On the complete integrability of the Raychaudhuri differential system in $\mathbb{R}^4$ and of a CRNT model in $\mathbb{R}^5$

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Abstract

We study the Darboux integrability of two differential systems with parameters: a relativistic model in $\mathbb{R}^4$ called the Raychaudhuri equation and a chemical reaction model in $\mathbb{R}^5$. For the first one we prove that it is completely integrable and that the first integrals are of Darboux type. This is the first four-dimensional realistic non-trivial model which is completely integrable with first integrals of Darboux type and for which for a full Lebesgue measure set of the values of the parameters the three linearly independent first integrals are rational. For the second one, we find all its Darboux polynomials and exponential factors and we prove that it is not Darboux integrable.

Keywords. Darboux polynomial; exponential factor; Darboux integrability; Raychaudhuri equation; chemical reaction network

1 Introduction and presentation of the systems

Consider a polynomial differential system of degree $d \in \mathbb{N}$

$$\dot{\mathbf{x}} = P(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \tag{1.1}$$

where $P(\mathbf{x}) = (P_1(\mathbf{x}), \ldots, P_n(\mathbf{x})) \in \mathbb{C}[\mathbf{x}]$ and the dot denotes derivative with respect to the independent variable t.

A function $H(\mathbf{x})$ is a *first integral* of system (1.1) if it is continuous and defined in a full Lebesgue measure subset $\Omega \subseteq \mathbb{R}^n$, is not locally constant on any positive Lebesgue measure subset of Ω and moreover is constant along each orbit of system (1.1) in Ω. If H is $\mathcal{C}^1$, then if $\mathcal{X}$ is the vector field associated to system (1.1), we have

$$\mathcal{X}(H) = P_1 \frac{\partial H}{\partial x_1} + \cdots + P_n \frac{\partial H}{\partial x_n} = 0.$$

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System (1.1) is $\mathcal{C}^k$-*completely integrable* in Ω if it has $n-1$ functionally independent C^k first integrals in Ω. Recall that k functions $H_1(\mathbf{x}), \ldots, H_k(\mathbf{x})$ are *functionally independent* in Ω if the matrix of gradients $(\nabla H_1, \ldots, \nabla H_k)$ has rank k in a full Lebesgue measure subset of Ω.

For an n-dimensional system of differential equations the existence of some first integrals reduces the complexity of its dynamics and the existence of $n-1$ functionally independent first integrals solves completely the problem (at least theoretically) of determining its phase portrait. In general for a given differential system it is a difficult problem to determine the existence or non-existence of first integrals.

During recent years the interest in the study of the integrability of differential equations has attracted much attention from the mathematical community. Darboux theory of integrability plays a central role in the integrability of the polynomial differential systems since it gives a sufficient condition for the integrability inside the family of rational functions. We highlight that it works for real or complex polynomial ordinary differential equations and that the study of complex algebraic solutions is necessary for obtaining all the real first integrals of a real polynomial differential equation.

A *Darboux polynomial* of (1.1) is a polynomial $f \in \mathbb{C}[\mathbf{x}]$ such that

$$\mathcal{X}(f) = P_1 \frac{\partial f}{\partial x_1} + \cdots + P_n \frac{\partial f}{\partial x_n} = kf, \tag{1.2}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $k \in \mathbb{C}[\mathbf{x}]$, which is called the *cofactor* of f, has degree at most $d-1$.

An *exponential factor* of (1.1) is a function $F = \exp(g/f)$, with $f, g \in \mathbb{C}[\mathbf{x}]$, such that

$$\mathcal{X}(F) = P_1 \frac{\partial F}{\partial x_1} + \cdots + P_n \frac{\partial F}{\partial x_n} = LF, \tag{1.3}$$

where $\mathbf{x} = (x_1, \ldots, x_n)$ and $L \in \mathbb{C}[\mathbf{x}]$, which is called the *cofactor* of F, has degree at most $d-1$. We note that in this case f is a Darboux polynomial of (1.1) and that

$$\mathcal{X}(g) = kg + Lf, \tag{1.4}$$

where k is the cofactor of f.

The Darboux theory of integrability relates the number of Darboux polynomials and exponentials factors with the existence of a Darboux first integral, see for example [13]. We recall that a Darboux first integral is a product of complex powers of Darboux polynomials and exponentials factors.

The main aim in this paper is to study the Darboux integrability of two differential systems. The first one belongs to $\mathbb{R}^4$ and plays an important role in relativity theory; the second one belongs to $\mathbb{R}^5$ and has an important contribution in the chemical reaction network.

Our first system is the so-called Raychaudhuri equation for a two dimensional curved surface of constant curvature:

$$\begin{aligned}\dot{x}_1 &= -\frac{x_1^2}{2} - \alpha x_1 - 2(x_2^2 + x_3^2 - x_4^2) - 2\beta,\\ \dot{x}_2 &= -(\alpha + x_1)x_2 - \gamma,\\ \dot{x}_3 &= -(\alpha + x_1)x_3 - \delta,\\ \dot{x}_4 &= -(\alpha + x_1)x_4,\end{aligned} \tag{1.5}$$

where $\alpha, \beta, \delta, \gamma$ are real parameters. See [14, 1, 12] and the references therein. In general relativity, the Raychaudhuri equation is a fundamental result describing the motion of nearby bits of matter. It is quite relevant since it is used as a fundamental lemma for the Penrose-Hawking singularity theorems (see [12] for details) and for the study of exact solutions in general relativity. However, it has an independent interest since it offers a simple and general validation of our intuitive expectation that gravitation should be a universal attractive force between any two bits of mass-energy in general relativity, as it is in Newton's theory of gravitation.

