DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **35**, Number **2**, February **2015**

doi:10.3934/dcds.2015.35.669

pp. **669–701**

# BIFURCATION VALUES FOR A FAMILY OF PLANAR VECTOR FIELDS OF DEGREE FIVE

Johanna D. García-Saldaña and Armengol Gasull

Dept. de Matemàtiques, Universitat Autònoma de Barcelona
Edifici C, 08193 Bellaterra, Barcelona, Spain

Hector Giacomini

Laboratoire de Mathématique et Physique Théorique
C.N.R.S. UMR 7350, Faculté des Sciences et Techniques
Université de Tours, Parc de Grandmont, 37200 Tours, France

Abstract. We study the number of limit cycles and the bifurcation diagram in the Poincaré sphere of a one-parameter family of planar differential equations of degree five $\dot{\mathbf{x}} = X_b(\mathbf{x})$ which has been already considered in previous papers. We prove that there is a value $b^* > 0$ such that the limit cycle exists only when $b \in (0, b^*)$ and that it is unique and hyperbolic by using a rational Dulac function. Moreover we provide an interval of length $27/1000$ where $b^*$ lies. As far as we know the tools used to determine this interval are new and are based on the construction of algebraic curves without contact for the flow of the differential equation. These curves are obtained using analytic information about the separatrices of the infinite critical points of the vector field. To prove that the Bendixson–Dulac Theorem works we develop a method for studying whether one-parameter families of polynomials in two variables do not vanish based on the computation of the so called double discriminant.

## 1. Introduction and main results.

**1. Introduction and main results.** Consider the one-parameter family of quintic differential systems

$$\begin{cases} \dot{x} &= y, \\ \dot{y} &= -x + (a - x^2)(y + y^3), \quad a \in \mathbb{R}. \end{cases} \tag{1}$$

Notice that without the term $y^3$, (1) coincides with the famous van der Pol system.

This family was studied in [30] and the authors concluded that it has only two bifurcation values, 0 and $a^*$, and exactly four different global phase portraits on the Poincaré disc. Moreover, they concluded that there exists $a^* \in (0, \sqrt[3]{9\pi^2/16}) \approx (0, 1.77)$, such that the system has limit cycles only when $0 < a < a^*$ and then if the limit cycle exists, is unique and hyperbolic. Later, it was pointed out in [17] that the proof of the uniqueness of the limit cycle had a gap and a new proof was presented.

---

2010 *Mathematics Subject Classification.* Primary: 34C07; Secondary: 34C23, 34C25, 37C27, 13P15.

*Key words and phrases.* Polynomial planar system, uniqueness of limit cycles, bifurcation, phase portrait on the Poincaré sphere, Dulac function, double discriminant.

The first two authors are partially supported by a MCYT-FEDER grant number MTM2008-03437 and by a CIRIT grant number 2009SGR 410. The first author is also supported by the grant AP2009-1189.