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On the absence of analytic integrability of the Bianchi Class B cosmological models

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We follow Bogoyavlensky's approach to deal with Bianchi class B cosmological models. We characterize the analytic integrability of such systems. [http://dx.doi.org/10.1063/1.4790828]

Bianchi models are cosmological models that describe space-times which are foliated by homogeneous hypersurfaces of constant time and are divided into two classes, Class A and Class B. There are many studies about the integrability of Class A. Here, we study the integrability of Class B. For the homogeneous cosmological models of Class B, Einstein's system of differential equations reduces to a dynamical system of dimension seven according to Bogoyavlensky's approach. We show that in order to study the integrability of such systems, it is sufficient to deal with homogeneous polynomial differential systems of dimension six. Concretely, Bianchi *V* is the simplest model and can be written as a homogeneous polynomial differential system of degree 2. Bianchi *IV* is dealt as a homogeneous polynomial differential system of degree 3 and the rest of the models, Bianchis III, VI, and VII are of degree 5. Due to the fact that all Bianchi class B models have been reduced to homogeneous polynomial differential systems, the study of their analytic integrability reduces to analyze their homogeneous polynomial first integrals. We show that Bianchi model *V* admits polynomial first integral, and we prove that the corresponding homogeneous polynomial differential systems that represent models Bianchi IV, III, VI, and VII do not admit polynomial first integrals. The fact that these Bianchi models are not completely integrable with analytic first integrals facilitates that they can have chaotic behavior.

I. INTRODUCTION AND STATEMENT OF THE RESULTS

Einstein's equations relate the geometry of the space-time with the properties of the matter which occupied it. The matter occupying the space-time is determined by the stress energy tensor of the matter. In our study, we follow[3] and we consider the hydrodynamical tensor of the matter. We will work with an equation of state of matter of the form $p = k\varepsilon$, where ε is the energy density of the matter, p is the pressure and $0 \le k < 1$.

We can found in the literature some methods in order to construct some Hamiltonians for the Bianchi class B models, see, for example, Ref. 4 and references therein. Here, we follow Bogoyavlensky's approach, see Ref. 3. So according to Bogoyavlensky, for the homogeneous cosmological models of Class B Einstein's system of equations reduces to the following dynamical system in the phase space $p_i, q_i, p_\varphi, \varphi$, $i = 1, 2, 3$,

$$\begin{aligned} \frac{dq_i}{d\tau} &= \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{d\tau} = -\frac{\partial H}{\partial q_i} - h_i,\\ \frac{d\varphi}{d\tau} &= \frac{\partial H}{\partial p_\varphi}, \quad \frac{dp_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi} - h_\varphi, \end{aligned} \tag{1}$$

where the function H is

$$H = \frac{1}{(q_1q_2q_3)^{\frac{1-k}{2}}}\left(T(p_iq_i) + V_G(q_i)\right), \tag{2}$$

with

$$T(p_i, q_i, p_\varphi) = 2\sum_{1\le i<j\le 3} p_ip_jq_iq_j - \sum_{i=1}^{3} p_i^2 q_i^2 - \frac{p_\varphi^2 q_1q_2}{(n_1q_1 - n_2q_2)^2},$$

$$V_G(q_i) = -\frac{1}{4}\left(12a^2q_1q_2 + (n_1q_1 - n_2q_2)^2\right),$$

and

$$h_1 = \frac{a^2q_2}{(q_1q_2q_3)^{\frac{1-k}{2}}}, \quad h_2 = \frac{a^2q_1}{(q_1q_2q_3)^{\frac{1-k}{2}}},$$

$$h_3 = \frac{-2a^2q_1q_2}{q_3(q_1q_2q_3)^{\frac{1-k}{2}}}, \quad h_\varphi = \frac{a(n_1q_1 - n_2q_2)^2}{(q_1q_2q_3)^{\frac{1-k}{2}}}.$$

The constants $a \ne 0, n_1, n_2$ determine the type of model according to Table I, see also Refs. 1–3. System (1) in an explicit form writes as

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