a planar non–collinear configuration. Then the set of equations* (2) *is equivalent to the set of equations*

$$f_{ij} = \sum_{\substack{k=1\\k\neq i,j}}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} = 0, \tag{3}$$

for $1 \le i < j \le n$, where $R_{ij} = 1/r_{ij}^3$ and $\Delta_{ijk} = (r_i - r_j) \wedge (r_i - r_k)$.

Note that Δ_{ijk} is twice the oriented area of the triangle formed by the bodies at r_i, r_j and r_k (see [11]). The $n(n-1)/2$ equations (3) are called the Dziobek–Laura–Andoyer equations or simply the Andoyer equations.

Using the notation of Lemma 7 we can state the main theorem of [3].

Theorem 8. *Consider a convex configuration of* 4 *bodies with positive masses m_1, m_2, m_3, m_4 and position vectors r_1, r_2, r_3, r_4 oriented counterclockwise like in Figure 2. Then the central configuration is symmetric with respect to the diagonal r_2r_4 if and only if $m_1 = m_3$. Also, $m_1 > m_3$ if and only if $\Delta_{124} > \Delta_{234}$.*

Of course an analogous result to Theorem 8 is true having a symmetry with respect to the other diagonal.

Without loss of generality we can assume $m \le \mu$. Moreover since we consider convex configurations, by the Perpendicular Bisector Theorem (see [19]), we also can assume that $x_4 < 0$, $x_3 > 0$, $y_3 > 0$ and $y_4 > 0$. See Figure 2.

The six Andoyer equations for our problem are

$$f_{12} = m\,\{(R_{13} - R_{23})\,\Delta_{123} + (R_{14} - R_{24})\,\Delta_{124}\} = 0, \tag{4}$$

$$f_{13} = \mu\,(R_{12} - R_{23})\,\Delta_{132} + m\,(R_{14} - R_{34})\,\Delta_{134} = 0, \tag{5}$$

$$f_{14} = \mu\,(R_{12} - R_{24})\,\Delta_{142} + m\,(R_{13} - R_{34})\,\Delta_{143} = 0, \tag{6}$$

$$f_{23} = \mu\,(R_{12} - R_{13})\,\Delta_{231} + m\,(R_{24} - R_{34})\,\Delta_{234} = 0, \tag{7}$$

$$f_{24} = \mu\,(R_{12} - R_{14})\,\Delta_{241} + m\,(R_{23} - R_{34})\,\Delta_{243} = 0, \tag{8}$$

$$f_{34} = \mu\,\{(R_{13} - R_{14})\,\Delta_{341} + (R_{23} - R_{24})\,\Delta_{342}\} = 0. \tag{9}$$

Since $m > 0$ and $\mu > 0$ if we define

$$G(x_3, y_3, x_4, y_4) = (R_{13} - R_{23})\,\Delta_{123} + (R_{14} - R_{24})\,\Delta_{124}, \tag{10}$$

$$H(x_3, y_3, x_4, y_4) = (R_{13} - R_{14})\,\Delta_{341} + (R_{23} - R_{24})\,\Delta_{342}, \tag{11}$$

then $f_{12} = 0$ if and only if $G = 0$ and $f_{34} = 0$ if and only if $H = 0$.

From equation (5), $R_{14} = R_{34}$ if and only if $R_{12} = R_{23}$. In this case, $\Delta_{124} = \Delta_{234}$. So, from equation (8) we have $m = \mu$. Then, from Theorem 8 the configuration must be a square with four equal masses at the vertices, which is a type of isosceles trapezoid. Hence in what follows we can assume that $R_{14} \neq R_{34}$ and $R_{12} \neq R_{23}$.

Again from equation (5), if $R_{34} < R_{14}$ then $R_{23} < R_{12}$, or equivalently, if $r_{34} > r_{14}$ then $r_{23} > r_{12}$, which implies that $\Delta_{124} < \Delta_{234}$. So, from Theorem 8 it follows that $m > \mu$, in contradiction with our hypothesis. Thus we must have $R_{34} > R_{14}$ which implies that $R_{23} > R_{12}$. A similar argument can be used to show that we must have $R_{34} > R_{23}$ which implies that $R_{14} > R_{12}$. In order to have a central configuration, taking out the case of the square and using Theorem 2 the following inequalities must hold

$$r_{13}, r_{24} > r_{12} > r_{23}, r_{14} > r_{34}. \tag{12}$$

Since $r_{12} = 2$ inequalities (12) imply that

$$\sqrt{4\sqrt{2} - 5} < y_4 < 2, \quad \sqrt{4\sqrt{2} - 5} < y_3 < 2, \quad -2 < x_4 < 0, \quad 0 < x_3 < 2 + x_4.$$

Without loss of generality we can assume that $y_4 \leq y_3$. Then from Theorem 8 the following inequalities must hold

$$\Delta_{123} \geq \Delta_{124} \geq \Delta_{234} \geq \Delta_{134}.$$

The explicit expressions for these areas are the following

$$\begin{array}{ll} \Delta_{123} = 2y_3, & \Delta_{134} = x_3 y_4 - x_4 y_3 - y_3 + y_4, \\ \Delta_{124} = 2y_4, & \Delta_{234} = x_3 y_4 - x_4 y_3 + y_3 - y_4. \end{array} \tag{13}$$

In the rest of this section we consider the hypotheses of Theorem 6. Thus the configuration is like described in Figure 2 satisfying (12). So we have the first lemma.

