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NORMAL FORMS FOR SINGULARITIES OF ONE DIMENSIONAL HOLOMORPHIC VECTOR FIELDS

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Abstract. We study the normal form of the ordinary differential equation $\dot{z} = f(z)$, $z \in \mathbb{C}$, in a neighbourhood of a point $p \in \mathbb{C}$, where f is a one-dimensional holomorphic function in a punctured neighbourhood of p. Our results include all cases except when p is an essential singularity. We treat all the other situations, namely when p is a regular point, a pole or a zero of order n. Our approach is based on a formula that uses the flow associated with the differential equation to search for the change of variables that gives the normal form.

1. Introduction and Main Result

This note provides a proof for the following result:

Theorem 1.1. *Let $f(z)$ be a one-dimensional holomorphic function in a punctured neighbourhood $\mathcal{U} \subset \mathbb{C}$ of a point p. Consider the ordinary differential equation*

$$\frac{dz}{dt} = f(z), \quad z \in \mathcal{U}, \quad t \in \mathbb{C}. \tag{1.1}$$

Then, in a neighbourhood of p, this equation is conformally conjugate, in a neighbourhood of the origin, to

(a) $\dot{z} = 1$, *if* $f(p) \neq 0$,
(b) $\dot{z} = f'(p)z$, *if p is a zero of f of order 1 (i.e. $f'(p) \neq 0$),*
(c) $\dot{z} = z^n/(1 + cz^{n-1})$, *where* $c = \operatorname{Res}(1/f, p)$, *if p is a zero of f of order $n > 1$,*
(d) $\dot{z} = 1/z^n$, *if p is a pole of order n.*

Statements (a), (b) and (c) are well-known, see for instance [1]. To our knowledge statement (d), as it is presented here, is new. Just topological conjugacy between equation (1.1) and $\dot{z} = 1/z^n$ has been proved in [4, 5, 6, 8]. In [3] a similar result is also given. Our approach is based in the one dimensional version of a nice formula, as far as we know, firstly introduced in the unpublished thesis of Pazzi [7] and recently used in [2, 9, 10]. This formula allows to look for the conjugacy between the original differential equation (1.1) and its normal form by using the flow of

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