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On the period function for a family of complex differential equations

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Abstract

We consider planar differential equations of the form $\dot{z} = f(z)g(\bar{z})$ being $f(z)$ and $g(z)$ holomorphic functions and prove that if $g(z)$ is not constant then for any continuum of period orbits the period function has at most one isolated critical period, which is a minimum. Among other implications, the paper extends a well-known result for meromorphic equations, $\dot{z} = h(z)$, that says that any continuum of periodic orbits has a constant period function.

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Keywords: Meromorphic vector fields; Periodic orbit; Period function; Critical period

1. Introduction

Analytic planar autonomous differential equations can be written in complex coordinates as

$$\dot{z} = h(z, \bar{z}), \quad z \in \mathbb{C},$$

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where the dot indicates the derivative with respect the real variable t. In particular, the holomorphic case, i.e. $h(z,\bar{z}) \equiv f(z)$ with f being holomorphic, has been widely studied. Among other properties, normal forms for the singularities, the non-existence of limit cycles or the non-existence of monodromic (stable or unstable) polycycles have been established, see for instance [1,8,9,11,12,15]. The case of f being meromorphic or having essential singularities has been also considered in some of the quoted papers.

This paper deals with the following family of planar vector fields:

$$\frac{dz}{dt} = f(z)g(\bar{z}), \tag{1}$$

where $f(z)$ and $g(z)$ are holomorphic maps, and $t \in \mathbb{R}$. Notice that it includes the holomorphic case. It is not difficult to see that any phase portrait of (1) coincides with the phase portrait of $\dot{z} = h(z)$, for some h meromorphic. This is true because by making the re-parameterization $ds/dt = |g(\bar{z})|^2$, Eq. (1) can be written as

$$\frac{dz}{ds} = \frac{f(z)}{\bar{g}(z)}, \tag{2}$$

which is of the form $\dot{z} = h(z)$, being $h = f/\bar{g}$, where $\bar{g}(z) := \overline{g(\bar{z})}$. Moreover by using the new re-parameterization $du/dt = |f(z)|^2$, Eq. (1) can be written as

$$\frac{dz}{du} = \frac{g(\bar{z})}{\bar{f}(\bar{z})}, \tag{3}$$

which is a Hamiltonian system wherever it is defined. In other words, outside some points, Eq. (1) is topologically equivalent to (2) and (3). These properties together with the results of the above-mentioned papers can be used to obtain the phase portraits of (1). For instance, if $f(p) = 0$ and $g(\bar{p}) \neq 0$ (resp. $g(\bar{p}) = 0$ and $f(p) \neq 0$), then p is a singular point of system (1) with positive (resp. negative) index and having no hyperbolic (resp. elliptic or parabolic) sectors, see [11,12] or [9]. When (1) is polynomial, we extend the equations at infinity (so we define them at $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$) by doing the change of variables $w \to 1/z$ and studding a neighborhood of $w = 0$.

Once we know the equivalence of the phase portraits of systems (1)–(3) whenever they are defined, we focus on the period function of the continuum of periodic orbits. It is well known that all the orbits of any continuum of periodic orbits of system (2) have the same period (see [9,11,12]) or Corollary 2.5. As we will prove, and this is the main goal of this paper, this is no longer true for system (1) (unless $g \equiv 1$). Before stating the main result we introduce some notation.

We say that a region $\mathcal{P} \subset \mathbb{C}$ is a *period annulus* if it is the largest open and doubly connected set fulfilled of periodic orbits. Notice that one of the boundaries could be $z = \infty$. More precisely, since it is convenient for later discussions, we distinguish between two kinds of period annulus. On the one hand, we denote by $\mathcal{P}_p$ when at least one of the boundaries of $\mathcal{P}$ is a single point p. The point p is usually called *center*, and consequently, $\mathcal{P}_p$ is the period annulus associated to the center p (here $p = \infty$ is possible). If $\hat{\mathbb{C}} \setminus \mathcal{P}_p$ reduces to two points (namely, p and ∞), we say that the period annulus is *global*. On the other hand, we denote by $\mathcal{P}_R$ when none of the boundaries of $\mathcal{P}$ is a point. In this case, the period annulus is a non-degenerate ring. As it will become clear the boundaries of $\mathcal{P}_R$ must be monodromic graphs with at least one "singularity" p such that $g(\bar{p}) = 0$ (a generalized saddle point).