Lemma 9. *Under the assumptions of Theorem 6, if $y_4 = y_3$ then the configuration must be an isosceles trapezoid.*

Proof. If $y_3 = y_4$, using (13), equations (4) and (9) can be written as

$$m\,(R_{13} + R_{14} - R_{23} - R_{24})\,\Delta_{124} = 0,$$
$$\mu\,(R_{13} - R_{14} + R_{23} - R_{24})\,\Delta_{134} = 0.$$

Since the areas are positive, these equations are satisfied if and only if $R_{13} = R_{24}$ and $R_{14} = R_{23}$. But in this case the configuration is an isosceles trapezoid. □

Thus henceforth consider $y_3 > y_4$.

Lemma 10. *Under the assumptions of Theorem 6, if $x_4 \in [-1, 0)$ and $x_3 \in (0, -x_4]$, then there are no positions satisfying $f_{12} = 0$.*

Proof. First consider the inequalities (12), in which we must have $r_{14} > r_{34}$, or equivalently

$$(1 + x_4)^2 + y_4^2 > (x_3 - x_4)^2 + (y_3 - y_4)^2.$$

In order that this inequality be satisfied for $x_3 > 0$ it is necessary that (x_4, y_4) belongs to the region (open and connected set) A_1 which is determined by the parabola $y_4^2 + 2x_4 + 1 = 0$ and the circles $r_{14} = 2$ and $r_{24} = 2$, see Figure 3.

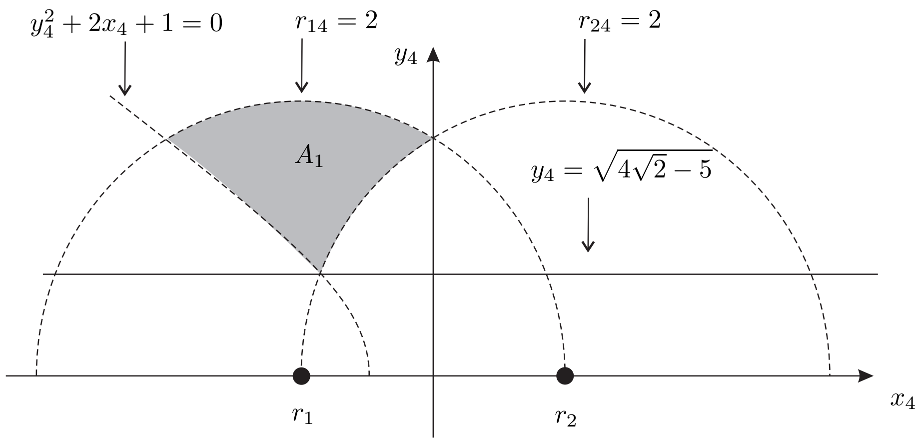

FIGURE 3. The admissible region A_1 is bounded by the parabola $y_4^2 + 2x_4 + 1 = 0$ and by the circles $r_{14} = 2$ and $r_{24} = 2$.

Now consider (x_4, y_4) fixed. Computing the partial derivative of G, defined in (10), with respect to x_3 we get

$$\frac{\partial G}{\partial x_3} = 6y_3\,(-(1 + x_3)Q_{13} - (1 - x_3)Q_{23}) < 0,$$

for $x_3 \in (0, -x_4]$ because $-x_4 \le 1$, where $Q_{ij} = r_{ij}^{-5}$.

Computing the partial derivative of G with respect to y_3 we get

$$\frac{\partial G}{\partial y_3} = -6y_3^2\,(Q_{13} - Q_{23}) + 2\,(R_{13} - R_{23})\,,$$

or equivalently

$$\begin{aligned}\frac{\partial G}{\partial y_3} &= 2\left[Q_{13}\left((1+x_3)^2-2y_3^2\right)-Q_{23}\left((1-x_3)^2-2y_3^2\right)\right]\\ &> 2\left[Q_{13}\left((1-x_3)^2-2y_3^2\right)-Q_{23}\left((1-x_3)^2-2y_3^2\right)\right]\\ &= 2(Q_{23}-Q_{13})\left(2y_3^2-(1-x_3)^2\right)>0.\end{aligned}$$

The last part of the above inequality arises from the fact that the points (x_3, y_3) must belong to the region A_2 symmetric to A_1 determined by the parabola $y_3^2 - 2x_3 + 1 = 0$ and the circles $r_{23} = 2$ and $r_{13} = 2$, where we have $Q_{23} > Q_{13}$ and $2y_3^2 - (1 - x_3)^2 > 0$. See Figure 4. Thus for $x_4 \in [-1, 0)$ and $x_3 \in (0, -x_4]$ the gradient of G points always northwest. Since $G(-x_4, y_4, x_4, y_4) = 0$ for all values of (x_4, y_4), $G > 0$ for all values of $(x_3, y_3) \in B_2$ characterized by the points of A_2 such that $y_3 > y_4$ and $x_3 \in (0, -x_4]$. See Figure 4. Thus $f_{12} > 0$ and this completes the proof. □

