NEW FAMILY OF CUBIC HAMILTONIAN CENTERS

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ABSTRACT. We characterize the 11 non topological equivalent classes of phase portraits in the Poincaré disc of the new family of cubic polynomial Hamiltonian differential systems with a center at the origin and Hamiltonian

$$H = \frac{1}{2}\left((x + ax^2 + bxy + cy^2)^2 + y^2\right),$$

with $a^2 + b^2 + c^2 \neq 0$.

1. Introduction

For a given family of real planar polynomial differential systems depending on parameters one of the main problems is the characterization of their centers and their phase portraits. The notion of *center* goes back to Poincaré in [16]. He defined it for differential systems on the real plane; i.e. a singular point surrounded by a neighborhood fulfilled of closed orbits with the unique exception of the singular point.

The classification of the centers of the real polynomial differential systems started with the quadratic ones with the works of Kapteyn [10, 11], Bautin [3], Vulpe [21], Schlomiuk [18, 19], Żołądek in [23], ... Schlomiuk, Guckenheimer and Rand in [20] described a brief history of the problem of the center in general, and it includes a list of 30 papers covering the history of the center for the quadratic polynomial differential systems (see pages 3, 4 and 13).

While the centers and all their phase portraits have been characterized for all the quadratic polynomial differential systems, this is not the case for the polynomial differential systems of degree larger than 2, but for such systems there are many partial results. Thus the centers for cubic polynomial differential systems of the form linear with homogeneous nonlinearities of degree 3 were classified by Malkin [12], and Vulpe and Sibirskii [22], and their phase portraits when they are Hamiltonian have been classified by Colak, Llibre and Valls in [6, 7]. Moreover for polynomial differential systems which are linear with homogeneous nonlinearities of degree $k > 3$ the centers are not classified, but there are partial results for $k = 4, 5$ see Chavarriga and Giné [4, 5], respectively.

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Nowdays we are very far from obtaining a complete classification of the centers for the class of all polynomial differential systems of degree 3. In any case some interesting results on some subclasses of cubic polynomial differential systems are the ones of Rousseau and Schlomiuk [17], and the ones of Żołądek [24, 25].

Our goal in this paper is to characterize the phase portraits in the Poincaré disc of the new family of cubic polynomial Hamiltonian differential systems with a center at the origin and Hamiltonian

$$H = \frac{1}{2}\left((x + ax^2 + bxy + cy^2)^2 + y^2\right), \tag{1}$$

with $a^2 + b^2 + c^2 \neq 0$, i.e. of the cubic polynomial Hamiltonian differential systems

$$\begin{aligned} \dot{x} &= -y - (bx + 2cy)(x + ax^2 + bxy + cy^2), \\ \dot{y} &= (1 + 2ax + by)(x + ax^2 + bxy + cy^2). \end{aligned} \tag{2}$$

Note that such cubic polynomial differential systems have terms of degree 1, 2 and 3 and consequently are not studied in the previous mentioned papers dedicated to the centers of cubic polynomial differential systems.

It is clear that the origin $(0, 0)$ is an isolated minimum of the Hamiltonian function H given in (1), and so the origin is a center of the Hamiltonian system (2) because near $(0, 0)$ the curves $H(x, y) =$ constant are closed.

In what follows we shall talk interchangeably of the cubic polynomial Hamiltonian differential system (2), or of its associated cubic polynomial Hamiltonian vector field

$$\begin{aligned} \mathcal{X}_H = \quad & \big(-y - (bx + 2cy)(x + ax^2 + bxy + cy^2)\big)\frac{\partial}{\partial x} + \\ & \big((1 + 2ax + by)(x + ax^2 + bxy + cy^2)\big)\frac{\partial}{\partial y}. \end{aligned} \tag{3}$$

Let $\mathcal{X}$ be a polynomial vector field. See information on the notations and definitions used in this paper, on the Poincaré disc and on the compactified vector field $p(\mathcal{X})$ associated to $\mathcal{X}$ in subsection 2.1.

We say that two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there is a homeomorphism on the Poincaré disc preserving the infinity and carrying orbits of the vector field $p(\mathcal{X})$ into orbits of the vector field $p(\mathcal{Y})$, preserving or reversing simultaneously the sense of all orbits.

Our main result can be stated as follows.

Theorem 1. *The phase portrait in the Poincaré disc of a cubic Hamiltonian differential systems* (2) *is topologically equivalent to one of the* 11 *phase portraits described in Figures 1 to 4.*

Theorem 1 is proved in section 5. In sections 3 and 4 we study the finite and infinite singular points of system (2).

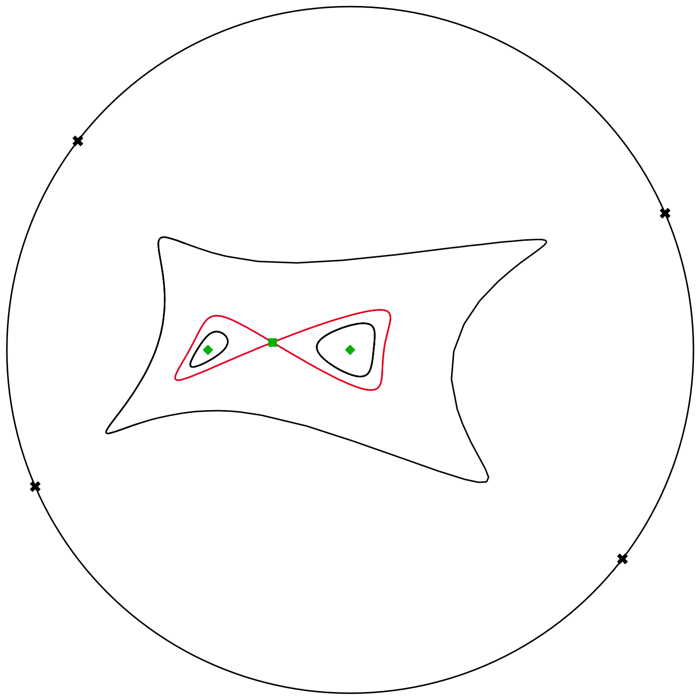

FIGURE 1. Phase portrait of system (2) when either $b(b^2-4ac) \neq 0$, $a(27ab^2+4(2a-c)^3) > 0$ and $b^2-4ac > 0$; or $c = 0$ and $ab \neq 0$; or $ac < 0$, $b = 0$ and $a(c-2a) \leq 0$. This phase portrait corresponds to the values $a = 1$, $b = -1$ and $c = -3$. This phase portrait has 3 canonical regions.

We must mention that our study only have used the program P4 for doing the pictures of the phase portraits, and that we studied the local phase portraits of finite and infinite singular points in the Poincaré disc, and the separatrices of thee system analytically. We also must say that we prefer to provide the phase portraits in the Poincaré disc, that we have studied analytically, drawn with the program P4 because then we have the analytical qualitative study of these phase portraits and also its quantitative study for the considered values of the parameters. Usually this is not possible to do because some separatrices become too much close and we cannot distinguish them, but for the Hamiltonian systems here studied, only two phase portraits (see Figures 3 and 10) have one center which become to close to the separatrices, but since we know analytically that they are centers, these two phase portraits can be well understood.