Here we further contribute to the understanding of the complexity, or more precisely of the topological structure of the dynamics of system (1.5) by studying its integrability. Darboux polynomials, analytic integrability and Darboux first integrals of this system were partially studied in [15, 16, 17]. We prove here that Raychaudhuri equation is completely Darboux integrable, and moreover that the three first integrals are, for almost all of the values of the parameters, rational. As far as the authors know, this is the first four-dimensional realistic non-trivial model which is completely integrable with first integrals of Darboux type for which for a full Lebesgue measure set of the values of the parameters the three linearly independent first integrals are rational.

Our second differential system comes from a chemical reaction model in $\mathbb{R}^5$. In Chemical Reaction Network Theory (CRNT), a reaction network $\mathcal{N} = (\mathcal{S}, \mathcal{C}, \mathcal{R})$ is defined as a set of species $\mathcal{S}$, a set of complexes $\mathcal{C}$ and a set of reactions $\mathcal{R}$ between complexes. Each complex is a combination of species. It is assumed that a reaction occurs according to mass-action kinetics, that is, at a rate proportional to the product of the species concentrations in the reactant or source complex. The set of reactions together with a rate vector give rise to a polynomial system of ordinary differential equations. We refer the reader to [9, 10] for more information about CRNT. For a concrete system of chemical reactions the parameter and state spaces are typically high-dimensional and one uses numerical methods to analyze the solutions. Due to high computational complexity this can be done only for a small set of values of system's parameters. Thus instead of studying quantitative aspects of the dynamics, recently there has been an increasing interest in studying *qualitative* properties of the CRN. For example in [2, 3, 4, 5, 6, 7] the authors considered the question of existence of single versus multiple steady states (also referred to as multistationary). The existence of first integrals of a polynomial differential system describing a CRN often provides essential qualitative information (the level sets are invariant under the flow) about the solution or can be used, as explained in the introduction, to reduce the dimension of the total state space. Since the computation of nonlinear conservation laws (i.e., first integrals) is highly nontrivial most of the results known by now related to the CRN dynamics provide only trivial linear first integrals. Hence, in this paper, our purpose is to show, by following an example (see system (1.6)), how to apply Darboux theory of integrability to obtain nontrivial and nonlinear algebraic and Darboux type first integrals.

More concretely, we deal with a particular system appearing in [9] coming from an enzymatic mechanism:

$$
\begin{aligned}
\dot{x}_1 &= -c_1 x_1 x_4,\\
\dot{x}_2 &= -c_3 x_2 x_4 + c_2 x_5,\\
\dot{x}_3 &= c_4 x_5,\\
\dot{x}_4 &= -c_1 x_1 x_4 - c_3 x_2 x_4 + (c_2 + c_4) x_5,\\
\dot{x}_5 &= c_1 x_1 x_4 + c_3 x_2 x_4 - (c_2 + c_4) x_5,
\end{aligned}
\tag{1.6}
$$

where c_1, c_2, c_3, c_4 are positive constants. We study the Darboux integrability of this system by characterizing its Darboux polynomials and its exponential factors.

2 Main results

In this section we provide the main results of the paper. Concerning system (1.5), we shall prove that it is completely integrable, and moreover we shall provide three functionally independent Darboux first integrals, distinguishing several cases depending on the parameters. We note that for almost all of the cases, these first integrals are rational functions, as we explained above. It is a nicely surprising result for a four-dimensional non-trivial system.

For system (1.6) we shall prove that there only exist two first integrals (already found in [9]), which are polynomial, one irreducible Darboux polynomial and one exponential factor. With this situation, we shall prove that the system is not Darboux integrable.

2.1 The Raychaudhuri equation

We can remove the parameter α in system (1.5) by the change $x_1 + \alpha \mapsto x_1$:

$$
\begin{aligned}
\dot{x}_1 &= -\frac{x_1^2}{2} - 2(x_2^2 + x_3^2 - x_4^2) - \gamma_0,\\
\dot{x}_2 &= -x_1 x_2 - \gamma,\\
\dot{x}_3 &= -x_1 x_3 - \delta,\\
\dot{x}_4 &= -x_1 x_4,
\end{aligned}
\tag{2.1}
$$

where $\gamma_0 = 2\beta - \alpha^2/2 \in \mathbb{R}$. It is clear that x_4 is a Darboux polynomial of systems (1.5) and (2.1) with cofactor $-x_1$, for all values of the parameters.

Depending on the values of γ and δ we distinguish three different systems. If $\gamma = \delta = 0$ then (2.1) becomes

$$
\begin{aligned}
\dot{x}_1 &= -\frac{x_1^2}{2} - 2(x_2^2 + x_3^2 - x_4^2) - \gamma_0,\\
\dot{x}_2 &= -x_1 x_2,\\
\dot{x}_3 &= -x_1 x_3,\\
\dot{x}_4 &= -x_1 x_4.
\end{aligned}
\tag{2.2}
$$

A first result about the Raychaudhuri equation is the following theorem concerning system (2.2). Its proof was given in [15].

Theorem 2.1. *System* (2.2) *is rationally completely integrable with the rational first integrals*

$$
H_1 = \frac{x_2}{x_3}, \quad H_2 = \frac{x_3}{x_4}, \quad H_3 = \frac{x_1^2 - 4(x_2^2 + x_3^2 - x_4^2) + 2\gamma_0}{x_4}.
$$

If $\gamma \neq 0$ and $\delta = 0$ then we can apply the change $x_2 \mapsto x_2\gamma$ to obtain

$$
\begin{aligned}
\dot{x}_1 &= -\frac{x_1^2}{2} - 2(\gamma^2 x_2^2 + x_3^2 - x_4^2) - \gamma_0,\\
\dot{x}_2 &= -x_1 x_2 - 1,\\
\dot{x}_3 &= -x_1 x_3,\\
\dot{x}_4 &= -x_1 x_4.
\end{aligned}
\tag{2.3}
$$

If $\gamma = 0$ and $\delta \neq 0$, then applying the change $x_3 \mapsto x_3\delta$ and swapping x_2 and x_3, and δ and γ we obtain system (2.3) again.