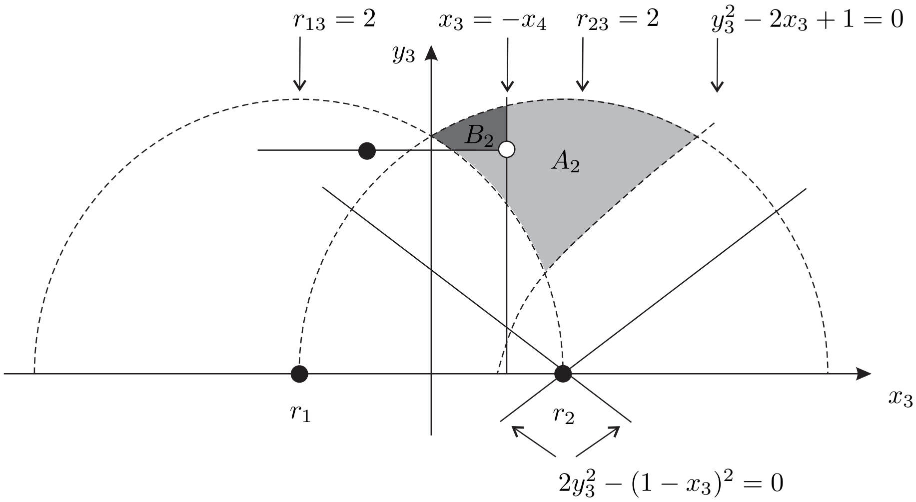

FIGURE 4. The region A_2 is bounded by the parabola $y_3^2 - 2x_3 + 1 = 0$ and by the circles $r_{23} = 2$ and $r_{13} = 2$. The set B_2 is defined by the points of A_2 such that $y_3 > y_4$ and $x_3 \in (0, -x_4]$.

Lemma 11. *Under the assumptions of Theorem 6, if* $x_4 \in (-2, -1)$ *then there are no positions satisfying* $f_{12} = 0$.

Proof. With (x_4, y_4) fixed, the zero level set of G is the set of points (x_3, y_3) such that

$$(R_{23} - R_{13})\, y_3 = (R_{14} - R_{24})\, y_4.$$

Since $y_3 > y_4$ we must have

$$R_{23} - R_{13} < R_{14} - R_{24},$$

which implies that

$$R_{23} - R_{14} < R_{13} - R_{24}.$$

Thus if $R_{13} - R_{24} < 0$ we must have $R_{23} - R_{14} < 0$. Analogously if $R_{23} - R_{14} > 0$ we must have $R_{13} - R_{24} > 0$.

Consider the point $(x_3, y_3) = (2 + x_4, y_4)$ in the circle $r_{23} = r_{14}$ (remember r_4 is fixed). Thus

$$G(2 + x_4, y_4, x_4, y_4) = (R_{13} - R_{24})y_4.$$

But at this point G is positive since the point $(x_3, y_3) = (2 + x_4, y_4)$ belongs to the interior of the circle $r_{13} = r_{24}$ (remember r_4 is fixed). Notice that the gradient of G remains pointing northwest as in Lemma 10, because $x_3 \in (0, 2 + x_4) \subset (0, 1)$. So $G > 0$ for all points in B_3, which is the subset of A_2 with $x_3 \in (0, 2 + x_4)$ and $y_3 > y_4$. See Figure 5. Thus $f_{12} > 0$. □

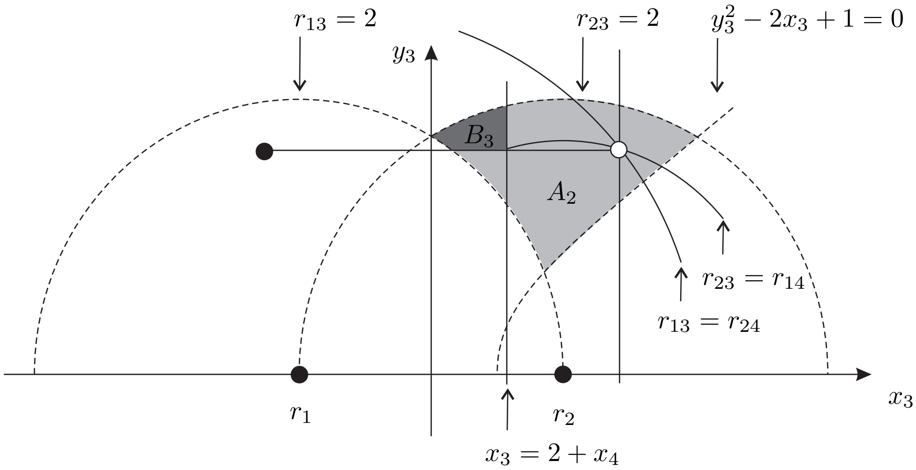

FIGURE 5. The set B_3 is defined by the points of A_2 (see Figure 4) such that $x_3 \in (0, 2 + x_4)$ and $y_3 > y_4$.

From the above calculations we only need to study the case where $x_4 \in [-1, 0)$ and $x_3 \in (-x_4, 2 + x_4)$. In order to satisfy (12) with $x_3 > -x_4$ we need that

$$r_{34}^2 < (-2x_4)^2 + (y_3 - y_4)^2 < (1 + x_4)^2 + y_4^2 = r_{14}^2.$$

Thus it is necessary that (x_4, y_4) belongs to the region A_3 determined by the hyperbola $y_4^2 - 3x_4^2 + 2x_4 + 1 = 0$ and the circles $r_{14} = 2$ and $r_{24} = 2$, see Figure 6. The intersection of the hyperbola $y_4^2 - 3x_4^2 + 2x_4 + 1 = 0$ with the circle $r_{24} = 2$ is the point

$$x_c = \frac{1}{2} - \frac{\sqrt{5}}{2}, \quad y_c = \sqrt{4 - \sqrt{5} - \frac{(1 - \sqrt{5})^2}{4}}. \tag{14}$$

Thus we must have $y_4 > y_c$.

