PHASE PORTRAITS OF ABEL QUADRATIC DIFFERENTIAL SYSTEMS OF SECOND KIND

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Abstract. We provide normal forms and the global phase portraits on the Poincaré disk of some Abel quadratic differential equations of the second kind. Moreover, we also provide the bifurcation diagrams for these global phase portraits.

1. Introduction and statements of the main results

Neils Henrick Abel, one of the most active mathematicians in his time, dedicated himself to integral equations where he defined the Abel integral and worked on methods to solve special integral equations, which were later known as *Abel integral equations* [11]. His research on integral equations and their solutions led him work on differential equations, where he verified the importance of the Wronskian determinant for a differential equation of order two [11]. His study of the theory of elliptic functions was what got him involved into the analysis of special differential equations, which are a generalized Riccati differential equation. Due to his seminal work on these differential equations, they are named Abel differential equations. The Abel differential equation of the first and second kind are both non-homogeneous differential equations of first order and are related between them by a substitution. Despite the fact that Abel equations of first kind are very well studied, this is not the case of Abel differential equations of second kind. In this paper we will focus on them.

An Abel differential equation of second kind has the form

$$y\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x), \tag{1}$$

with $A(x), B(x), C(x) \in \mathbb{R}(x, y)$. This differential equation can be written equivalently as the polynomial differential system

$$\dot{x} = d(x)y, \quad \dot{y} = a(x)y^2 + b(x)y + c(x),$$

where $A(x) = a(x)/d(x)$, $B(x) = b(x)/d(x)$ and $C(x) = c(x)/d(x)$, with polynomials $a(x), b(x), c(x)$ and $d(x)$. In this paper we are interested in studying the *Abel quadratic polynomial differential systems*, i.e. the differential systems of the form

$$\dot{x} = d(x)y := (d_0 + d_1x)y, \quad \dot{y} = a_0y^2 + (b_0 + b_1x)y + c_0 + c_1x + c_2x^2 \tag{2}$$

coming from the Abel differential equation of second kind (1). Here all the parameters are real. We assume that $\dot{x}$ and $\dot{y}$ do not have a common factor, that is $c_0^2+c_1^2+c_2^2 \neq 0$. Moreover $a_0 \neq 0$, otherwise this is not the Abel equation of second kind, and $b_1^2 + c_1^2 + c_2^2 + d_1^2 \neq 0$ to have a quadratic system, otherwise the system would not depend on x and hence would not be of our interest.

In this work we provide the global phase portraits of all the Abel differential systems of second kind of degree two given by system (2) with $d_1 = 0$. We assume then that $d_0 \neq 0$.

We shall use the well-known Poincaré compactification of polynomial vector fields. For more details, if needed, see chapter V in [6]. We say that two polynomial vector fields on the Poincaré disk are *topologically equivalent* if there exists a homeomorphism from one

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