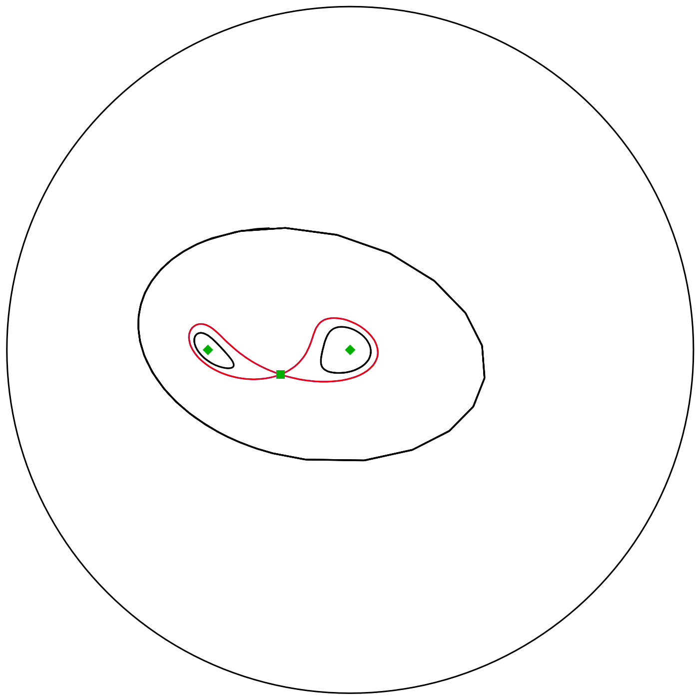

FIGURE 2. Phase portrait of system (2) when $b(b^2 - 4ac) \neq 0$, $a(27ab^2 + 4(2a - c)^3) > 0$ and $b^2 - 4ac < 0$. This phase portrait corresponds to the values $a = 1$, $b = 1$ and $c = 3$. This phase portrait has 3 canonical regions.

2. Notations and basic results

In this section we recall the basic definitions, notations and results that we will need for the analysis of the local phase portraits of the finite and infinite singular points of the polynomial Hamiltonian systems (2), and for doing their phase portraits in the Poincaré disc.

We denote by $\mathcal{P}_3(\mathbb{R}^2)$ the set of polynomial vector fields in $\mathbb{R}^2$ of the form $\mathcal{X}(x, y) = (P(x, y), Q(x, y))$ where P and Q are real polynomials in the variables x and y such that the maximal degree of P and Q is 3.

2.1. **Poincaré compactification.** In this subsection we provide a brief summary of the Poincaré compactification of a polynomial vector field $\mathcal{X}$ of degree 3, presenting the formulas that we need for studying the phase portraits of system (2). For the proof of these formulas and all the details on the Poincaré compactification see for instance Chapter 5 of [8].

The *Poincaré disc* is the image of the north hemispheri of the *Poincaré sphere* $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$ into the plane $y_3 = 0$

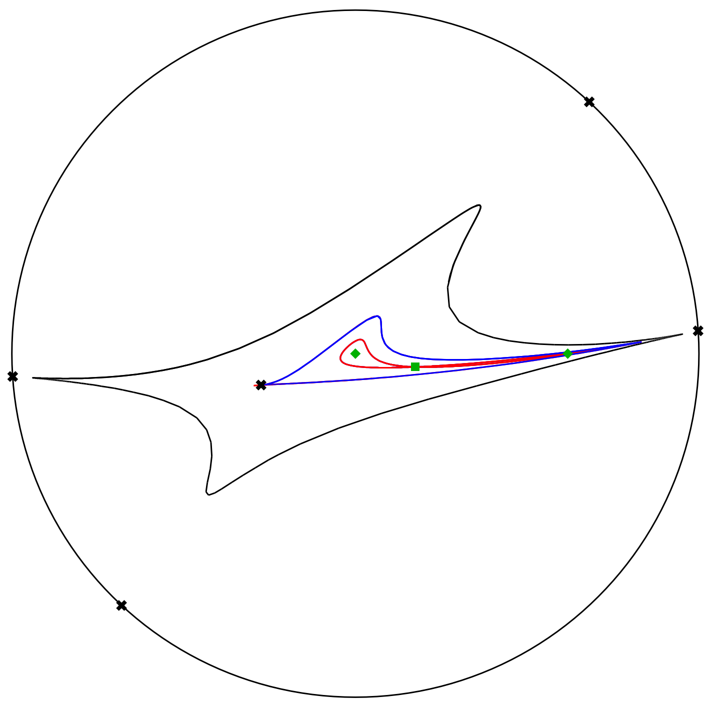

FIGURE 3. Phase portrait of system (2) when $ab(b^2 - 4ac) \neq 0$, $27ab^2 + 4(2a - c)^3 = 0$ and $b^2 - 4ac > 0$. This phase portrait corresponds to the values $a = -1/2$, $b = 8$ and $c = -7$. It has two centers, one saddle and one cusp. One of the centers is difficult to see, but it is clear where it is. We do not draw a periodic orbit in each canonical region having by inner boundary a center because there is no sufficient space. This phase portrait has 4 canonical regions.

under the projection $(y_1, y_2, y_3) \mapsto (y_1, y_2)$. The interior of the Poincaré disc is diffeomorphic to the plane $\mathbb{R}^2$ where we have a copy $\mathcal{X}'$ of our polynomial vector field $\mathcal{X}$ through the mentioned diffeomorphism, and the boundary of this disc $\mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$. There is a unique analytic extension of the vector field $\mathcal{X}'$ to the Poincaré disc called the *Poincaré compactification* $p(\mathcal{X})$ of $\mathcal{X}$. The infinity $\mathbb{S}^1$ is invariant under the flow of $p(\mathcal{X})$.

On the sphere $\mathbb{S}^2$ we take for $i = 1, 2, 3$ the six local charts $U_i = \{y_2 \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{y_2 \in \mathbb{S}^2 : y_i < 0\}$. We denote by (u, v) the value of $F_i(y)$ or $G_i(y)$ for any $i = 1, 2, 3$ (thus (u, v) means different coordinates in the

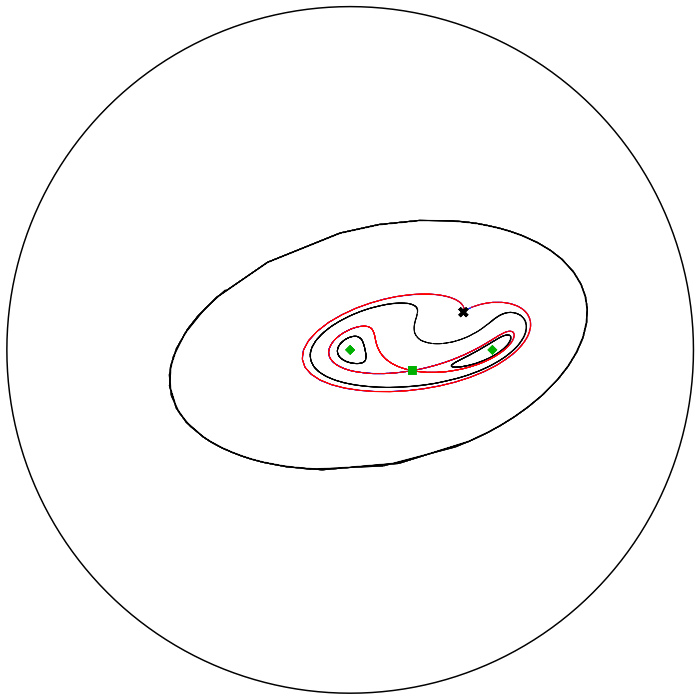

FIGURE 4. Phase portrait of system (2) when $ab(b^2-4ac) \neq 0$, $27ab^2+4(2a-c)^3 = 0$ and $b^2-4ac < 0$. This phase portrait corresponds to the values $a=-1$, $b=2$ and $c=-5$. This phase portrait has 4 canonical regions.

distinct local charts). Easy computations give for $p(\mathcal{X})$ the next expressions:

$$v^3\Delta(u,v)\left(Q\left(\frac{1}{v},\frac{u}{v}\right)-uP\left(\frac{1}{v},\frac{u}{v}\right),-vP\left(\frac{1}{v},\frac{u}{v}\right)\right) \quad \text{in } U_1, \tag{4}$$

$$v^3\Delta(u,v)\left(P\left(\frac{u}{v},\frac{1}{v}\right)-uQ\left(\frac{u}{v},\frac{1}{v}\right),-vQ\left(\frac{u}{v},\frac{1}{v}\right)\right) \quad \text{in } U_2, \tag{5}$$

$$v^3\Delta(u,v)\left(P\left(u,v\right),Q\left(u,v\right)\right) \quad \text{in } U_3,$$

where $\Delta(u,v) = \left(u^2+v^2+1\right)^{-\frac{3}{2}}$.