Since we are considering values of $x_3 \in (-x_4, 2 + x_4)$ and $y_3 > y_4$, for a fixed pair (x_4, y_4) the region of interest for (x_3, y_3) is the region B_4 defined by the points of A_2 where $x_3 \in (-x_4, 2 + x_4)$ and $y_3 > y_4$, see Figure 7.

Now define the region A_4 bounded by the hyperbola $y_3^2 - 3x_3^2 - 2x_3 + 1 = 0$ and the circles $r_{23} = 2$ and $r_{13} = 2$, see again Figure 7. Note that the points on the straight line $x_3 = -x_4$ between the line $y_3 = y_4$ and the circle $r_{23} = 2$ always belong to A_4. Note also that in region B_4 we have $r_{13} > r_{24}$.

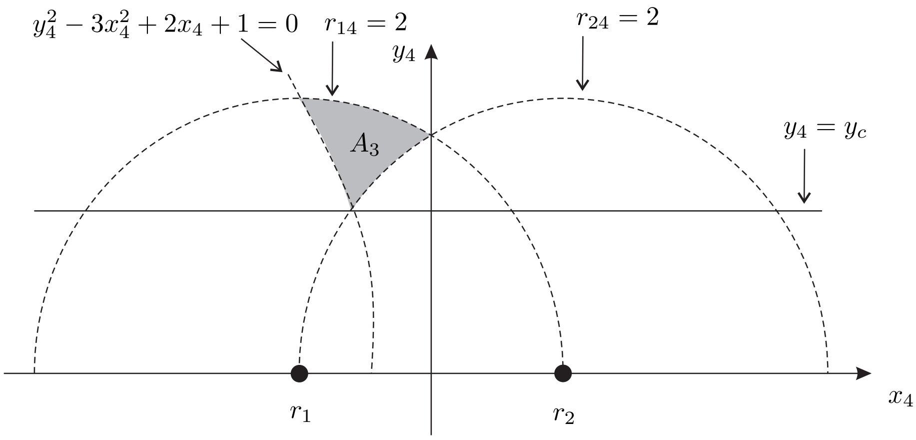

FIGURE 6. The admissible region A_3 is bounded by the hyperbola $y_4^2 - 3x_4^2 + 2x_4 + 1 = 0$ and by the circles $r_{14} = 2$ and $r_{24} = 2$.

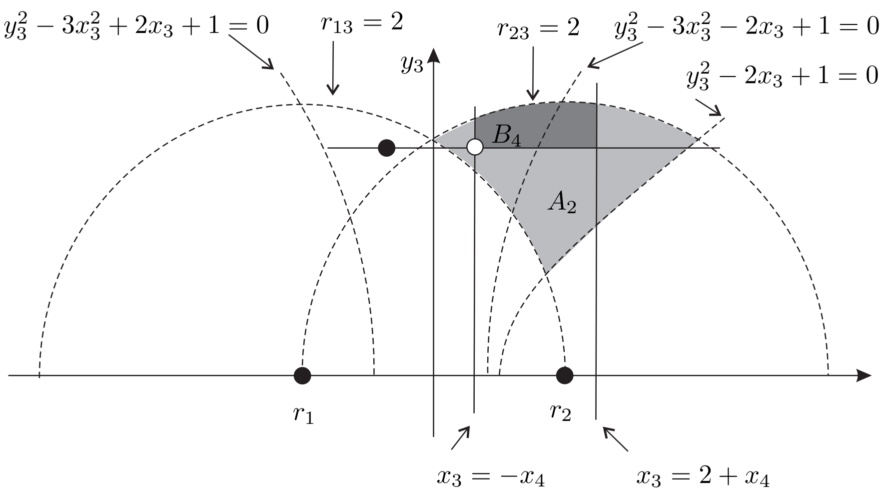

FIGURE 7. The region B_4 defined by the points of A_2 where $x_3 \in (-x_4, 2 + x_4)$ and $y_3 > y_4$. The region A_4 is bounded by the hyperbola $y_3^2 - 3x_3^2 - 2x_3 + 1 = 0$ and by the circles $r_{23} = 2$ and $r_{13} = 2$.

For a fixed pair (x_4, y_4) consider the function

$$T(x_3, y_3) = y_3(1 - x_4) - y_4(1 + x_3).$$

The zero level set of this function, denoted by T_0, is the straight line passing through $(x_3, y_3) = (-1, 0)$ and $(x_3, y_3) = (-x_4, y_4)$.

The sum of equation (6) multiplied by Δ_{243} and equation (7) multiplied by Δ_{143} gives

$$\mu\left[(R_{12} - R_{24})\Delta_{124}\Delta_{234} - (R_{12} - R_{13})\Delta_{123}\Delta_{134}\right] + m\left[R_{13} - R_{24}\right]\Delta_{134}\Delta_{234} = 0.$$

In the region B_4 the coefficient of m is always negative, so in order to satisfy this equation the coefficient of μ must be positive. We define the following function

$$L(x_3, y_3) = (R_{12} - R_{24})\Delta_{124}\Delta_{234} - (R_{12} - R_{13})\Delta_{123}\Delta_{134}. \tag{15}$$

In Lemmas 12 and 14 we use the sets defined below

$$B_{41} = B_4 \cap \left\{ (x_3, y_3) : y_3 \leq \frac{y_4(1+x_3)}{1-x_4} \right\},$$

$$B_{42} = B_4 \cap \left\{ (x_3, y_3) : y_3 \geq \frac{y_4(1+x_3)}{1-x_4} \right\}.$$

Thus, $T \leq 0$ in B_{41} and $T \geq 0$ in B_{42}. See Figure 8.