Finally, if $\gamma, \delta \neq 0$ then the changes $x_2 \mapsto x_2\gamma$ and $x_3 \mapsto x_3\delta$ first and $(x_2, x_3) \mapsto (x_2 + x_3/2, x_2 - x_3/2)$ afterwards lead to

$$
\begin{aligned}
\dot{x}_1 &= -\frac{x_1^2}{2} - 2\left(\gamma_1^2 x_2^2 + \frac{\gamma_1^2}{4} x_3^2 + \gamma_2 x_2 x_3 - x_4^2\right) - \gamma_0,\\
\dot{x}_2 &= -x_1 x_2 - 1,\\
\dot{x}_3 &= -x_1 x_3,\\
\dot{x}_4 &= -x_1 x_4,
\end{aligned}
\tag{2.4}
$$

where $\gamma_1^2 = \gamma^2 + \delta^2 > 0$ and $\gamma_2 = \gamma^2 - \delta^2 \in \mathbb{R}$.

We note that after the change $x_3 \mapsto 2x_3/\gamma_1$ and setting $\gamma_2 = 0$, $\gamma_1 = \gamma$, system (2.4) is system (2.3). Therefore the cases $\delta = 0, \gamma \neq 0$ and $\delta \neq 0, \gamma = 0$ can be obtained from the case $\delta, \gamma \neq 0$. In short, from now on we shall only study system (2.4).

The following theorem is our first main result. It completes the study of the integrability of the Raychaudhuri equation by characterizing completely the existence of first integrals for any value of the parameters $\gamma, \delta \neq 0$.

Theorem 2.2. *System* (2.4) *is completely integrable, with at least two of the three first integrals being rational. Indeed:*

(a) If $4\gamma_1^2 - \gamma_0^2 \neq 0$ *then we have the following functionally independent rational first integrals:*

$$H_1 = \frac{x_3}{x_4}, \quad H_2 = \frac{F}{x_3\, x_4}, \quad H_3 = \frac{T_1\, T_2}{T_3\, T_4},$$

where F *and* T_i *are Darboux polynomials.*

(b) If $4\gamma_1^2 - \gamma_0^2 = 0$ *then we have the following functionally independent Darboux first integrals:*

$$H_1 = \frac{x_3}{x_4}, \quad H_2 = \frac{\bar{T}_1\, \bar{T}_2}{x_3\, x_4}, \quad H_3 = \frac{\bar{T}_2\, G^{4\sqrt{-\gamma_0}}}{\bar{T}_1}.$$

where $\bar{T}_1$ *and* $\bar{T}_2$ *are Darboux polynomials and* G *is an exponential factor.*

The expressions of the Darboux polynomials and of the exponential factor appearing in Theorem 2.2 are provided in the proof of Theorem 2.2, see Subsection 3.1.

2.2 The chemical reaction model

The second main result of this paper deals with system (1.6).

Theorem 2.3. *The following results hold for system* (1.6)*.*

(a) *It has the polynomial first integrals*

$$H_1 = x_1 + x_2 + x_3 + x_5, \quad H_2 = x_4 + x_5.$$

Any other polynomial first integral is a polynomial function of H_1, H_2.

(b) *The unique irreducible Darboux polynomial is* x_1.

(c) *The unique exponential factors are of the form* $e^{\mu x_3}$ *for any* $\mu \in \mathbb{C}$ *and* $\exp(f - \lambda/c_4 x_3)$ *where* $f - \lambda/c_4 x_3$ *is a rational function of* H_1, H_2 *for any* $\lambda \in \mathbb{C}$.

(d) *It is not Darboux integrable.*

3 Proof of the main results

We consider two subsections. The first one deals with the main result related to the Raychaudhuri equation, while the second one deals with the main result related to the Chemical reaction system.

3.1 The Raychaudhuri equation: proof of Theorem 2.2

To prove Theorem 2.2 we first state and prove some auxiliary results. The first one simplifies the expression of the general cofactor of a Darboux polynomial of system (2.1). We denote by f_i the homogeneous part of degree i of a polynomial f.

Proposition 3.1. *Let f be a Darboux polynomial of degree $m \in \mathbb{N}$ of system* (2.1) *with cofactor k. Then $k = k_0 + k_1 x_1$, with $k_1 \in [-m, -m/2] \cap \mathbb{Z}$.*

Proof. Let $k = k_0 + k_1 x_1 + k_2 x_2 + k_3 x_3 + k_4 x_4$ be the cofactor of f. Taking the homogeneous part of degree $m+1$ of the equation $\mathcal{X}(f) = kf$ and using the Euler theorem of homogeneous functions for f_m we get the equation

$$\begin{aligned}\big(2k_1x_1^2 + 2x_1(k_2x_2 + k_3x_3 + k_4x_4) + m(x_1^2 + 4(x_2^2 + x_3^2 - x_4^2))\big) f_m \\ + (x_1^2 - 4(x_2^2 + x_3^2 - x_4^2)) \left(x_2 \frac{\partial f_m}{\partial x_2} + x_3 \frac{\partial f_m}{\partial x_3} + x_4 \frac{\partial f_m}{\partial x_4} \right) = 0.\end{aligned}$$

The general solution of this equation is

$$\begin{aligned} f_m(x_1, x_2, x_3, x_4) =& x_2^{-2k_1 - m} \left(x_1 - 2\sqrt{x_2^2 + x_3^2 - x_4^2} \right)^{k_1 + m + \frac{k_2x_2 + k_3x_3 + k_4x_4}{2\sqrt{x_2^2 + x_3^2 - x_4^2}}} \\ & \left(x_1 + 2\sqrt{x_2^2 + x_3^2 - x_4^2} \right)^{k_1 + m - \frac{k_2x_2 + k_3x_3 + k_4x_4}{2\sqrt{x_2^2 + x_3^2 - x_4^2}}} C_m(x_1, x_3/x_2, x_4/x_2), \end{aligned}$$

where C_m is an arbitrary function. Since this is to be a polynomial of degree m, we must take $k_2 = k_3 = k_4 = 0$. Moreover C_m cannot depend on x_1. Then

$$f_m(x_1, x_2, x_3, x_4) = x_2^{-2k_1 - m} (x_1^2 - 4(x_2^2 + x_3^2 - x_4^2))^{k_1 + m} C_m(x_3/x_2, x_4/x_2).$$

Since the exponents must be non-negative, the proposition follows. □

Lemma 3.2. *The following statements hold concerning system* (2.4).

