Facultat de Matemàtiques
Universitat de Barcelona
Dept. de Matemàtica Aplicada i Anàlisi
Bienni 1998-2000

Iteration of certain families of transcendental maps and phase portraits of complex differential equations

Antonio Garijo Real

Advisors: Núria Fagella Rabionet and Xavier Jarque Ribera

To Lourdes

Preface

This thesis is organized in three parts. In Part I we study the iteration of the entire transcendental maps

$$F_{\lambda,m}(z) = \lambda z^m \exp(z), \qquad \text{where } m \geq 2 \text{ and } \lambda \in \mathbb{C}.$$

This part is organized in five chapters. In Chapter 1 we introduce some fundamental concepts of complex dynamics and present the main results. In Chapter 2 we recall some well known tools in complex dynamics which we will use in this part. To deal with the dynamics generated by the iterates of the map $F_{\lambda,m}$, we consider three different points of view. First, we take an *entire transcendental approach*, trying to solve the problems staying within the type of functions that we study. Second we take a *polynomial approach*, that is we study a family of polynomials that approximate (in a sense to be determined) $F_{\lambda,m}$, and try to transfer the obtained polynomial properties to the limit. Third and last, we consider a family of functions of $\mathbb{C}^*$ that relate to $F_{\lambda,m}$ via a surgery procedure. We call this the *surgical approach* and although we do not obtain new information about $F_{\lambda,m}$, it does open several very interesting questions. We devote one chapter to each one of these approaches. Chapters 3, 4 and 5 have a common structure. The first section is an introduction; in the second section we introduce specific tools used in the corresponding chapter, and finally, in the rest of the sections we prove the corresponding results.

In Part II we present a transition from discrete to continuous complex dynamical system. We connect them using Euler's method. This part is organized in one chapter. The first section of Chapter 6 is an introduction where we present this connection. In the second section of Chapter 6 we explain the relation between the relaxed Newton's method and Newton's flow. In the third section we explain some general properties of Newton's method.

In Part III we investigate in the complex first order differential equation

$$\frac{dz}{dt} = f(z), \qquad z \in \mathbb{C}, \quad t \in \mathbb{R},$$

where f is an analytic function in $\mathbb{C}$ except, possibly, at isolated singularities. This part is organized in seven chapters. In Chapter 7 we introduce the equation above and we present the main results. In Chapter 8 we present specific tools which we will use in this part. Chapter 9 is devoted to study the local normal forms of a complex differential equation near a singularity. In Chapter 10 we investigate the phase portrait near periodic orbits or graphs. In Chapter 11 we present an application of Chapter 9 and 10. In Chapter 12 we study the period function of a family of complex differential equations related to $dz/dt = f(z)$. In the final chapter, we include two appendixes where we present alternatives proofs of two results of this part.

ACKNOWLEDGEMENTS. I want to thank my advisors Núria Fagella and Xavier Jarque for introducing me into the subject. I am very grateful for many discussions on the topics of this thesis and for giving important comments to earlier versions.

Then I thank Armengol Gasull for his support in the last three years; in particular, I am very grateful for many discussions, comments and improvements on the third part in this thesis. Moreover, I thank very much Pascale Roesch for explain me the concept of Yoccoz's puzzles. She also explain me how to apply this technique in the polynomial approach. Also, I thank Christian Henriksen for many discussions on the concept of Holomorphic motions, he explains me the main idea to apply this concept to the family $F_{\lambda,m}$. I thank Jordi Villadelprat for his support and comments on the concept of Period function. Finally, I thank Adrien Douady for his support in many steps of this work, including poetry lectures, I also thank him for his useful comments on earlier versions of this thesis.

Contents