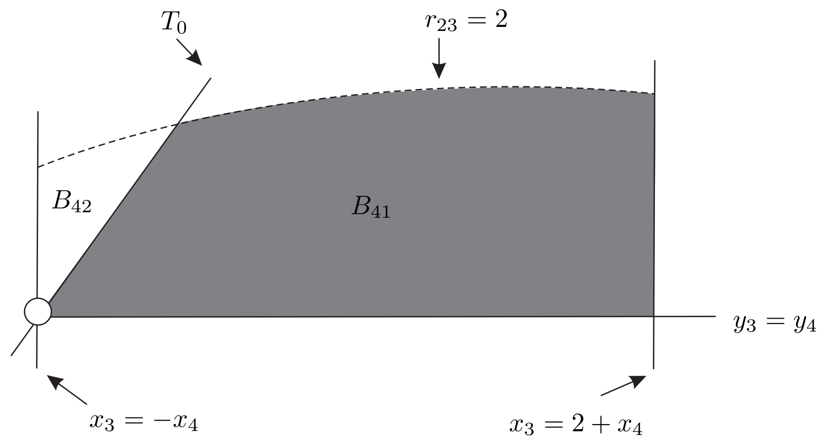

FIGURE 8. The sets B_{41} and B_{42}.

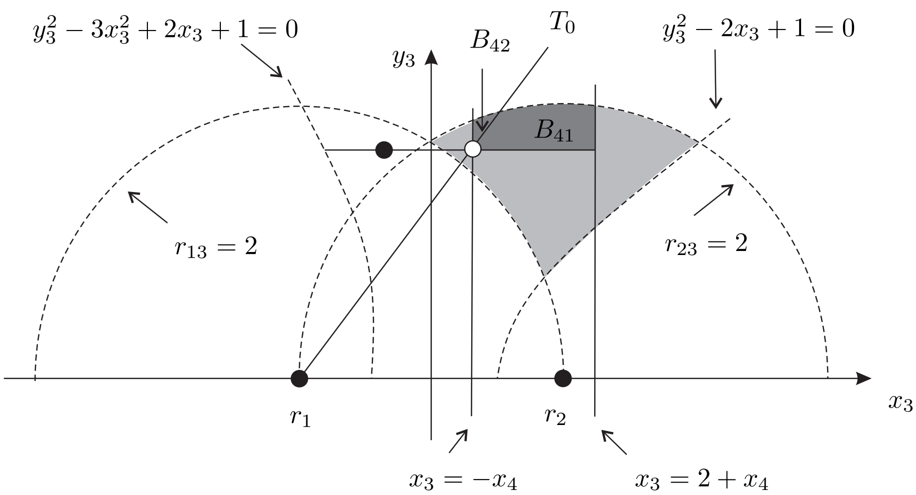

FIGURE 9. L is negative in B_{41}.

Lemma 12. *Under the assumptions of Theorem 6, if* $(x_3, y_3) \in B_{41}$ *then the function* L *is negative. See Figure 9. Thus the equations* $f_{14} = 0$ *and* $f_{23} = 0$ *are not satisfied simultaneously.*

Proof. Consider the function L restricted to T_0. Note that $L(-x_4, y_4) = 0$. Using equation (13) and the definition of R_{ij} the expression (15) restricted to T_0 can be written as

$$L|_{T_0} = (R_{12} - R_{24})4x_3y_4^2 + \left[R_{12} - R_{24}\frac{y_4^3}{y_3^3}\right]4x_4y_3^2.$$

Solving $T = 0$ for x_3 and replacing the result in the last equation we have

$$L|_{T_0} = 4\frac{(y_3 - y_4)}{y_3}\left[y_3^2x_4R_{12} + (-R_{24}y_4 + R_{12}y_4 + R_{24}y_4x_4)y_3 + y_4^2R_{24}x_4\right].$$

Note that the expression between the brackets is a function P of y_3 whose graph is a parabola concave downward. We will compare the position of the roots of P with $y_3 = y_4$ in order to study the sign of L restricted to T_0. Evaluating P at $y_3 = y_4$ we get

$$y_4^2(R_{12}(1 + x_4) + R_{24}(-1 + 2x_4)).$$

From the last equation, define

$$L_1(x_4, y_4) = R_{12}(1 + x_4) + R_{24}(-1 + 2x_4).$$

The zero level set of L_1 is given by

$$y_4^2 = \frac{2x_4^2 - 1 - x_4^4 + 4\left((1 + x_4)^2(1 - 2x_4)\right)^{\frac{2}{3}}}{(1 + x_4)^2}.$$

Thus the zero level set of L_1 for $y_4 > 0$ is a function of x_4 passing through the point $(0, \sqrt{3})$ and going to $+\infty$ when x_4 goes to -1^+. So the zero level set of L_1 crosses the circles $r_{14} = 2$ and $r_{24} = 2$ just at the point $(0, \sqrt{3})$.