1. *It has two Darboux polynomials of degree one.*
2. *If $4\gamma_1^2 - \gamma_0^2 \neq 0$, then it has four Darboux polynomials of degree two and one Darboux polynomial of degree four.*
3. *If $4\gamma_1^2 - \gamma_0^2 = 0$, then it has two Darboux polynomials of degree two and an exponential factor.*

Proof. Clearly x_3 and x_4 are Darboux polynomials. Both have cofactor $-x_1$. In particular x_3/x_4 is a rational first integral of (2.4).

Straightforward computations show that

$$T = k_0^4 - 4\gamma_1^2 + 2k_0^3 x_1 + k_0^2 x_1^2 - 4\gamma_1^2 k_0^2 x_2^2 + 4\gamma_2 k_0 x_3 - \gamma_1^2 k_0^2 x_3^2 + 8\gamma_1^2 k_0 x_2 - 4\gamma_2 k_0^2 x_2 x_3 + 4k_0^2 x_4^2$$

is a Darboux polynomial of degree two of (2.4) with cofactor $k_0 - x_1$, with k_0 a solution of $K_0 := 4\gamma_1^2 + 2\gamma_0 k_0^2 + k_0^4 = 0$. These Darboux polynomials were already found in [11]. The discriminant of K_0 is $2^{10}\gamma_1^2(4\gamma_1^2 - \gamma_0^2)$. We recall that $\gamma_1 \neq 0$. If $4\gamma_1^2 - \gamma_0^2 \neq 0$ then we have four Darboux polynomials, say T_1, T_2, T_3, T_4, one for each solution of $K_0 = 0$. Otherwise we have just two, say $\bar{T}_1, \bar{T}_2$, as $K_0 = 0$ has two double solutions.

The polynomial

$$
\begin{aligned}
F = \left(x_1^2 + 2\gamma_0 - 4\gamma_1^2 x_2^2 - 4\gamma_2 x_2 x_3 - \gamma_1^2 x_3^2 + 4x_4^2\right)^2 \\
+ 4\left(\gamma_1^2(-4 - 8x_1x_2 + 8\gamma_0 x_2^2 + \gamma_0 x_3^2) - 4(\gamma_2(x_1 - 2\gamma_0 x_2)x_3 + \gamma_0 x_4^2)\right)
\end{aligned}
$$

is a Darboux polynomial of degree four of system (2.4) with cofactor $-2x_1$. We note that it coincides with $\bar{T}_1\bar{T}_2$ in the case $4\gamma_1^2 - \gamma_0^2 = 0$, for which we have the exponential factor

$$
G = \exp\left\{\frac{2(-\gamma_0)^{3/2} + \gamma_0 x_1 + \gamma_0^2 x_2 + 2\gamma_2 x_3}{\bar{T}_1}\right\}.
$$

We notice that the cofactor of G is $-1/2$. □

Proof of Theorem 2.2. It follows immediately from Lemma 3.2 and from the fact that the corresponding linear combination of the cofactors of the Darboux polynomials and of the exponential factor in the expressions of the functions in Theorem 2.2 is zero. □

3.2 The Chemical reaction: proof of Theorem 2.3

Statement (d) follows immediately from statements (a), (b) and (c), since there is no way to construct two Darboux first integrals functionally independent of H_1, H_2. In particular, it is clear that the unique rational first integrals of system (1.6) are the rational functions in the variables H_1, H_2. Hence, we need to prove only statements (a), (b) and (c).

As in the proof of Theorem 2.2, we start the study of system (1.6) simplifying the general expression of the cofactor of a Darboux polynomial.

Proposition 3.3. *Let f be a Darboux polynomial of degree $m \in \mathbb{N}$ of system* (1.6) *with cofactor k. Then $k = k_0 + k_4 x_4$.*

Proof. Let $k = k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5$ be the cofactor of f. Taking the homogeneous part of degree $m+1$ of the equation $\mathcal{X}(f) = kf$ and using the Euler theorem of homogeneous functions for f_m we get the equation

$$
\begin{aligned}
-(k_1x_1 + k_2x_2 + k_3x_3 + (k_4 + c_1 m)x_4 + k_5x_5)f_m + (c_1 - c_3)x_2x_4\frac{\partial f_m}{\partial x_2} + c_1x_3x_4\frac{\partial f_m}{\partial x_3} \\
+ x_4(-c_1x_2 - c_3x_2 + c_1x_4)\frac{\partial f_m}{\partial x_4} + x_4(c_1x_1 + c_3x_2 + c_1x_5)\frac{\partial f_m}{\partial x_5} = 0.
\end{aligned}
\tag{3.1}
$$

The general solution of this equation is

$$f_m(x_1,x_2,x_3,x_4,x_5) = e^{\frac{k_2x_2}{(c_1-c_3)x_4}} x_2^{\frac{k_1x_1+k_3x_3+(k_4+c_1m)x_4+k_5x_5}{(c_1-c_3)x_4}}$$
$$C_m\left(x_1, x_2^{-\frac{c_1}{c_1-c_3}}x_3, x_2^{-\frac{c_1}{c_1-c_3}}(x_1+x_2-x_4), x_2^{-\frac{c_1}{c_1-c_3}}(x_1+x_2+x_5)\right),$$

where C_m is an arbitrary function. We must take $k_2 = 0$; moreover the exponent of x_2 cannot depend on the variables, hence $k_1 = k_3 = k_5 = 0$. Then the proposition follows. □

We prove the statements of Theorem 2.3 separately.