The expression for V_i is the same as that for U_i except for a change of sign. In the local charts with subindices $i=1,2$, $v=0$ always denotes the points of $\mathbb{S}^1$, i.e. the points of the infinity. In what follows we shall omit the factor $\Delta(z)$ doing a convenient scaling of the vector field $p(\mathcal{X})$. Thus we have a polynomial vector field for $p(\mathcal{X})$ in every local chart.

The singular points of $p(\mathcal{X})$ which are in the interior of the Poincaré disc are called the *finite* singular points, which correspond with the singular

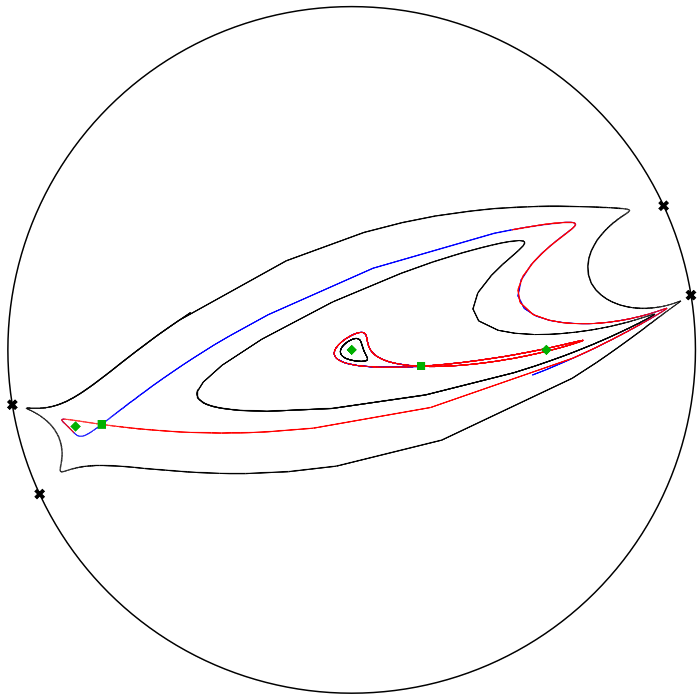

FIGURE 5. Phase portrait of system (2) when $b(b^2 - 4ac) \neq 0$, $a(27ab^2 + 4(2a - c)^3) < 0$ and $b^2 - 4ac > 0$. This phase portrait corresponds to the values $a = -3/5$, $b = 5$ and $c = -8$, and it has three centers, and two saddles as finite singular points. One of the centers is difficult to see in the figure, but it is clear where it is. We do not draw a periodic orbit in the canonical region having by inner boundary the left center because there is no sufficient space. This phase portrait has 5 canonical regions.

points of $\mathcal{X}$, and the singular points of $p(\mathcal{X})$ which are in $\mathbb{S}^1$ are called the *infinite* singular points of $\mathcal{X}$.

We note that studying the infinite singular points of the local chart U_1, we obtain also the ones of the local chart V_1, and only remains to see if the origin of the local chart U_2, and consequently the origin of the local chart V_2, are infinite singular points.

2.2. **Singular points.** Let $\mathcal{X}_H$ be the vector field given in (3) with $a^2+b^2+c^2 \neq 0$, and let $p(\mathcal{X}_H)$ be its Poincaré compactification. Let q be a singular point of $p(\mathcal{X}_H)$.

If some of the two eigenvalues λ_1 and λ_2 of the linear part of the vector field $p(\mathcal{X}_H)$ at the singular point q is not zero, then this singular is *elementary*.

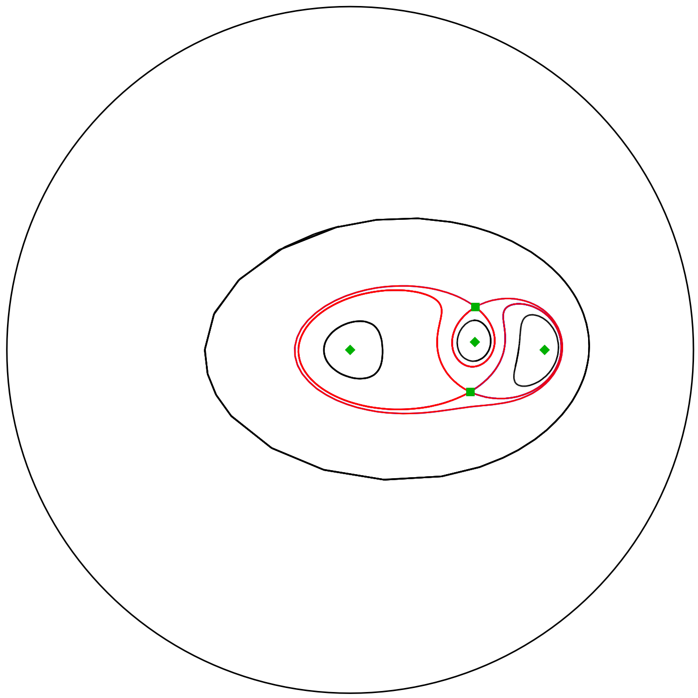

FIGURE 6. Phase portrait of system (2) when $b(27ab^2 + 4(2a - c)^3) < 0$ and $b^2 - 4ac < 0$. This phase portrait corresponds to the values $a = -3/5$, $b = 1/10$ and $c = -2$. We do not draw a periodic orbit in the canonical region between the curves having an eight shape because there is no sufficient space. This phase portrait has 5 canonical regions.

The local phase portrait of a finite elementary singular point q with both eigenvalues non-zero for a Hamiltonian system is well known, it is a saddle if $\lambda_1\lambda_2 \neq 0$ (which in fact only can be $\lambda_1\lambda_2 < 0$), and a center if $\text{Re}\lambda_1 = \text{Re}\lambda_2 = 0$, for more details see for instance [2]. We recall that the flow of a Hamiltonian system in the plane preserves the area, so their finite singular points cannot have parabolic and elliptic sectors, only hyperbolic sectors, or they must be centers.

The local phase portrait of an infinite elementary singular point q with both eigenvalues non-zero can studied using, for instance, Theorem 2.15 of [8].

Let q be an elementary singular point with an eigenvalue zero. Then q is called a *semi-hyperbolic* singular point. The local phase portrait of a semi-hyperbolic singular point of a Hamiltonian system only can be a saddle, see again for exemple [2].