Evaluating the derivative of P with respect to y_3 at $y_3 = y_4$ we get

$$y_4(2x_4R_{12} - R_{24} + R_{12} + x_4R_{24}).$$

From the last equation, define

$$K_1(x_4, y_4) = 2x_4R_{12} - R_{24} + R_{12} + x_4R_{24}.$$

The zero level set of K_1 is given by

$$y_4^2 = -(x_4 - 1)^2 + \frac{4\left((1 - x_4)^2(2x_4 + 1)^2\right)^{\frac{2}{3}}}{(2x_4 + 1)^2}.$$

Thus the zero level set of K_1 for $y_4 > 0$ is a function of x_4 passing through the point $(0, \sqrt{3})$ and going to $+\infty$ when x_4 goes to $-(1/2)^+$. See Figure 10. In conclusion, K_1 is negative in the region A_3 and this implies that L_1 is negative in the region A_3. So the function L restricted to T_0 is always negative when $y_3 > y_4$.

To see that the function L is negative in B_{41} we compute the partial derivative of L with respect to x_3

$$\frac{\partial L}{\partial x_3} = (R_{12} - R_{24})2y_4^2 - (R_{12} - R_{13})2y_3y_4 - 3(1 + x_3)Q_{13}\Delta_{123}\Delta_{134}.$$

Denote the first two terms in the above expression by the following function

$$L_2(x_3, y_3) = (R_{12} - R_{24})2y_4^2 - (R_{12} - R_{13})2y_3y_4.$$

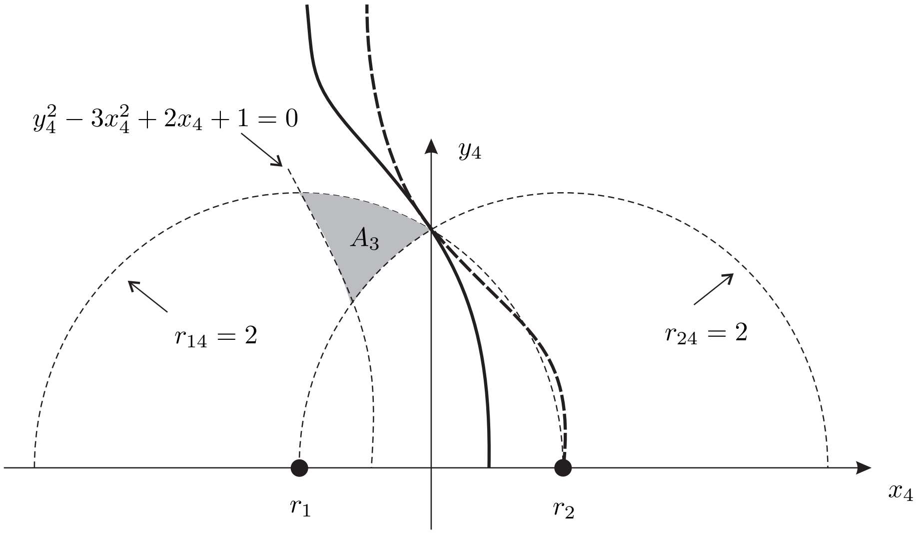

FIGURE 10. The solid curve indicates the zero level set of L_1 while the dashed one indicates the zero level set of K_1. Both L_1 and K_1 are negative in the region A_3.

This function vanishes at the point $(x_3, y_3) = (-x_4, y_4)$ and its gradient points southwest in B_4. In fact

$$\frac{\partial L_2}{\partial x_3} = -3(1 + x_3)Q_{13}y_3y_4 < 0$$

and

$$\frac{\partial L_2}{\partial y_3} = -2(R_{12} - R_{13})y_4 - 6Q_{13}y_3^2y_4 < 0.$$

Thus the partial derivative of L with respect to x_3 is always negative when $y_3 > y_4$. See Figure 9. So the function L is always negative in B_{41}. In short, the equations $f_{14} = 0$ and $f_{23} = 0$ are not satisfied simultaneously in B_{41}. □

For a fixed pair $(x_4, y_4) \in A_3$, define the following two functions

$$H_1(x_3, y_3) = R_{23}y_3 - R_{14}y_4,$$
$$H_2(x_3, y_3) = R_{13}y_3 - R_{24}y_4.$$

In the next lemmas we prove some properties of the above functions.

Lemma 13. *Under the assumptions of Theorem 6, if (x_3, y_3) in B_4 then the function H_2 is negative.*

Proof. Note that $H_2(-x_4, y_4) = 0$. The derivative of H_2 with respect to x_3 is given by

$$\frac{\partial H_2}{\partial x_3} = -3(1 + x_3)Q_{13}y_3 < 0,$$

while the derivative of H_2 with respect to y_3 is given by

$$\frac{\partial H_2}{\partial y_3} = -3Q_{13}y_3^2 + R_{13}.$$

The zero level set of this last derivative is formed by the two straight lines

$$y_3 = \pm\frac{\sqrt{2}}{2}(1+x_3).$$

The derivative of the function H_2 with respect to y_3 is negative in the region A_4. Thus the function H_2 is negative in the region B_4. □

Lemma 14. *Under the assumptions of Theorem 6, if $(x_3, y_3) \in B_{42}$ then the function H_1 is negative.*