Let $\mathcal{P}$ be an arbitrary period annulus (of any kind). We define the *period function* associated to $\mathcal{P}$ as the function

$$T : \mathcal{P} \to \mathbb{R}$$

that gives to any point $z \in \mathcal{P}$ the period of the periodic orbit passing through z. Since all the z-points in $\mathcal{P}$, belonging to the same periodic orbit γ, have the same period we may denote by $T(\gamma)$ the period of the periodic orbit. We say that T is an increasing (resp. decreasing) function if for any couple of periodic orbits γ_0 and γ_1 in $\mathcal{P}$ with γ_0 contained in γ_1 we have $T(\gamma_1) - T(\gamma_0) > 0$ (resp. < 0). Here *contained* means that γ_0 belongs to the bounded region associated to γ_1 as a Jordan curve in $\mathbb{C}$. A local maximum or minimum of the period function is called a *critical period.*

When studying the period function, two different approaches appear. On the one hand, some works deal with its behavior (asymptotic expansion) near the boundaries of $\mathcal{P}$. Based on this, we can quote, for instance, the results in [3] where it is considered the period function in a neighborhood of a non-degenerated center, or the ones in [5] when the center is degenerated. In particular, from the results of this later paper, we get that, when p is a degenerated center, the period function tends to infinity when the periodic orbits approach to p. Finally, in [14], the authors describe the asymptotic expansion of the period function for the same family as [3], but now close to the connected component of the boundary of the period annulus different to the center point. Applying all together it is possible to give some partial information of the period function in the whole annulus.

On the other hand, apart from the local study, one can also deal with giving global information of the period function. These results are devoted to study the number of critical periods of this function, and in most cases to study its monotonicity. Without pretending to be exhaustive we list some of the different approaches to this global study and we give, in each of these approaches, a representative reference.

- To show that the period function satisfies a Picard–Fuchs system; see [4].
- To prove that the differential equation can be transformed into a potential Hamiltonian system; see [2].
- To prove that some parametrization of the period function has positive derivatives; see [10].
- To use a normalizer of the vector field having the continuum of periodic orbits to get an expression of the derivative of the period function; see [6].

The proof of Theorem A, see below, uses the last approach. It is also worth mentioning that among all families of vector fields for which the study of the period function is done, the Hamiltonian one is the most treated. Some of the reasons for this fact are as follows: (a) there are no focus in that family and hence any monodromic point is a center (so it is not necessarily a primer study on the *center conditions*); (b) the Hamiltonian function associated to the vector field may be used to parameterize the periodic orbits as well as, using the level curves, to compute properties of the period function or, if possible, the periodic function itself. As we have already mentioned, Eq. (1) is somehow related to Hamiltonian systems, but it is not included in this family. Fortunately, it is not difficult to solve the center-focus problem for system (1); see Propositions 2.6 and 2.8.

Our main result is the following theorem. Part (ii) is a well-known result (see [9,11,12]) that we state for the sake of completeness.

Theorem A. *Let $\mathcal{P}$ be a period annulus of Eq.* (1)

(i) *If g is not a constant function, then the period function in $\mathcal{P}$ has at most one critical period, which whenever exists, is a minimum.*
(ii) *If g is a constant, then the period function is constant in $\mathcal{P}$.*

Precisely, our proof of (i) in the above theorem shows that the period function is a *convex map* once we have fixed a convenient parametrization of the period annulus, that is, the period function has an increasing first derivative (so the second derivative is positive) with respect to that parametrization. However, although it is true that the sign of the first derivative does not depend on the parametrization, the second derivative does. In an appendix we include a totally different approach to the proof of Theorem A. This approach is merely analytic and looks simpler, but unfortunately it only works for period annulus which are associated to a critical point, i.e. it does not work for period annulus of the form $\mathcal{P}_R$.

