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Bifurcation diagram and stability for a one-parameter family of planar vector fields

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ABSTRACT

We consider the one-parameter family of planar quintic systems, $\dot{x} = y^3 - x^3$, $\dot{y} = -x + my^5$, introduced by A. Bacciotti in 1985. It is known that it has at most one limit cycle and that it can exist only when the parameter m is in $(0.36, 0.6)$. In this paper, using the Bendixson–Dulac theorem, we give a new unified proof of all the previous results. We shrink this interval to $(0.547, 0.6)$ and we prove the hyperbolicity of the limit cycle. Furthermore, we consider the question of the existence of polycycles. The main interest and difficulty for studying this family is that it is not a semi-complete family of rotated vector fields. When the system has a limit cycle, we also determine explicit lower bounds of the basin of attraction of the origin. Finally, we answer an open question about the change of stability of the origin for an extension of the above systems.

1. Introduction and main results

A. Bacciotti, during a conference about the stability of analytic dynamical systems held in Florence in 1985, proposed to study the stability of the origin of the following quintic system

$$\begin{cases} \dot{x} = y^3 - x^3, \\ \dot{y} = -x + my^5, \quad m \in \mathbb{R}. \end{cases} \tag{1}$$

Two years later, Galeotti and Gori in [10] published an extensive study of (1). They proved that system (1) has no limit cycles when $m \in (-\infty, 0.36] \cup [0.6, \infty)$, otherwise, it has at most one. Their proofs are mainly based on the study of the stability of the limit cycles which is controlled by the sign of its characteristic exponent, together with a transformation of the system using a special type of adapted polar coordinates. Their proof of the uniqueness of the limit cycle does not cover its hyperbolicity.

In this paper we refine the above results. To guess which is the actual bifurcation diagram we first did a numerical study, obtaining the following results. It seems that there exists a value $m^* > 0$ such that:

(i) System (1) has no limit cycles if $m \in (-\infty, m^*] \cup [0.6, \infty)$. Moreover, for $m = m^*$ it has a heteroclinic polycycle formed by the separatrices of the two saddle points located at $(\pm m^{-1/4}, \pm m^{-1/4})$.

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