3.2.1 Proof of statement (a)

Lemma 3.4. *The polynomial functions* $H_1 = x_1 + x_2 + x_3 + x_5$ *and* $H_2 = x_4 + x_5$ *are first integrals of system* (1.6).

Proof. It follows after direct computations. □

Statement (a) follows after next proposition.

Proposition 3.5. *Any polynomial first integral of system* (1.6) *is a polynomial function of* H_1, H_2.

Proof. Let f be a polynomial first integral of (1.6). Then f satisfies the equation $\mathcal{X}(f) = 0$; that is,

$$-c_1x_1x_4\frac{\partial f}{\partial x_1} + (-c_3x_2x_4 + c_2x_5)\frac{\partial f}{\partial x_2} + c_4x_5\frac{\partial f}{\partial x_3}$$
$$+ (-c_1x_1x_4 - c_3x_2x_4 + (c_2+c_4)x_5)\left(\frac{\partial f}{\partial x_4} - \frac{\partial f}{\partial x_5}\right) = 0.$$

Write $f = g(x_1,x_2,x_3,x_4) + H_2^j h(x_1,x_2,x_3,x_4,x_5)$, with $H_2 \nmid h$ and $j \in \mathbb{N}$. Then on $H_2 = 0$ we have, after some simplifications,

$$c_1x_1\frac{\partial g}{\partial x_1} + (c_2 + c_3x_2)\frac{\partial g}{\partial x_2} + c_4\frac{\partial g}{\partial x_3} + (c_1x_1 + c_3x_2 + c_2 + c_4)\frac{\partial g}{\partial x_4} = 0,$$

taking into account that $g = f|_{H_2=0}$. The solution of this equation is

$$g(x_1,x_2,x_3,x_4) = F\left(\frac{x_1^{c_3}}{(c_2+c_3x_2)^{c_1}}, x_3 - \frac{c_4}{c_1}\log x_1, x_1 + x_2 - x_4 + \frac{c_4}{c_1}\log x_1\right),$$

with F an arbitrary function. Then $g = g(x_1^{c_3}(c_2 + c_3x_2)^{-c_1}, H_1)$. Since $c_1c_3 > 0$ and we want g to be a polynomial, we have $g = g(H_1)$.

At this time we have $f = g(H_1) + H_2^j h$, thus h is either a polynomial first integral or a constant, since

$$0 = \mathcal{X}(f) = \mathcal{X}(g(H_1)) + \mathcal{X}(H_2^j)h + H_2^j\mathcal{X}(h) = H_2^j\mathcal{X}(h).$$

Indeed we can assume it is a constant, otherwise we apply the above arguments repeatedly until we obtain a constant. So f is a function of H_1 and H_2 and thus the proposition follows. □

Remark 3.6. We note that from the proof of Proposition 3.5 we get that any polynomial first integral of system (1.6) restricted to $H_2 = 0$ is a polynomial function of H_1.

Remark 3.7. We notice that $x_1^{-c_3}(c_2+c_3x_2)^{c_1}$ is a Darboux first integral of system (1.6) restricted to $H_2 = 0$.

3.2.2 Proof of statement (b)

We need two lemmas and one proposition to prove statement (b).

Lemma 3.8. *The unique Darboux polynomial of degree one of system* (1.6) *is* x_1. *Its cofactor is* $k = -c_1x_4$.

Proof. It follows after easy computations. $\square$

Lemma 3.9. *Let f be an irreducible Darboux polynomial of system* (1.6) *of degree greater than one with cofactor k. Then $k = k_4x_4$.*

Proof. From Proposition 3.3 we have $k = k_0 + k_4x_4$. We want to prove that $k_0 = 0$.

Suppose that $k_0 \neq 0$. Let $\bar{f} = f|_{x_1=0}$. Since f is irreducible it is clear that $x_1 \nmid f$, hence $\bar{f} \not\equiv 0$. Moreover $\bar{f}$ satisfies

$$(-c_3x_2x_4 + c_2x_5)\frac{\partial \bar{f}}{\partial x_2} + c_4x_5\frac{\partial \bar{f}}{\partial x_3} + (-c_3x_2x_4 + (c_2+c_4)x_5)\left(\frac{\partial \bar{f}}{\partial x_4} - \frac{\partial \bar{f}}{\partial x_5}\right) = (k_0 + k_4x_4)\bar{f}. \quad (3.2)$$

We first suppose that $H_2 \nmid \bar{f}$. Let $\tilde{f} = \bar{f}|_{H_2=0} \not\equiv 0$. We note that $\frac{\partial \tilde{f}}{\partial x_4} = \frac{\partial \bar{f}}{\partial x_4} - \frac{\partial \bar{f}}{\partial x_5}$. Then $\tilde{f}$ satisfies equation (3.2) on $H_2 = 0$, that is

$$-x_4\left((c_3x_2 + c_2)\frac{\partial \tilde{f}}{\partial x_2} + c_4\frac{\partial \tilde{f}}{\partial x_3} + (c_3x_2 + c_2 + c_4)\frac{\partial \tilde{f}}{\partial x_4}\right) = (k_0 + k_4x_4)\tilde{f}. \quad (3.3)$$

Since we are assuming that $k_0 \neq 0$, we have $x_4|\tilde{f}$. Hence $\tilde{f} = x_4^{\ell}g \not\equiv 0$, where $\ell \in \mathbb{N}$ and $g \in \mathbb{C}[x_2, x_3, x_4]$ is such that $x_4 \nmid g$. Moreover, g satisfies the equation

$$-x_4\left((c_3x_2 + c_2)\frac{\partial g}{\partial x_2} + c_4\frac{\partial g}{\partial x_3} + (c_2 + c_4 + c_3x_2)\frac{\partial g}{\partial x_4}\right) = \big(k_0 + \ell(c_2 + c_4) + \ell c_3x_2 + k_4x_4\big)g.$$

Since $\ell c_3 \neq 0$ and $x_4 \nmid g$, we have $g \equiv 0$, which is a contradiction.