Proof. Note that $H_1(-x_4, y_4) = 0$ and $H_1(2+x_4, y_4) = 0$. Computing the derivative of H_1 with respect to y_3 we obtain

$$\frac{\partial H_1}{\partial y_3} = -3Q_{23}y_3^2 + R_{23}.$$

The zero level set of this derivative is formed by the two straight lines

$$y_3 = \pm\frac{\sqrt{2}}{2}(1-x_3).$$

In the region B_4 the derivative of H_1 with respect to y_3 is negative. Thus the zero level set of H_1 is a function of x_3 which has implicit derivative given by

$$a_1 = \frac{dy_3}{dx_3} = \frac{3y_3(1-x_3)}{2y_3^2-(1-x_3)^2}.$$

The slope of the straight line T_0 is given by

$$a_0 = \frac{y_4}{1-x_4}.$$

Now we study the function

$$a_0 - a_1 = \frac{y_4(1-x_3)^2 - 2y_4y_3^2 + 3y_3(1-x_3)(1-x_4)}{(1-x_4)\left((1-x_3)^2-2y_3^2\right)}.$$

The denominator of the above expression is negative in B_4 according to the previous analysis. The numerator of the above expression vanishes on the straight lines

$$y_3 = \frac{\left(3(1-x_4) \pm \sqrt{9-18x_4+9x_4^2+8y_4^2}\right)(1-x_3)}{4y_4}. \tag{16}$$

See Figure 11. Thus in the set B_4 the difference $a_0 - a_1$ is positive. Since the zero level set of H_1 passes through the point $(-x_4, y_4)$, it means that the zero level set of H_1 belongs to the set B_{41}. Therefore the function H_1 is negative on B_{42}. □

Now we state the last lemma of this section.

Lemma 15. *Under the assumptions of Theorem 6, if $(x_3, y_3) \in B_{42}$ then the equation $f_{34} = 0$ is not satisfied.*

Proof. Consider the function H defined in (11) for a fixed pair $(x_4, y_4) \in A_3$, that is

$$H(x_3, y_3) = (R_{13} - R_{14})\Delta_{134} + (R_{23} - R_{24})\Delta_{234}.$$

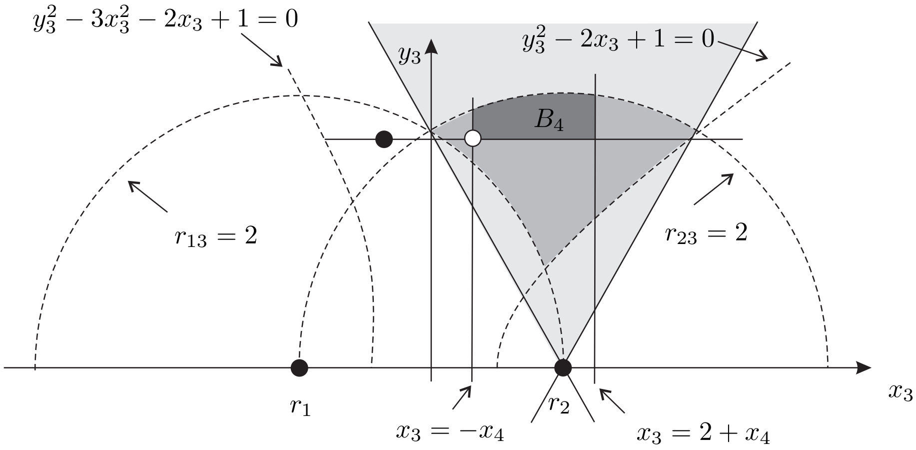

FIGURE 11. The numerator of $a_0 - a_1$ is negative in the cone defined by the straight lines (16) containing the region B_4.

Note that $H(-x_4, y_4) = 0$. By Lemmas 13 and 14 in the set B_{42} we have $R_{13}y_3 < R_{24}y_4$ and $R_{23}y_3 < R_{14}y_4$. So in B_{42}

$$H(x_3, y_3) < \left(R_{24}\frac{y_4}{y_3} - R_{14}\right)\Delta_{134} + \left(R_{14}\frac{y_4}{y_3} - R_{24}\right)\Delta_{234} = \frac{(y_3 - y_4)}{y_3}h(x_3, y_3),$$

where

$$h(x_3, y_3) = \left((1 + x_4)R_{14} - (1 - x_4)R_{24}\right)y_3 - y_4\,(R_{14} + R_{24})\,x_3 + y_4(R_{14} - R_{24}).$$

Since $y_3 > y_4$ we will prove that $H(x_3, y_3) < 0$ in B_{42} by proving that $h(x_3, y_3) < 0$ in this set. Note that the zero level set of h is the straight line given by

$$y_3 = \frac{y_4\,(R_{14} + R_{24})\,x_3 - y_4(R_{14} - R_{24})}{(1 + x_4)R_{14} - (1 - x_4)R_{24}}.$$

This straight line always pass through $(x_3, y_3) = (-x_4, -y_4)$. Thus in order to complete the proof we need to analyze the slope of this straight line which is

$$\frac{y_4\,(R_{14} + R_{24})}{(1 + x_4)R_{14} - (1 - x_4)R_{24}}. \tag{17}$$

The numerator of this last expression is positive so the sign of the slope is given by the denominator

$$(1 + x_4)R_{14} - (1 - x_4)R_{24}.$$

The zero level set of this expression for $y_4 > 0$ is a function of x_4 given by

$$y_4^2 = (1 - x_4)^{4/3}(1 + x_4)^{2/3} + (1 - x_4)^{2/3}(1 + x_4)^{4/3}$$

whose graph is depicted in Figure 12. Thus for all points in the region A_3 the sign of the slope is negative. Therefore the function H is always negative in B_{42} and this implies that the equation $f_{34} = 0$ is not satisfied in B_{42}. $\square$