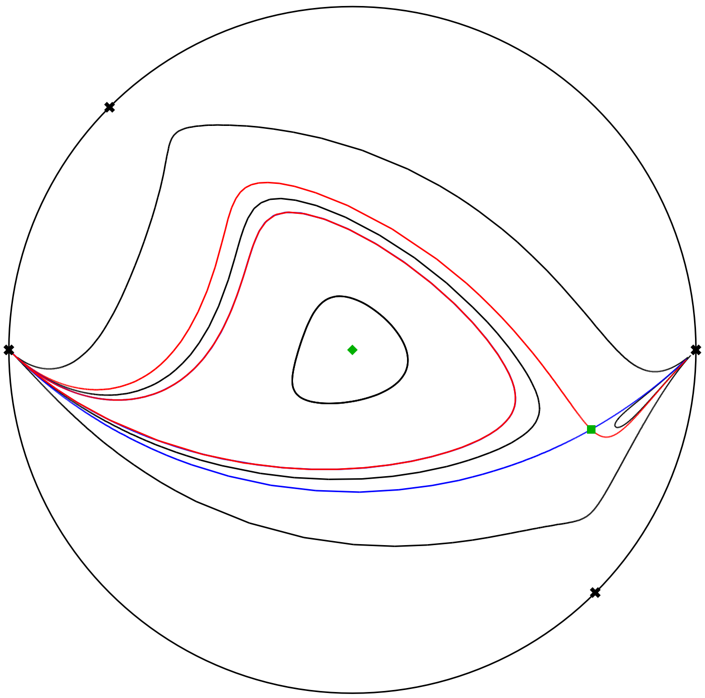

FIGURE 7. Phase portrait of system (2) when $a = 0$ and $bc \neq 0$. This phase portrait corresponds to the values $a = 0$, $b = 1$ and $c = 1$. This phase portrait has 5 canonical regions.

The local phase portrait of an infinite semi-hyperbolic singular point q can described using, for instance, Theorem 2.19 of [8].

When both eigenvalues are zero but the linear part of $p(\mathcal{X}_H)$ at the singular point q is not the zero matrix, we say that q is a *nilpotent* singular point. The local phase portrait of a nilpotent singular point can be studied using Theorem 3.5 of [8]. If q is nilpotent and finite, then it only can be a saddle, a cusp, or a center.

Finally, if the Jacobian matrix of $p(\mathcal{X}_H)$ at the singular point q is identically zero, and q is isolated inside the set of all singular points, then we say that q is a *linearly zero* singular point. The study of the local phase portraits of such singular points needs special changes of variables called blow-ups, see for more details Chapter 3 of [8], or [1].

2.3. **Local phase portraits of the singular points.** We want to classify in the Poincaré disc, modulo topological equivalence, the phase portraits of $p(\mathcal{X}_H)$ being $\mathcal{X} - H$ the polynomial Hamiltonian vector fields given in (3). For doing that we must start by classifying the local phase portraits at all

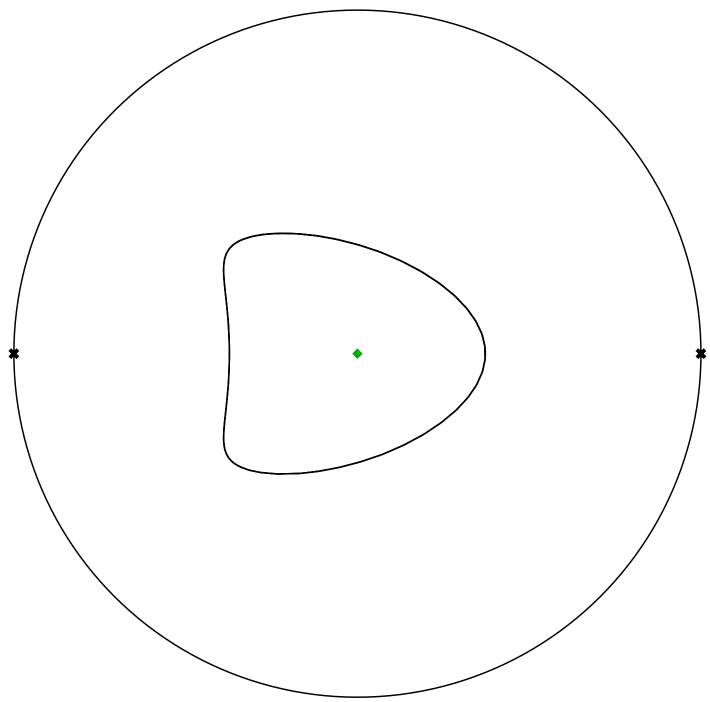

FIGURE 8. Phase portrait of system (2) when $a = b = 0$ and $c \neq 0$. This phase portrait corresponds to the values $a = 0$, $b = 0$ and $c = 1$. This phase portrait has 1 canonical region.

finite and infinite singular points of $p(\mathcal{X}_H)$ in the Poincaré disc. This will be made by using the techniques described in subsection 2.2.

2.4. **Phase portraits in the Poincaré disc.** In this subsection we shall see how to characterize the phase portraits of $p(\mathcal{X}_H)$ in the Poincaré disc for the polynomial Hamiltonian vector fields $\mathcal{X}_H$ of (3).

A *separatrix* of $p(\mathcal{X}_H)$ is an orbit which is either a singular point, or a trajectory which lies in the boundary of a hyperbolic sector of a finite or infinite singular point, or any orbit contained in the infinity $\mathbb{S}^1$. Neumann [14] proved that the set formed by all separatrices of $p(\mathcal{X}_H)$, denoted by $S(p(\mathcal{X}_H))$ is closed.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}_H))$ are called canonical regions of $\mathcal{X}_H$ or of $p(\mathcal{X}_H)$. A *separatrix configuration* is the union of $S(p(\mathcal{X}_H))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}_{H_1}))$ and $S(p(\mathcal{X}_{H_2}))$ are *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}_{H_1}))$ into the trajectories of $S(p(\mathcal{X}_{H_2}))$. The following

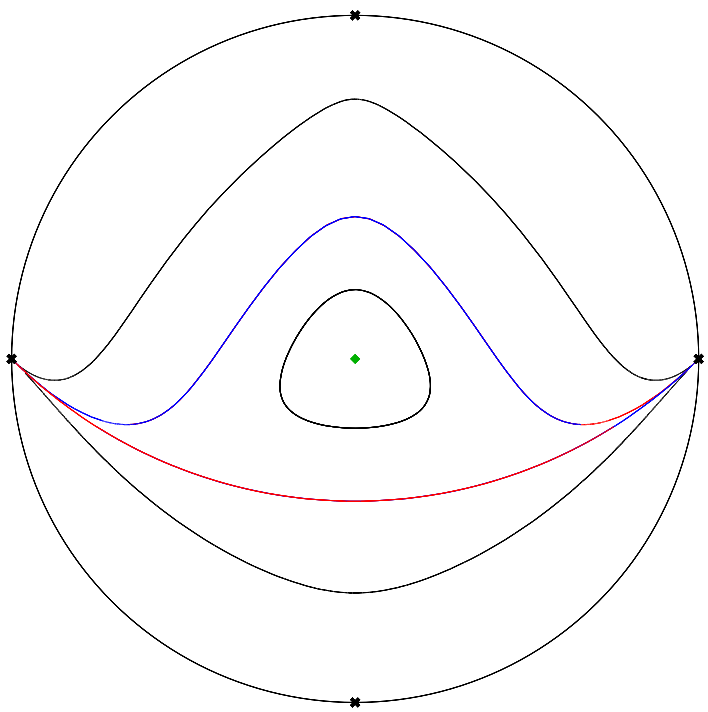

FIGURE 9. Phase portrait of system (2) when $a = c = 0$ and $b \neq 0$. This phase portrait corresponds to the values $a = 0$, $b = 1$ and $c = 0$. This phase portrait has 3 canonical regions.

result is due to Markus [13], Neumann [14] and Peixoto [15], that find it independently.