Thus the case $H_2 \nmid \bar{f}$ cannot happen. We write $\bar{f} = H_2^{\ell}\tilde{f}$, with $\ell \in \mathbb{N}$, $H_2 \nmid \tilde{f}$ and $\tilde{f} \not\equiv 0$. Let $\hat{f} = \tilde{f}|_{H_2=0} \not\equiv 0$. We notice that $\hat{f}$ satisfies (3.3). Then, proceeding as above, we conclude that $\hat{f} = 0$, which is again a contradiction. This concludes the proof of the lemma, since the assumption $k_0 \neq 0$ leads to contradiction.

$\square$

In view of Lemma 3.9 the cofactor of any Darboux polynomial of (1.6) has the form $k = k_4x_4$.

Proposition 3.10. *System* (1.6) *has no irreducible Darboux polynomials other than* x_1.

Proof. Suppose that f is an irreducible Darboux polynomial of system (1.6) of degree greater than one and cofactor $k = k_4x_4$, see Lemma 3.9. Since $x_1 \nmid f$, we can write

$$f = f_0(x_2, x_3, x_4, x_5) + x_1^j g_1(x_1, x_2, x_3, x_4, x_5),$$

with $j \in \mathbb{N}$ and $f_0 \not\equiv 0$. From the equation $\mathcal{X}(f) - kf = 0$ restricted to $x_1 = 0$ we have

$$(-c_3x_2x_4+c_2x_5)\frac{\partial f_0}{\partial x_2}+c_4x_5\frac{\partial f_0}{\partial x_3}-(c_3x_2x_4-(c_2+c_4)x_5)\left(\frac{\partial f_0}{\partial x_4}-\frac{\partial f_0}{\partial x_5}\right)-k_4x_4f_0 = 0. \quad (3.4)$$

We first suppose that $H_2 \nmid f_0$. Let $\bar{f} = f_0|_{H_2=0} \not\equiv 0$. Notice that $\frac{\partial \bar{f}}{\partial x_4} = \frac{\partial f_0}{\partial x_4} - \frac{\partial f_0}{\partial x_5}$. We have $f_0 = \bar{f}(x_2, x_3, x_4) + H_2^\ell g_0(x_2, x_3, x_4, x_5)$, with $\ell \in \mathbb{N}$, $H_2 \nmid g_0$ and $g_0 \not\equiv 0$. On $H_2 = 0$ equation (3.4) becomes

$$(c_3x_2 + c_2)\frac{\partial \bar{f}}{\partial x_2} + c_4\frac{\partial \bar{f}}{\partial x_3} + (c_3x_2 + c_2 + c_4)\frac{\partial \bar{f}}{\partial x_4} + k_4\bar{f} = 0, \quad (3.5)$$

from which we obtain

$$\bar{f}(x_2, x_3, x_4) = (c_3x_2 + c_2)^{-k_4/c_3} F(x_2 + x_3 - x_4),$$

where $F \not\equiv 0$ is an arbitrary function such that $\bar{f}$ is a polynomial. We must take $k_4 = -c_3n$, with $n \in \mathbb{N}$. We note that $n \neq 0$ since f is not a first integral.

Now equation (3.4) writes as

$$\begin{aligned} H_2^\ell\Big[&(-c_3x_2x_4 + c_2x_5)\frac{\partial g_0}{\partial x_2} + c_4x_5\frac{\partial g_0}{\partial x_3} \\ &\qquad -(c_3x_2x_4 - (c_2 + c_4)x_5)\left(\frac{\partial g_0}{\partial x_4} - \frac{\partial g_0}{\partial x_5}\right) + c_3nx_4g_0\Big] \\ &\qquad\qquad - c_3n(c_3x_2 + c_2)^{n-1}(c_2x_4 + 2c_3x_2x_4 - c_2x_5)F = 0. \end{aligned}$$

Since $\ell > 0$ we take $H_2 = 0$ to obtain $F \equiv 0$, and hence $\bar{f} \equiv 0$, a contradiction.

Thus the case $H_2 \nmid \bar{f}_0$ cannot happen and we have $f_0 = H_2^\ell f_1 \not\equiv 0$ with $\ell \in \mathbb{N}$ and $H_2 \nmid f_1$. Let $\tilde{f} = f_1|_{H_2=0} \not\equiv 0$. We notice that $\tilde{f}$ satisfies (3.5). Therefore $\tilde{f} \equiv 0$, which is again a contradiction. This concludes the proof of the lemma. $\square$

After Lemma 3.8 and Proposition 3.10, statement (b) of Theorem 2.3 follows.

3.2.3 Proof of statement (c)

We divide the proof of statement (c) into different partial results.

Lemma 3.11. *The function* e^{x_3} *is an exponential factor of system* (1.6)*. It has cofactor* c_4x_5.

Proof. It follows from direct computations. □

We notice that, since system (1.6) has only one Darboux polynomial and two polynomial first integrals, if (1.6) has an exponential factor, then it must be of the form

$$\exp(g/(x_1^n Q(H_1, H_2))), \tag{3.6}$$

with $n \in \mathbb{N} \cup \{0\}$ and $Q \in \mathbb{C}[H_1, H_2]$. Next lemma will be useful later on to finish the proof of statement (c).

