Weak periodic solutions of $x\ddot{x}+1=0$ and the Harmonic Balance Method

J D García-Saldaña[1,3] and A Gasull [2]

[1,2] Departament de Matemàtiques, Universitat Autònoma de Barcelona
Edifici C. 08193 Bellaterra, Barcelona. Spain.

E-mail: jgarcias@ucsc.cl, gasull@mat.uab.cat

Abstract. We prove that the differential equation $x\ddot{x}+1=0$ has continuous weak periodic solutions and compute their periods. Then, we use the Harmonic Balance Method until order six to approximate these periods and to illustrate how the accuracy of the method increases with the order. Our computations rely on the Gröbner basis approach.

1. Introduction and main results

The nonlinear differential equation

$$x\ddot{x}+1=0 \tag{1}$$

appears in the modeling of certain phenomena in plasma physics. More concretely, studying a certain electron beam injected into a plasma tube, where the magnetic field is cylindrical and increases towards the axis in inverse proportion to the radius, see [1, 8]. In [11] and [13, Sec. 3.2.2], the author calculates the period of its periodic orbits and use the N-th order Harmonic Balance Method (HBM), for $N=1,2$, to obtain approximations of these periodic solutions and of their corresponding periods. Similarly, the same functions are approximated in [8]. However, strictly speaking, it can be seen that neither equation (1), nor its associated planar system

$$\begin{cases} \dot{x}=y, \\ \dot{y}=-\frac{1}{x}, \end{cases} \tag{2}$$

which is singular at $x=0$, have smooth periodic solutions. For system (2) this is so, because it has no critical points and it is well-known ([9]) that periodic orbits of planar autonomous systems must surround a critical point. For equation (1) it is not difficult to see that any periodic solution $x(t)$ must vanish for some $t^*\in\mathbb{R}$, that is $x(t^*)=0$. Then $\lim_{t\to t^*}\ddot{x}(t)=\infty$ and as a consequence this equation cannot have $\mathcal{C}^2$-periodic solutions.

We start giving two different interpretations of the computations of [8, 11] of the period function. The first one states their results in terms of weak (or generalized) solutions, where in this work a weak solution is a function satisfying the differential equation (1) on an open and dense set, but which is only continuous ($\mathcal{C}^0$) at some isolated points.

[3] Current address: Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción. Casilla 297, Concepción, Chile.