Theorem 2. *The phase portraits in the Poincaré disc of two compactified polynomial differential systems $p(\mathcal{X}_{H_1})$ and $p(\mathcal{X}_{H_2})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}_{H_1}))$ and $S(p(\mathcal{X}_{H_2}))$ are topologically equivalent.*

In summary, for drawing the complete phase portrait of a Hamiltonian vector field $\mathcal{X}_H$ in the Poincaré disc we must draw all the separatrices of $p(\mathcal{X}_H)$ plus one orbit in each canonical region.

3. The finite singular points

In the next lemma we characterize the finite singular points of our cubic Hamiltonian system (2).

Lemma 3. *The following statements hold.*

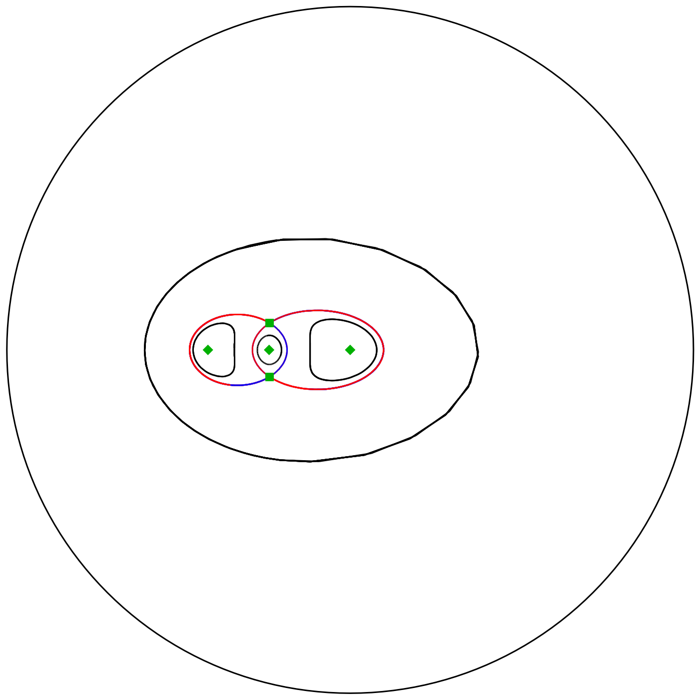

FIGURE 10. Phase portrait of system (2) when $ac \neq 0$, $b = 0$ and $a(c-2a) > 0$. This phase portrait corresponds to the values $a = 1$, $b = 0$ and $c = 3$. This phase portrait has 4 canonical regions.

(a) *If $b(b^2 - 4ac) \neq 0$ and $a(27ab^2 + 4(2a-c)^3) > 0$, system (2) has three finite singular points:*

$$p_1 = (0,0), \quad p_2 = \left(-\frac{1}{a}, 0\right),$$

and the unique real solution q_1 of the system

$$\begin{aligned} -y - (bx + 2cy)(x + ax^2 + bxy + cy^2) &= 0, \\ 1 + 2ax + by &= 0. \end{aligned} \tag{6}$$

Moreover, p_1 and p_2 are centers, and q_1 is a saddle.

(b) *If $ab(b^2 - 4ac) \neq 0$ and $27ab^2 + 4(2a-c)^3 = 0$, system (2) has four finite singular points: p_1, p_2 and the two real solutions q_1 and q_2 of system (6). More precisely,*

$$b = \pm 2\sqrt{\frac{(c-2a)^3}{27a}}, \quad \textit{and} \quad b^2 - 4ac = -\frac{4(8a-c)(a+c)^2}{27a}.$$

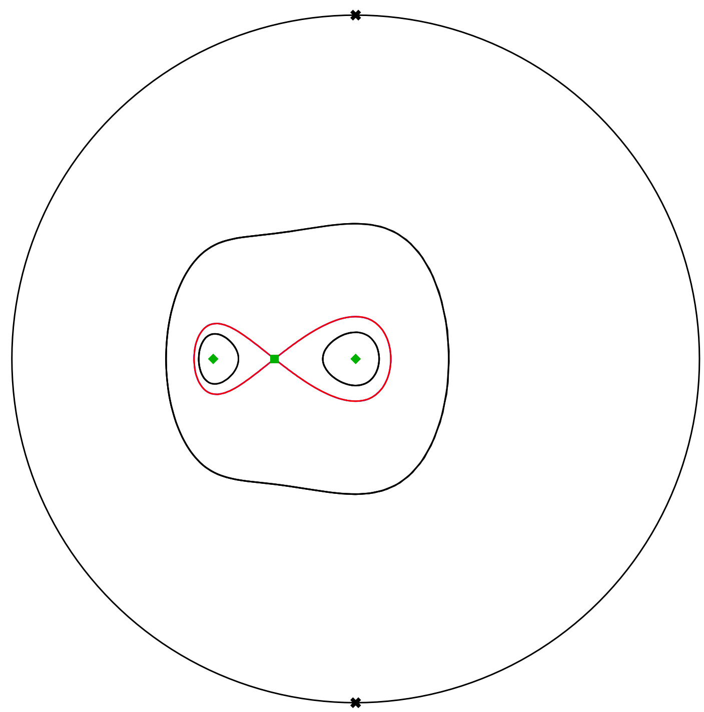

FIGURE 11. Phase portrait of system (2) when $a \neq 0$ and $b = c = 0$. This phase portrait corresponds to the values $a = 1$, $b = 0$ and $c = 0$. This phase portrait has 3 canonical regions.

So $a(c-2a) > 0$ *and* $(8a-c)(a+c) \neq 0$. *Then*

$$q_1 = \left(\frac{3(a-2c)}{2(a+c)^2}, \frac{3\sqrt{3a(c-2a)}}{(a+c)^2} \right),$$

$$q_2 = \left(\frac{3(4a+c)}{2(a+c)(c-8a)}, \frac{3\sqrt{3a(c-2a)^3}}{2(2a-c)(8a-c)(a+c)} \right).$$

Moreover p_1 *and* p_2 *are centers,* q_1 *is a saddle, and* q_2 *is a cusp.*

(c) *If* $b(b^2-4ac) \neq 0$ *and* $a(27ab^2+4(2a-c)^3) < 0$*, system* (2) *has five finite singular points:* p_1*,* p_2 *and the three real solutions* q_1*,* q_2 *and* q_3 *of system* (6)*. Moreover,* p_1*,* p_2 *and a* q_i *are centers, the* q_j *with* $j \neq i$ *are saddles.*

(d) *If* $a = 0$ *and* $bc \neq 0$*, then system* (2) *has two singular points* p_1 *and*

$$q_1 = \left(\frac{b^2+2c^2}{b^2 c}, -\frac{1}{b} \right),$$

being p_1 *a center and* q_1 *a saddle.*

(e) *If* $a = 0$, $bc = 0$ *and* $b^2 + c^2 \neq 0$, *then system* (2) *has only one singular point, the center* $p_1 = (0, 0)$.

(f) *If* $b = 0$ *and* $ac \neq 0$ *the finite singular points of system* (2) *are* p_1, p_2 *and*

$$q_1 = \left(-\frac{1}{2a}, 0\right), \quad q_{4,5} = \left(-\frac{1}{2a}, \pm\frac{1}{2c}\sqrt{\frac{c-2a}{a}}\right),$$

if they exist.