In light of Theorem A, if g is not a constant function, the period function of any continuum of periodic orbits is either increasing, decreasing or it has a unique critical period (minimum). As we will see at the end of Section 3 and the Appendix all cases are possible.

The paper is organized as follows. We give basic tools and the center conditions in Section 2. We prove Theorem A in Section 3. In Section 4 we show, by means of several examples, the shape of the period function. In particular we consider the intersection between the Hamiltonian polynomial family and Eq. (1). Finally the Appendix discusses an alternative approach to the proof of Theorem A.

2. Preliminaries

Here we give the basic notations and state the main tools we use later on.

The *Lie bracket* $[X, Y]$ associated to two $\mathcal{C}^1$ vector fields is defined as $[X, Y] = (DX)\,Y - (DY)\,X$, where D denotes the differential matrix, see [16]. If the two vector fields are written as $\dot{z} = F(z, \bar{z})$ and $\dot{z} = G(z, \bar{z})$, it is easy to verify that

$$[F, G] = FG_z - F_zG + \bar{F}G_{\bar{z}} - \bar{G}F_{\bar{z}}, \tag{4}$$

where the subscripts denote the corresponding derivatives of F and G with respect to z and $\bar{z}$, respectively. Next theorem is proved for the case of a period annulus associated to a center in [6]. Here we generalize it to include all type of period annulus. Since its proof follows with minor changes, we do not include it here.

Theorem 2.1. *Let $\mathcal{P}$ denote the period annulus of a C^1 vector field. Let U be a $\mathcal{C}^1$ vector field defined on $\mathcal{P}$, transversal to X. Assume that $[X, U] = \mu X$, with μ being a $\mathcal{C}^1$ function defined on $\mathcal{P}$. Then, if $\psi(s)$ is a trajectory of U, for any s_0 such that $\psi(s_0) \in \mathcal{P}$ it holds*

$$T'(s_0) = \int_0^{T(s_0)} \mu(x(t; s_0), y(t; s_0))\, dt, \tag{5}$$

where $(x(t; s_0), y(t; s_0))$ is the periodic orbit of X of period $T(s_0)$ passing though $\psi(s_0)$.

The proof of Theorem A is based on computing the integral given by (5) for a suitable function μ. To this end, we use Green's Theorem (see [7]) to an annulus given by two periodic orbits in the interior of the period annulus. For completeness, we also include here the statement of Green's Theorem in the $(z, \bar{z})$-coordinates (see, for instance, [7]).

Theorem 2.2. *Let D be a bounded domain in the plane whose boundary ∂D consists of a finite number of disjoint piecewise smooth closed positively oriented* (*with respect to D*) *curves. If $h(z, \bar{z})$ is a smooth function on $D \cup \partial D$, then*

$$\int_{\partial D} h(z, \bar{z})\, dz = 2i \iint_D \frac{\partial h(z, \bar{z})}{\partial \bar{z}}\, dx\, dy.$$

Remark 2.3. Notice, for instance, that if $h(z, \bar{z}) \equiv h(z)$ is holomorphic, then the above equality gives the well-known result $\int_{\partial D} h(z)\, dz = 0$.

Finally we study the center conditions for Eq. (1). Before this we state, following [8] (see also [1,11,12,15,17]), the normal forms for an equation of type (2).