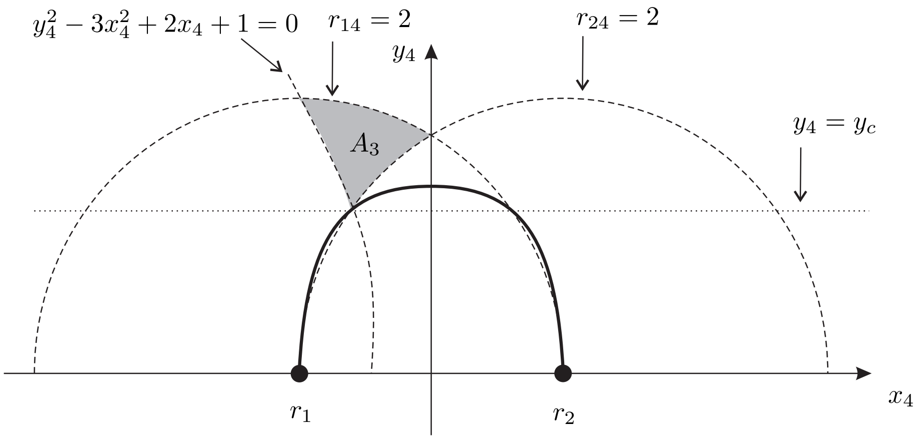

FIGURE 12. The solid curve represents the zero level set of the slope given in (17). Note that this zero level set passes through the point (x_c, y_c) given in (14) and is negative in the region A_3.

3. Proof of Theorem 6

In this section we give the proof of Theorem 6. We will prove that the symmetry in the masses implies the symmetry in the positions in order to satisfy all the Andoyer equations. Thus we will be under the hypotheses of MacMillan and Bartky Theorem, that is of Theorem 3. In other words, if we have symmetry in the masses and the positions then the uniqueness follows from that theorem.

Consider the position vectors $r_1 = (-1, 0)$, $r_2 = (1, 0)$, $r_3 = (x_3, y_3)$, $r_4 = (x_4, y_4)$ and masses $m_1 = m_2 = \mu$ and $m_3 = m_4 = m$ with $m \leq \mu$. Thus the Andoyer equations (3) are

$$
\begin{aligned}
f_{12} &= m\,\{(R_{13} - R_{23})\,\Delta_{123} + (R_{14} - R_{24})\,\Delta_{124}\} = 0,\\
f_{13} &= \mu\,(R_{12} - R_{23})\,\Delta_{132} + m\,(R_{14} - R_{34})\,\Delta_{134} = 0,\\
f_{14} &= \mu\,(R_{12} - R_{24})\,\Delta_{142} + m\,(R_{13} - R_{34})\,\Delta_{143} = 0,\\
f_{23} &= \mu\,(R_{12} - R_{13})\,\Delta_{231} + m\,(R_{24} - R_{34})\,\Delta_{234} = 0,\\
f_{24} &= \mu\,(R_{12} - R_{14})\,\Delta_{241} + m\,(R_{23} - R_{34})\,\Delta_{243} = 0,\\
f_{34} &= \mu\,\{(R_{13} - R_{14})\,\Delta_{341} + (R_{23} - R_{24})\,\Delta_{342}\} = 0.
\end{aligned}
$$

As mentioned before the necessary conditions for these equations be satisfied are the inequalities (12). Since $r_{12} = 2$ those inequalities imply that

$$\sqrt{4\sqrt{2} - 5} < y_4 < 2, \quad 1 < y_3 < 2, \quad -2 < x_4 < 0, \quad 0 < x_3 < 2 + x_4.$$

Without loss of generality we can assume that $y_4 \leq y_3$. Thus for a fixed pair (x_4, y_4), by Lemma 9, we have that if $y_3 = y_4$ then the configuration is an isosceles trapezoid. So, consider henceforth $y_3 > y_4$.

Note that if $(x_3, y_3) = (-x_4, y_4)$ we have an isosceles trapezoid and the equations $f_{12} = 0$ and $f_{34} = 0$ are already satisfied.

The aim of the proof is to show that, if $x_3 \neq -x_4$ and $y_3 \neq y_4$, then at least one of the Andoyer equations will not be satisfied.

If $x_4 \in [-1, 0)$ and $x_3 \in (0, -x_4]$ then, by Lemma 10, $f_{12} = 0$ is not satisfied. Thus we do not have a central configuration.

If $x_4 \in (-2, -1)$ then, by Lemma 11, $f_{12} = 0$ is not satisfied. Thus we do not have a central configuration.

If $x_4 \in [-1, 0)$ and $x_3 \in (-x_4, 2 + x_4)$ then $(x_3, y_3) \in B_4$ and, by Lemma 12, equations $f_{14} = 0$ and $f_{23} = 0$ are not satisfied simultaneously in B_{41}. Thus we do not have a central configuration.

If $x_4 \in [-1, 0)$ and $x_3 \in (-x_4, 2+x_4)$ then $(x_3, y_3) \in B_4$ and, by Lemma 15, the equation $f_{34} = 0$ is not satisfied in B_{42}. Thus we do not have a central configuration.

From the previous analyses a necessary condition to satisfy all the Andoyer equations is the symmetry $(x_3, y_3) = (-x_4, y_4)$, that is the quadrilateral must be an isosceles trapezoid, see Theorem 3. This completes the proof of Theorem 6.

For a modern and very well written work about the isosceles trapezoid central configuration, see Cors and Roberts [8].

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