(f.1) *If* $a(c - 2a) > 0$, *then* p_1, p_2 *and* q_1 *are centers, and* q_4 *and* q_5 *saddles.*

(f.2) *If* $a(c-2a) \leq 0$, *then* p_1 *and* p_2 *are centers,* q_1 *is a saddle, and* q_2 *and* q_3 *do not exist.*

(g) *If* $b = c = 0$ *and* $a \neq 0$, *then system* (2) *has three singular points* p_1, p_2 *and* $q_1 = (-1/(2a), 0)$, *being* p_1 *and* p_2 *centers, and* q_1 *a saddle.*

(h) *If* $ab \neq 0$ *and* $b^2 - 4ac = 0$, *then system* (2) *has three finite singular points:*

$$p_1 = (0, 0), \;\; p_2 = \left(-\frac{1}{a}, 0\right), \;\; p_3 = \left(\frac{-8a^2 - b^2}{4a(4a^2 + b^2)}, \frac{-b}{2(4a^2 + b^2)}\right),$$

where p_1 *and* p_2 *are centers, and* p_3 *is a saddle.*

Proof. The singular points of system (2) are the solutions of the system

$$\begin{aligned} -y - (bx + 2cy)(x + ax^2 + bxy + cy^2) &= 0, \\ (1 + 2ax + by)(x + ax^2 + bxy + cy^2) &= 0. \end{aligned}$$

The singular points p_1 and p_2 are obtained solving the system

$$y = 0, \quad x + ax^2 + bxy + cy^2 = 0.$$

For solving system (6) we first eliminate the variable x between the previous two equations we get

$$-a\big(b + (8a^2 + 3b^2 - 4ac)y + 3b(b^2 - 4ac)y^2 + (b^2 - 4ac)^2 y^3\big) = 0,$$

while eliminating the variable y we obtain

$$b^2 + 2c^2 + (-b^2c + 2a(b^2 + 6c^2))x + 6ac(-b^2 + 4ac)x^2 + a(b^2 - 4ac)^2 x^3 = 0.$$

The discriminant of these two cubic equations are

$$\Delta_1 = -64a^7(b^2 - 4ac)^2(27ab^2 + 4(2a - c)^3),$$

and

$$\Delta_2 = -ab^6(b^2 - 4ac)^2(27ab^2 + 4(2a - c)^3),$$

respectively. Hence if $b(b^2 - 4ac)^2 \neq 0$ and $a(27ab^2 + 4(2a - c)^3)$ is positive, zero, or negative, then it follows that system (6) has a unique solution, two solutions, or three solutions, respectively. See for more details [9]. Adding to these solutions the two solutions p_1 and p_2 we obtain the singular points which appear in statements (a), (b) and (c) of the lemma, respectively.

The singular points of the remaining statements of the lemma follow easily by direct computations.

The local phase portraits of the singular points which appear in all the statements of the lemma have been studied easily using the results of subsection 2.2. □

4. The infinite singular points

Let q be an infinite singular point of some polynomial vector field and let h be a hyperbolic sector of q. We say that h is *degenerated* if its two separatrices are contained in the infinity (i.e. in $\mathbb{S}^1$ using the notation of subsection 2.1), otherwise h is called *non-degenerated*.

In the following lemma we characterize the infinite singular points of our cubic Hamiltonian system (2), we are using the notations of subsection 2.1.

Lemma 4. *The following statements hold.*

(a) *If $ac \neq 0$ and $b^2 - 4ac > 0$, then there are two pairs of infinite singular points both contained in the local charts $U_1 \cup V_1$. Moreover the local phase portrait on the Poincaré sphere at any of these singular points is formed by two degenerate hyperbolic sectors (see for instance Figure 1).*

(b) *If $c \neq 0$ and $b^2 - 4ac = 0$, then there is one pair of infinite singular points contained in the local charts $U_1 \cup V_1$, and the local phase portrait on the Poincaré sphere at these singular points is formed by two degenerate hyperbolic sectors (see for instance Figure 8).*

(c) *If $c \neq 0$ and $b^2 - 4ac < 0$, then there are no infinite singular points. So the infinity is a periodic orbit.*

(d) *If $c = 0$ and $ab \neq 0$, then there are two pairs of infinite singular points, one contained in the local charts $U_1 \cup V_1$, and the other pair is at the origins of the local charts U_2 and V_2. Moreover the local phase portrait on the Poincaré sphere at any of these singular points is formed by two degenerate hyperbolic sectors (see Figure 1 for a qualitative topologically equivalent infinity).*

(e) *If $c = a = 0$ and $b \neq 0$, then there are two pairs of infinite singular points, one pair at the origins of the local charts U_1 and V_1, and the local phase portrait of the singular points of this pair on the Poincaré sphere is formed by two non-degenerate hyperbolic sectors separated by two parabolic sectors, these two parabolic sectors contain the infinity. The other pair is at the origins of the local charts U_2 and V_2, and the local phase portrait of the singular points of this pair on the Poincaré sphere is formed by two degenerate hyperbolic sectors (see Figure 9).*

(f) *If $c = b = 0$ and $a \neq 0$, then there is only one pair of infinite singular points the origins of the local charts U_2 and V_2. The local phase* *portrait on the Poincaré sphere at these singular points is formed by two degenerate hyperbolic sectors (see Figure 11).*

(g) *If $a = 0$ and $bc \neq 0$, then there are two pairs of infinite singular points both contained in the local charts $U_1 \cup V_1$. The local phase portrait on the Poincaré sphere at the singular points of one of these pairs is formed by two degenerate hyperbolic sectors. The local phase portrait on the Poincaré sphere at the singular points of the other pair is formed by one non-degenerate hyperbolic sector and one elliptic sector separated by two parabolic sectors, these two parabolic sectors contain the infinity (see Figure 7).*

Proof. From (4) the Hamiltonian system (2) in the local chart U_1 becomes

$$\begin{aligned} \dot{u} = \ & 2a^2 + 4abu + 3av + 2(b^2 + 2ac)u^2 + 3buv + v^2 + 4bcu^3 + \\ & 3cu^2v + 2c^2u^4 + u^2v^2, \\ \dot{v} = \ & v(ab + (b^2 + 2ac)u + bv + 3bcu^2 + 2cuv + 2c^2u^3 + uv^2). \end{aligned} \tag{7}$$

Therefore, if $c \neq 0$ and $b^2 - 4ac > 0$ this system has two infinite singular points in this chart, namely

$$\left(\frac{-b \pm \sqrt{b^2 - 4ac}}{2c}, 0 \right).$$

If $c \neq 0$ and $b^2 - 4ac = 0$ this system has only one infinite singular point

$$\left(-\frac{b}{2c}, 0 \right).$$

in the chart U_1. If $c = 0$ and $b \neq 0$, again the system has only one infinite singular point

$$\left(-\frac{a}{b}, 0 \right),$$

in U_1. Finally if $b^2 - 4ac < 0$, then there are no infinite singular points in U_1.

The charts U_1 and its symmetric V_1 cover all the infinity except the origins of the local charts U_2 and V_2.

From (5) the Hamiltonian system (2) in the local chart U_2 is

$$\begin{aligned} \dot{u} = \ & -2c^2 - 4bcu - 2(b^2 + 2ac)u^2 - 3cuv - v^2 - 4abu^3 - \\ & 3bu^2v - 2a^2u^4 - 3au^3v - u^2v^2, \\ \dot{v} = \ & v(-bc - (b^2 + 2ac)u - cv - 3abu^2 - 2buv - 2a^2u^3 - \\ & 3au^2v - uv^2). \end{aligned} \tag{8}$$

So the origin of the chart U_2 is an infinite singular point if and only if $c = 0$.

All the infinite singular points have linear part identically zero. So in order to know its local phase portrait we must do blow ups, see subsection 2.2.