Theorem 2.4. *Let $h(z)$ be a one-dimensional holomorphic function in a punctured neighborhood $\mathcal{U} \subset \mathbb{C}$ of a point p and consider the ordinary differential equation*

$$\frac{dz}{dt} = h(z), \quad z \in \mathcal{U}, \quad t \in \mathbb{R}. \tag{6}$$

Then, near p, Eq. (6) *is conformally conjugate, near* 0, *to*

(a) $\dot{z} = 1$, *if* $h(p) \neq 0$,
(b) $\dot{z} = h'(p)z$, *if p is a zero of h of order* 1 (*i.e.*, $h(p) = 0$ *and* $h'(p) \neq 0$),

(c) $\dot{z} = z^n/(1 + cz^{n-1})$, *where* $c = \operatorname{Re} s(1/h, p)$, *if* p *is a zero of* h *of order* $n > 1$,
(d) $\dot{z} = 1/z^n$, *if* p *is a pole of order* n.

As a corollary of this result we get the phase portrait in a neighborhood of a singular point or a pole.

Corollary 2.5. *Under the hypothesis of the previous theorem we have*

(a) *if* h *is a holomorphic map at* $p \in \mathbb{C}$, *and it is verified that* $h(p) = 0, h'(p) \neq 0$ *and* $\operatorname{Re}(h'(p)) = 0$ (*respectively*, $\operatorname{Re}(h'(p)) \neq 0$), *then* p *is a center* (*respectively*, *focus*),
(b) *if* h *is a holomorphic map at* $p \in \mathbb{C}$, *and it is verified that* $h^{k)}(p) = 0$, $k = 0, \dots n-1$ *for some* $n > 1$, *then* p *is the union of* $2(n-1)$ *elliptic sectors*, *and*
(c) *if* h *has a pole of order* $n \geqslant 1$ *at* $p \in \mathbb{C}$, *then* p *is the union of* $2(n+1)$ *hyperbolic sectors.*

As we have already mentioned, by taking a time re-parameterization, Eq. (1) can be transformed (without modifying the phase portrait) into Eq. (2), or equivalently into Eq. (6) with $h(z) := f(z)/\bar{g}(z)$. So, Corollary 2.5 applied to $h(z) := f(z)/\bar{g}(z)$ gives the center conditions for Eq. (1).

Proposition 2.6. *Let* $p \in \mathbb{C}$. *Eq.* (1) *has a center at* $p \in \mathbb{C}$ *if and only if there exist a natural index* $k \geqslant 0$ *such that the following conditions hold*:

(a) $f^{k)}(p) = 0$ *and* $f^{i)}(p) = \bar{g}^{\,i)}(p) = 0$, $\quad i = 0, \dots, k-1$, *and* $f^{k+1)}(p)\,\bar{g}^{k)}(p) \neq 0$.
(b) $\operatorname{Re}\left(\frac{f^{k+1)}(p)}{\bar{g}^{k)}(p)}\right) = 0$.

Proof. Of course p is a center for Eq. (1) if and only if p is a center for Eq. (2). Thus, all we need to do is to visualize the conditions of Corollary 2.5 when $h(z) = f(z)/\bar{g}(z)$. $\square$

The above proposition uses, from the point of view of the phase portrait, Eqs. (1) and (6) with $h(z) = f(z)/\bar{g}(z)$, which are indistinguishable. However, we know from Theorem 2.4 that each center at $z = p$ of Eq. (6) is linear, so *non-degenerate*, while Eq. (1) can have a center at $z = p$ with $g(\bar{p}) = 0$, i.e., $k > 0$ in the previous proposition. If this is the case, we will say that $z = p$ is a *degenerate* center for Eq. (1). Although there is no topological difference between degenerate and non-degenerate centers it is well known that the behavior of the period function in a neighborhood of the center itself varies. It is not difficult to see the following result, see also [5].

Lemma 2.7. *Assume that system* (1) *has a finite center at* $z = p$. *Then*, *the value of the period function as we approach the point* p *tends*, *either*

(a) *to* $\frac{2\pi}{|Im(f'(p)g(\bar{p})|)}$, *if* p *is a non-degenerate center*, *or*,
(b) *to* ∞, *if* p *is a degenerate center.*

Finally, we discuss the case of $z = \infty$ being a center when $f(z)$ and $g(\bar{z})$ are polynomials.