Lemma 3.12. *System* (1.6) *on* $x_1 = 0$ *has no Darboux polynomials.*

Proof. Let $g \in \mathbb{C}[x_2, x_3, x_4, x_5]$ be a Darboux polynomial of degree $m \in \mathbb{N}$ of system (1.6) restricted to $x_1 = 0$, which will be called $\mathcal{Y}$ in this proof. Let $k = k_0 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5$ be its cofactor. We write $g = \sum_{i=0}^m g_i$, where g_i is a homogeneous polynomial of degree i in the variables. From the terms of degree $m+1$ of equation $\mathcal{Y}(g) = kg$ we have, using the Euler Theorem of homogeneous functions for g_m,

$$g_m(x_2, x_3, x_4, x_5) = x_3^{\frac{k_2+k_4+k_5}{c_3}+m} x_4^{-\frac{k_2}{c_3}-\frac{k_3x_3+k_5(x_4+x_5)}{c_3(x_2-x_4)}} C_m\left(x_2, \frac{x_2-x_4}{x_3}, \frac{x_2+x_5}{x_3}\right),$$

where C_m is an arbitrary function. Since g_m is a homogeneous polynomial of degree m we must take $k_3 = k_5 = 0$ and thus

$$g_m(x_2, x_3, x_4, x_5) = x_2^{-\frac{k_4}{c_3}} x_3^{\frac{k_2+k_4}{c_3}+m} x_4^{-\frac{k_2}{c_3}} C_m\left(\frac{x_2-x_4}{x_3}, \frac{x_2+x_5}{x_3}\right),$$

which rewrites as

$$g_m(x_2, x_3, x_4, x_5) = x_2^{m_1} x_3^{m-m_1-m_2-m_3-m_4} x_4^{m_2} (x_2-x_4)^{m_3} (x_2+x_5)^{m_4},$$

with $m_i \in \mathbb{N} \cup \{0\}$ and $k_4 = -m_1c_3$, $k_2 = -m_2c_3$.

From the homogeneous equation of degree m we obtain

$$\begin{aligned} g_{m-1}(x_2, x_3, x_4, x_5) = & \frac{1}{c_3} x_2^{m_1-1} x_3^{m-m_1-m_2-m_3-m_4-1} x_4^{m_2-1} (x_2-x_4)^{m_3-2} (x_2+x_5)^{m_4-1} \\ & \times \Big[L_1 \log x_2 + L_2 \log x_4 + x_3(x_2-x_4)(x_2+x_5)(c_2m_1x_4(x_2+x_5) \\ & - (c_2+c_4)m_2x_2(x_4+x_5)) \Big] + x_2^{m_1} x_4^{m_2} C_{m-1}(x_2-x_4, x_2+x_5, x_3), \end{aligned}$$

where C_{m-1} is an arbitrary function and L_1, L_2 are some homogeneous polynomials of degree 5. We must take $L_1 = L_2 = 0$. The equations corresponding to the coefficients of L_1 and L_2 equaled to zero lead to

$$k_0 = -\frac{c_2(c_2+c_4)}{2c_2+c_4} m, \quad m_1 = \frac{c_2+c_4}{2c_2+c_4} m, \quad m_2 = \frac{c_2}{2c_2+c_4} m, \quad m_3 = m_4 = 0.$$

We notice that $m|k$. Moreover,

$$g_{m-1}(x_2,x_3,x_4,x_5) = -\frac{c_2(c_2+c_4)}{c_3(2c_2+c_4)}\, m x_2^{\frac{c_2+c_4}{2c_2+c_4}m-1} x_4^{\frac{c_2}{2c_2+c_4}m-1} x_5 + x_2^{\frac{c_2+c_4}{2c_2+c_4}m} x_4^{\frac{c_2}{2c_2+c_4}m} C_{m-1}(x_2-x_4, x_2+x_5, x_3).$$

Since g_{m-1} has degree $m-1$, we must take $C_{m-1}=0$.

From the homogeneous equation of degree $m-1$ we obtain

$$g_{m-2}(x_2,x_3,x_4,x_5) = x_2^{\frac{c_2+c_4}{2c_2+c_4}m-2} x_4^{\frac{c_2}{2c_2+c_4}m-2}\Bigg(x_2^2x_4^2 C_{m-2}(x_2-x_4, x_2+x_5, x_3) + \frac{P_{m-2}}{(x_2-x_4)^3} - \frac{c_2c_4(c_2+c_4)m(x_4+x_5)(2x_2+x_4+3x_5)}{c_3^2(2c_2+c_4)(x_2-x_4)^4} x_2^2x_4^2 \log\frac{x_2}{x_4}\Bigg),$$

where C_{m-2} is an arbitrary function and P_{m-2} is some homogeneous polynomial. Since g_{m-2} is to be a polynomial, we must take $m=0$ to remove the logarithm, and hence no such g can exist. This ends the proof of the lemma. $\square$

Next we prove that the expression of an exponential factor (3.6) cannot contain a power of x_1 in the denominator of the exponent.

Lemma 3.13. *Suppose system* (1.6) *has the exponential factor* $E = \exp(g/(x_1^n Q(H_1,H_2))$*, with* $g \in \mathbb{C}[x_1,x_2,x_3,x_4,x_5]$*,* $n \in \mathbb{N}\cup\{0\}$*,* $x_1 \nmid g$ *and* $Q \in \mathbb{C}[H_1,H_2]$*. Then* $n=0$*.*

Proof. Suppose that $n>0$. Let L be the cofactor of E. Since $\mathcal{X}(Q(H_1,H_2))=0$, we have

$$LE = \mathcal{X}(E) = \frac{\mathcal{X}(g)\cdot x_1^n - g\cdot \mathcal{X}(x_1^n)}{x_1^{2n}Q(H_1,H_2)}E.$$