From the different statements of the lemma we see that there are only two different topological kinds of local phase portraits at the infinite singular points:

(i) Either the local phase portrait on the Poincaré sphere at an infinite singular point is formed by two degenerate hyperbolic sectors.

(ii) or the local phase portrait on the Poincaré sphere at an infinite singular point is formed by one hyperbolic and one elliptic sectors separated by two parabolic ones, these two parabolic sectors contain the infinity;

(iii) or the local phase portrait on the Poincaré sphere at an infinite singular point is formed by two non-degenerate hyperbolic sectors separated by two parabolic sectors, these two parabolic sectors contain the infinity.

We only prove using blow ups one local phase portrait of type (i), the remainder local phase portraits of this type which appear in the statements of the lemma are proved in a similar way.

For proving one local phase portrait of type (i) we have chosen the origin of the local chart U_2 when $a = c = 0$ and $b \neq 0$ (see Figure 9), thus we must study the local phase portrait at the origin of the system

$$\begin{aligned} \dot{u} &= -2b^2u^2 - v^2 - 3bu^2v - u^2v^2, \\ \dot{v} &= -b^2uv - 2buv^2 - uv^3. \end{aligned} \tag{9}$$

Since the linear part at the origin is identically zero, for studying its local phase portrait we do a directional blow up in the direction of the v-axis, i.e. $(u, v) \mapsto (u, w)$ with $w = v/u$, then system (9) in the new variables (u, w), after eliminating a common factor u between the components $(\dot{u}, \dot{w})$ doing a scaling of the independent variable, becomes

$$\begin{aligned} \dot{u} &= -u(2b^2 + 3buw + w^2 + u^2w^2), \\ \dot{w} &= w(b^2 + buw + w^2). \end{aligned} \tag{10}$$

We note that this change of variables has blow up the origin of system (9) to the whole w-axis. The unique singular point of system (10) on the w-axis is the origin which is a saddle. Going back through the changes of coordinates, the scaling and the blow up, and taking into account that for system (9) we have $\dot{u}|_{u=0} = -v^2$, the local phase portrait at the origin is either the one that we want to prove, i.e. the one described in (i), or it has some hyperbolic sector with both separatrices tangent to the v-axis. In order to eliminate this second possibility we do another directional blow up but now in the direction of the v-axis, i.e. $(u, v) \mapsto (w, v)$ with $w = u/v$, then system (9) in the new variables (w, v), after eliminating a common factor w between the components $(\dot{w}, \dot{v})$ doing a scaling of the independent variable, writes

$$\begin{aligned} \dot{w} &= -2b^2 - w^2 + b^2wv - 3bw^2v + 2bwv^2 + wv^3 - w^4v^2, \\ \dot{v} &= -v^2(b + v)^2. \end{aligned}$$

This system has no singular points on the w–axis. This implies when we go back through the changes of variables, the scaling and the blow up, that the mentioned second possibility does not occur. This completes the proof of the local phase portrait (i).

The local phase portrait of type (ii) only appears in statement (g) when $a = 0$ and $bc \neq 0$, and at the origin of the local chart U_1 (see Figure 7), thus we must study the local phase portrait at the origin of the system

$$\begin{aligned} \dot u &= 2b^2u^2 + 3buv + v^2 + 4bcu^3 + 3cu^2v + 2c^2u^4 + u^2v^2, \\ \dot v &= b^2uv + bv^2 + 3bcu^2v + 2cuv^2 + 2c^2u^3v + uv^3. \end{aligned} \tag{11}$$

Since the origin is a linearly zero singular point we do the directional blow up $(u, v) \mapsto (u, w)$ with $w = v/u$, then system (11) in the new variables (u, w), after eliminating a common factor u between the components $(\dot u, \dot w)$ doing a scaling of the independent variable, becomes

$$\begin{aligned} \dot u &= u(2b^2 + 4bcu + 3bw + 2c^2u^2 + 3cuw + w^2 + u^2w^2), \\ \dot w &= -w(b + w)(b + cu + w). \end{aligned} \tag{12}$$

We note that this change of variables has blow up the origin of system (11) to the whole w-axis. There are two singular points of system (12) on the w-axis, the origin which is a saddle and the point $(-b, 0)$ which is linearly zero.

For studying the local phase portrait of the singular point $(-b, 0)$, first we translate it to the origin of coordinates doing the change $(u, w) \mapsto (u, W = w + b)$. So system (12) in the variables (u, W) is

$$\begin{aligned} \dot u &= u(bcu + bW + (b^2 + 2c^2)u^2 + 3cuW + W^2 - 2bu^2W + u^2W^2), \\ \dot W &= -W(-b + W)(cu + W). \end{aligned} \tag{13}$$

We do the directional blow up $(u, W) \mapsto (u, z)$ with $z = W/u$, then system (13) in the new variables (u, z), after eliminating a common factor u^2 between the components $(\dot u, \dot W)$ doing a scaling of the independent variable, becomes

$$\begin{aligned} \dot u &= bc + (b^2 + 2c^2)u + bz + 3cuz - 2bu^2z + uz^2 + u^3z^2, \\ \dot z &= -z(b^2 + 2c^2 + 4cz - 2buz + 2z^2 + u^2z^2). \end{aligned}$$

This system has no singular points on the z-axis, but it has a contact point on this axis at the point $(0, -c)$, more precisely in that point one orbit is tangent to the z-axis and does not cross it. Going back through the changes of coordinates, the scaling and the blow up, we get that the local phase portrait at the origin of system (13) has one elliptic sector, one hyperbolic sector, in opposite quadrats of the plane (u, W) separated by two parabolic sectors. Consequently this is the local phase portrait at the singular point $(-b, 0)$ of system (12). Now going back through the change of coordinates from system (12) to system (11) we obtain the local phase portrait (ii).

The local phase portrait of type (iii) only appears in statement (e) when $c = a = 0$ and $b \neq 0$, and at the origin of the local chart U_1 (see Figure 9), thus we must study the local phase portrait at the origin of the system

$$\begin{aligned} \dot{u} &= 2b^2u^2 + 3buv + v^2 + u^2v^2, \\ \dot{v} &= b^2uv + bv^2 + uv^3. \end{aligned} \tag{14}$$

Since the origin is a linearly zero singular point we do the directional blow up $(u,v) \mapsto (u,w)$ with $w = v/u$, then system (14) in the new variables (u,w), after eliminating a common factor u between the components $(\dot{u},\dot{w})$ doing a scaling of the independent variable, becomes

$$\begin{aligned} \dot{u} &= u(2b^2 + 3bw + w^2 + u^2w^2), \\ \dot{w} &= -w(b+w)^2. \end{aligned} \tag{15}$$

We note that this change of variables has blow up the origin of system (14) to the whole w-axis. There are two singular points of system (15) on the w-axis, the origin which is a saddle and the point $(-b,0)$ which is linearly zero.