Proposition 2.8. *Let f and g be polynomials in z and $\bar{z}$ write as $f(z) = a_0 + a_1 z^n + \cdots + a_n z^n$ and $g(\bar{z}) = b_0 + b_1 \bar{z} + \cdots + b_m \bar{z}^m$, with $a_n b_m \neq 0$. Then, $z = \infty$ is a center of Eq.* (1) *if and only if $n = m + 1$ and* $\operatorname{Re}\left(a_n / \overline{b_m}\right) = 0$.

Proof. First we notice that the change of variable $w = 1/z$ transform the infinity of the z-plane to the origin of the w-plane. Therefore to guarantee that a neighborhood of infinity is fulfilled by periodic orbits (formally we say that $z = \infty$ is a center) we need to show that $w = 0$ is a (finite) center for the corresponding transformed equation. We notice that, although we can always do the transformation $w = 1/z$, it can be possible that $w = 0$ becomes a essential singularity of the new equation. This will not be the case when assuming that f and g in Eq. (1) are polynomials. Indeed, Eq. (1) in this case becomes,

$$\dot{z} = \frac{f(z)}{\bar{g}(z)} = \frac{a_n z^n + \cdots + a_0}{\overline{b_m} z^m + \cdots + \overline{b_0}} \quad \text{where } a_n \overline{b_m} \neq 0.$$

Considering the change of variables $w = 1/z$ we obtain

$$\dot{w} = -w^2 \frac{f(1/w)}{\bar{g}(1/w)} = -w^{2+m-n} \frac{a_n + \cdots + a_0 w^n}{\overline{b_m} + \cdots + \overline{b_0} w^m} = -w^{2+m-n} \left(\frac{a_n}{\overline{b_m}} + O(1) \right). \tag{7}$$

From Corollary 2.5, Eq. (7) has a center at $w = 0$ if and only if $2 + m - n = 1$ and $\operatorname{Re}\left(a_n / \overline{b_m}\right) = 0$. $\square$

3. Proof of Theorem A

To compute the derivative of the period function through Theorem 2.1, we need to obtain a transversal generator U for a Lie symmetry for Eq. (1). This is done in the following lemma.

Lemma 3.1. *Let $X = f(z) g(\bar{z})$ be the planar vector field defined by Eq.* (1)*. Then, if $U = i \frac{f(z)}{\bar{g}(z)}$ and $\mu(z) = 2 \operatorname{Im}\left(\frac{f(z)\bar{g}'(z)}{\bar{g}^2(z)}\right)$, it is satisfied that $[X, U] = \mu X$. Furthermore, outside the singularities of X, both vector fields are transversal.*

Proof. Using the expression of the Lie bracket in the $(z, \bar{z})$ coordinates (see, Eq. (4)), we obtain the expression of the Lie bracket of $f(z) g(\bar{z})$ and $i \frac{f(z)}{\bar{g}(z)}$

$$\left[f(z) g(\bar{z}), i \frac{f(z)}{\bar{g}(z)} \right] = f(z) g(\bar{z}) \frac{\partial}{\partial z} \left(i \frac{f(z)}{\bar{g}(z)} \right) - i \frac{f(z)}{\bar{g}(z)} \frac{\partial}{\partial z} (f(z) g(\bar{z})) + i \frac{\overline{f(z)}}{g(\bar{z})} \frac{\partial}{\partial \bar{z}} (f(z) g(\bar{z}))$$

$$= -i \frac{f^2(z) g(\bar{z}) \bar{g}'(z)}{\bar{g}^2(z)} + i \frac{\overline{f(z)} f(z) g'(\bar{z})}{g(\bar{z})}.$$