Hence

$$\mathcal{X}(g)x_1^n + nc_1x_4gx_1^n = Lx_1^{2n}Q(H_1,H_2),$$

see Lemma 3.8. Therefore

$$\mathcal{X}(g) + nc_1x_4g = Lx_1^nQ(H_1,H_2). \tag{3.7}$$

Let $\bar{g} = g|_{x_1=0} \not\equiv 0$. Since $n>0$, equation (3.7) on $x_1=0$ writes

$$(-c_3x_2x_4 + c_2x_5)\frac{\partial \bar{g}}{\partial x_2} + c_4x_5\frac{\partial \bar{g}}{\partial x_3} - (c_3x_2x_4 - (c_2+c_4)x_5)\left(\frac{\partial \bar{g}}{\partial x_4} - \frac{\partial \bar{g}}{\partial x_5}\right) = -nc_1x_4\bar{g},$$

which means that $\bar{g}$ is a Darboux polynomial of system (1.6) restricted to $x_1=0$. In view of Lemma 3.12, this is a contradiction, which comes from the assumption $n \neq 0$. Therefore $n=0$ and the lemma follows. $\square$

The following result completes the proof of statement (c).

Proposition 3.14. *Let $E = \exp(g/Q(H_1, H_2))$, with $Q \in \mathbb{C}[H_1, H_2]$, be an exponential factor of system* (1.6)*, where $g \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$. Then $g - k_5/c_4 x_3 Q(H_1, H_2)$ is a polynomial function in the variables H_1, H_2, where k_5 is the coefficient of x_5 in the expression of the cofactor of E.*

Proof. Let $k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 + k_5x_5$ be the cofactor of $\exp(g/Q(H_1, H_2))$. Consider the alternative exponential factor $\exp(h/Q(H_1, H_2))$, with $h = g - k_5/c_4x_3Q(H_1, H_2)$, see Lemma 3.11. We notice that its cofactor is $k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4$. We shall prove that h is a polynomial function of H_1, H_2.

Since $\exp(h/Q(H_1, H_2))$ is an exponential factor, h satisfies

$$\mathcal{X}(h) = (k_0 + k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4)Q(H_1, H_2), \tag{3.8}$$

see (1.4). Evaluating (3.8) on $x_1 = x_2 = x_5 = 0$, we get

$$0 = (k_0 + k_3x_3 + k_4x_4)Q(x_3, x_4).$$

Since $Q \not\equiv 0$ we get that $k_0 = k_3 = k_4 = 0$. Now evaluating (3.8) on $x_4 = x_5 = 0$, we obtain

$$0 = (k_1x_1 + k_2x_2)Q(x_1 + x_2 + x_3, 0).$$

If $H_2 \nmid Q$, then from the relation above we get $k_1 = k_2 = 0$. Thus from (3.8) we get that h is a polynomial first integral. Hence $h = h(H_1, H_2)$ in view of Lemma 3.5 and thus the proposition follows.

We are left with the case $Q(x_1 + x_2 + x_3, 0) = 0$. In this case $H_2 | Q(H_1, H_2)$, hence we write $Q(H_1, H_2) = H_2^\ell \bar{Q}(H_1, H_2)$, for some $\ell > 0$ and $\bar{Q}$ a polynomial such that $H_2 \nmid \bar{Q}(H_1, H_2)$. From (3.8) we have

$$\mathcal{X}(h) = (k_1x_1 + k_2x_2)H_2^\ell \bar{Q}(H_1, H_2). \tag{3.9}$$

Let $\bar{h} = h|_{H_2=0}$. Evaluating (3.9) on $H_2 = 0$ we get that $\bar{h}$ is a polynomial first integral of system (1.6) restricted to $H_2 = 0$. By Remark 3.6 we conclude that $\bar{h} = \bar{h}(H_1)$. Hence, we can write $h = \bar{h}(H_1) + H_2^j \tilde{h}$, where $j > 0$ and $\tilde{h} \in \mathbb{C}[x_1, x_2, x_3, x_4, x_5]$ is such that $H_2 \nmid \tilde{h}$. Since H_1 and H_2 are first integrals of (1.6), from (3.9) we get

$$\mathcal{X}(\tilde{h}) = (k_1x_1 + k_2x_2)H_2^{\ell - j} \bar{Q}(H_1, H_2). \tag{3.10}$$

Since $\mathcal{X}(\tilde{h})$ is a polynomial and $H_2 \nmid \bar{Q}$ we have $j \leq \ell$. We consider two different cases.

Case 1*:* $j = \ell$. Evaluating (3.10) on $x_4 = x_5 = 0$ we get

$$(k_1x_1 + k_2x_2)\bar{Q}(x_1 + x_2 + x_3, 0) = 0.$$

Since $\bar{Q}(H_1, 0) \not\equiv 0$ we have $k_1 = k_2 = 0$. Then, from (3.8) h is a polynomial first integral. Hence $h = h(H_1, H_2)$ in view of Lemma 3.5 and thus the proposition follows.

Case 2*:* $j < \ell$. Let $\hat{h} = \tilde{h}|_{H_2=0}$. From (3.10), $\hat{h}$ is a polynomial first integral of system (1.6) restricted to $H_2 = 0$. Regarding Remark 3.6 we have $\hat{h} = \hat{h}(H_1)$. Hence $\tilde{h} = \hat{h}(H_1) + H_2^m \check{h}$ where $m \in \mathbb{N}$ and $\check{h} \in \mathbb{C}[x_1, x_2, x3, x_4, x_5]$ satisfies (3.10) with $j - \ell - m$. Now proceeding inductively we can repeat this process until we obtain a polynomial $\breve{h}$ such that $\mathcal{X}(\breve{h}) = (k_1x_1 + k_2x_2)\bar{Q}(H_1, H_2)$, and hence proceeding as in Case 1 we conclude that $h = h(H_1, H_2)$, as we wanted to prove. Hence the proposition follows also in this case. □

After these results, statement (c) follows.

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