For studying the local phase portrait of the singular point $(-b,0)$, first we translate it to the origin of coordinates doing the change $(u,w) \mapsto (u, W = w+b)$. So system (12) in the variables (u,W) is

$$\begin{aligned} \dot{u} &= u(bW + b^2u^2 + W^2 - 2bu^2W + u^2W^2), \\ \dot{W} &= -W^2(W-b). \end{aligned} \tag{16}$$

We do the directional blow up $(u,W) \mapsto (u,z)$ with $z = W/u$, then system (16) in the new variables (u,z), after eliminating a common factor u^2 between the components $(\dot{u},\dot{W})$ doing a scaling of the independent variable, becomes

$$\begin{aligned} \dot{u} &= b^2u + bz - 2bu^2z + uz^2 + u^3z^2, \\ \dot{z} &= -z(b^2 - 2buz + 2z^2 + u^2z^2). \end{aligned}$$

The unique singular point of this system on the z-axis is the origin, which is a saddle. Now we assume that $b > 0$, the proof in the case $b < 0$ is analogous. This saddle has the two unstable separatrices on the u-axis, and the two stable ones tangent to a straight line through the origin which negative slope. Going back through the changes of coordinates, the scaling and the blow up, we get that the local phase portrait at the origin of system (16), which in $W \geq 0$ has un unstable parabolic sector with the orbits tangent to the W-axis with the exception of the two ones on the u-axis, and in $W \leq 0$ it has two hyperbolic sectors with both separatrices tangent to the u-axis, separated by one parabolic sector. Therefore this is the local phase portrait at the singular point $(-b,0)$ of system (15). Now going back through the change of coordinates from system (15) to system (14) we obtain the local phase portrait (iii). □

5. Global phase portraits

Taking into account the results on the finite and infinite singular points given in sections 3 and 4 respectively, and drawing the curves $H(x,y) = h$ passing through the finite singular points, we shall obtain the different phase portraits of the Hamiltonian system (2) that we describe in what follows. We recall that the figures of the phase portraits in the Poincaré disc are drawn with the program P4 described in Chapters 9 and 10 of [8], in this way as we have mention in section 1 we have the analytical qualitative study of the phase portraits in the Poincaré disc that we have done and also its quantitative study for the considered values of the parameters.

We separate the proof of Theorem 1 in nine cases.

Case 1: Assume that $b(b^2 - 4ac) \neq 0$ and $a(27ab^2 + 4(2a - c)^3) > 0$. Then, by Lemma 3(a), system (2) has three finite singular points, two centers and one saddle. We need to distinguish two subcases according the infinite singular points. First the subcase $b^2 - 4ac > 0$. If $c \neq 0$ then, by Lemma 4(a) the system has two pairs of infinite singular points in the local charts $U_1 \cup V_1$, and each of these infinite singular points is formed by two degenerate hyperbolic sectors. If $c = 0$ then, by Lemma 4(d), the system has two pairs of infinite singular points one pair in $U_1 \cup V_1$ and the other at the origins of the charts U_2 and V_2, and again all the local phase portraits at these infinite singular points are formed by two degenerate hyperbolic sectors. Taking into account the shape of the level curves of the Hamiltonian H on the finite saddle, we get that in both cases the phase portrait of system (2) is topologically equivalent to the one of Figure 1. Second, the subcase $b^2 - 4ac < 0$. By Lemma 4(c) the system has no infinite singular points. Using the same arguments of the previous case, we obtain that the phase portrait of the system now is topologically equivalent to the one of Figure 2.

Case 2: Assume that $ab(b^2 - 4ac) \neq 0$ and $27ab^2 + 4(2a - c)^3 = 0$. Then, by Lemma 3(b), system (2) has four finite singular points, two centers, one saddle and one cusp. The singular points at infinity are the same than in Case 1. Therefore in this case the phase portrait of the system is topologically equivalent to the one of Figure 3 if $b^2 - 4ac > 0$, and to the one of Figure 4 if $b^2 - 4ac < 0$.

Case 3: Assume that $b(b^2 - 4ac) \neq 0$ and $a(27ab^2 + 4(2a - c)^3) < 0$. Then by Lemma 3(c) system (2) has five finite singular points, three centers and two saddles. We note that the values of the Hamiltonian H are different in both saddles. Again we need to distinguish two subcases according the infinite singular points. First the subcase $b^2 - 4ac > 0$, then $c \neq 0$ otherwise $a(27ab^2 + 4(2a - c)^3) > 0$. So the behavior at the infinite singular points is as in the first subcase of Case 1. Hence the phase portrait of the system is topologically equivalent to the one of Figure 5. Second the subcase $b^2 - 4ac < 0$ with no infinite singular points, then the phase portrait of the system is topologically equivalent to the one of Figure 6.

Case 4: Assume that $a = 0$ and $bc \neq 0$. Then, by Lemma 3(d), system (2) has two finite singular points, one center and one saddle. By Lemma 4(f) the system has two pairs of infinite singular points in the local charts $U_1 \cup V_1$, the local phase portrait of the singular points of one of these pairs is formed by two degenerate hyperbolic sectors, and the local phase portrait at the singular points of the other pair is formed by one non-degenerate hyperbolic sector and one elliptic sector separated by two parabolic sectors, these two parabolic sectors contain the infinity. So, first evaluating the level curve γ of the Hamiltonian H on the saddle, and after the level curve near the saddle but inside the region limited by γ encircling the center, we obtain that the phase portrait of the system is topologically equivalent to the one of Figure 7.

Case 5: Assume that $a = 0$, $bc = 0$ and $b^2 + c^2 \neq 0$. Then, by Lemma 3(e), system (2) has a unique finite singular point, a center at the origin of coordinates. If $b = 0$, by Lemma 4(b), the system has a unique pair of infinite singular points in $U_1 \cup V_1$ and their local phase portraits are formed by two degenerate hyperbolic sectors. So the phase portrait of the system is topologically equivalent to the one of Figure 8. If $c = 0$, by Lemma 4(e), we have two pairs of infinite singular points, one pair at the origins of the local charts U_1 and V_1, and the local phase portrait of the singular points of this pair is formed by two non-degenerate hyperbolic sectors separated by two parabolic sectors, these two parabolic sectors contain the infinity. The other pair is at the origins of the local charts U_2 and V_2, and the local phase portrait of the singular points of this pair is formed by two degenerate hyperbolic sectors. Consequently the phase portrait of the system is topologically equivalent to the one of Figure 9.

Case 6: Assume that $ac \neq 0$, $b = 0$ and $a(c - 2a) > 0$. Then, by Lemma3(f) system (2) has five finite singular points, three centers and two saddles. We note that now the two saddles belongs to the same level curve of the Hamiltonian H. Since $a(c - 2a) > 0$ it follows that $ac > 0$. Then, by Lemma 4(c) there are no infinite singular points, and the phase portrait of the system is topologically equivalent to the one of Figure 10.

Case 7: Assume that $ac \neq 0$, $b = 0$ and $a(c - 2a) \leq 0$. Then, by Lemma3(f) system (2) has three finite singular points, two centers and one saddle. If $ac > 0$, by Lemma 4(c) there are no infinite singular points, and the phase portrait is topologically equivalent to the one of Figure 2). If $ac < 0$ then, by Lemma 4(a) there is a pair of infinite singular points in $U_1 \cup V_1$, and the local phase portrait at each of these infinite singular points is formed by two degenerate hyperbolic sectors. Hence the phase portrait of system (2) is topologically equivalent to the one Figure 1.

Case 8: Assume that $a \neq 0$ and $b = c = 0$. Then, by Lemma 3(g) system (2) has three finite singular points, two centers and one saddle. By Lemma 4f the system has only one pair of infinite singular points, the origins of the local charts U_2 and V_2. The local phase portrait at these infinite singular points is formed by two degenerate hyperbolic sectors. Consequently the phase portrait of the system is topologically equivalent to the one of Figure 11.

Case 9: Assume that $ab \neq 0$ and $b^2 - 4ac = 0$. Then, by Lemma 3(h) system (2) has three finite singular points, two centers and one saddle. By Lemma 4(c) the system has a unique pair of infinite singular points in $U_1 \cup V_1$, and the local phase portraits at these infinite singular points are formed by two degenerate hyperbolic sectors. So the phase portrait of this system is topologically equivalent to the one of Figure 11. This completes the proof of Theorem 1.

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