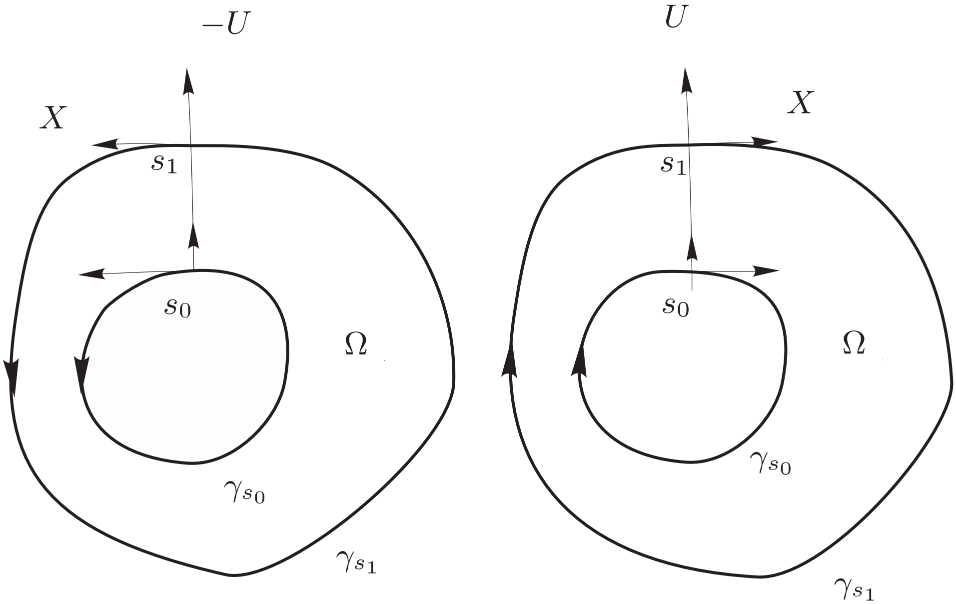

Fig. 1. Sketch of the relevant objects in the proof of Theorem A. On the left-hand side if the orbits in the period annulus $\mathcal{P}$ are positively oriented, in this case we observe that $\partial\Omega = \gamma_{s_1} - \gamma_{s_0}$. On the right-hand side, the orbits are negatively oriented, so $\partial\Omega = \gamma_{s_0} - \gamma_{s_1}$.

Hence, this shows that $\left[f(z)g(\bar{z}), i\frac{f(z)}{\bar{g}(z)}\right] = \mu(z,\bar{z})f(z)g(\bar{z})$ is verified with,

$$\mu(z,\bar{z}) = -i\,\frac{f(z)\bar{g}'(z)}{\bar{g}^2(z)} + i\,\frac{\overline{f(z)}g'(\bar{z})}{g^2(\bar{z})} = 2\,\mathrm{Im}\left(\frac{f(z)\bar{g}'(z)}{\bar{g}^2(z)}\right),$$

as we wanted to see. The proof that X and U are transversal is straightforward. □

Remark 3.2. Notice that, if X has a periodic orbit γ positively oriented with respect its interior, then U, as defined in the above lemma, is a transversal vector field *going* from the exterior to the interior of γ. In contrast, if X has a periodic orbit γ negatively oriented with respect its interior, then U, as defined in the above lemma, is a transversal vector field *going* from the interior to the exterior of γ. Of course, from the definition of the Lie bracket (see, Eq. (4)), if we use $-U$ instead of U in the above lemma, we have $[X, -U] = -\mu X$.

Proof of Theorem A. We want to prove that the first derivative is a non-decreasing function, once we have fixed a suitable parametrization. Notice that this is sufficient to prove the theorem since the number and character of the critical periods do not depend on the particular parametrization used.

Let $\mathcal{P}$ be an arbitrary period annulus of the vector field X defined by Eq. (1). Fix a periodic orbit γ in $\mathcal{P}$. Let us assume γ, parameterized by $z(t)$, is a negatively oriented periodic orbit of period T in the interior of $\mathcal{P}$, passing through p. Of course all orbits in $\mathcal{P}$ have the same orientation. The case positively oriented will be conveniently discussed at the end of the proof.

Let U and μ be as in Lemma 3.1. Take a transversal section Σ (on $\mathcal{P}$) parameterized by

$$g : (-\varepsilon, \varepsilon) \to \Sigma,$$

$g(s)$ being the solution of the differential equation $x' = U(x)$, such that $g(0) = p$. Notice that, because γ is negatively oriented, $g'(0) = U(p)$ is compatible with the parametrization of Σ by s.

Let $-\varepsilon < s_0 < s_1 < \varepsilon$. Let us denote by γ_{s_0} and γ_{s_1} the periodic orbits, parameterized by $z_0(t)$ and $z_1(t)$, of period $T(s_0)$ and $T(s_1)$ passing through s_0 and s_1, respectively. We denote by Ω the open annulus bounded by γ_{s_0} and γ_{s_1}.

We need to show that $T'(s_0) - T'(s_1) \geqslant 0$. Because of Theorem 2.1 the derivative of the period function can be formulated as the integral of μ. Precisely we have,

$$\begin{aligned} T'(s_1) - T'(s_0) &= 2\int_0^{T(s_1)} \mathrm{Im}\left(\frac{f(z_1(t))\bar{g}'(z_1(t))}{\bar{g}^2(z_1(t))}\right) dt \\ &\quad -2\int_0^{T(s_0)} \mathrm{Im}\left(\frac{f(z_0(t))\bar{g}'(z_0(t))}{\bar{g}^2(z_0(t))}\right) dt. \end{aligned} \tag{8}$$

In order to apply Green's Theorem (to the domain Ω), we have to write the above integrals as a line integral such that all the closed curve involved are positively oriented with respect to Ω. Consider $\partial\Omega = \gamma_0 - \gamma_1$. So, Eq. (8), under the change of variable $w = z(t)$ becomes

$$\begin{aligned} T'(s_0) - T'(s_1) &= -2\,\mathrm{Im}\left(\int_{\partial\Omega} \frac{\bar{g}'(w)}{\bar{g}^2(w)g(\bar{w})}\, dw\right) \\ &= -4\,\mathrm{Im}\left(i\int\int_{\Omega} \frac{\bar{g}'(w)}{\bar{g}^2(w)}\frac{\partial}{\partial\bar{w}}\left(\frac{1}{g(\bar{w})}\right) dx\, dy\right), \end{aligned}$$

where the second equality follows from Green Theorem. Evaluating the derivative we finally get

$$T'(s_1) - T'(s_0) = 4\,\mathrm{Im}\left(i\int\int_{\Omega}\left|\frac{\bar{g}'(w)}{\bar{g}^2(w)}\right|^2 dx\, dy\right) = 4\int\int_{\Omega}\left|\frac{\bar{g}'(w)}{\bar{g}^2(w)}\right|^2 dx\, dy \geqslant 0, \tag{9}$$

as desired. Note also that the integral is zero only when g is a constant function and thus in this situation the period function is constant.

To end the first part of the proof we suppose γ (and all orbits in P) are positively oriented. In this case, the transversal section Σ by s is exactly opposite to the direction of U. Hence we need to consider $-U$, and so, $-\mu$ (see Remark 3.2). The same proof follows straightforward by noting that in this case $\partial\Omega = \gamma_1 - \gamma_0$. □

From the proof of Theorem A and Corollary 2.7 we easily conclude the following result for the period function of a non-degenerate center.

Corollary 3.3. *Assume $\mathcal{P} = \mathcal{P}_p$ with $p \in \hat{\mathbb{C}}$ being a non-degenerate center. Then, if $p \in \mathbb{C}$ (respectively, $p = \infty$) the period function is a strictly increasing (respectively, strictly decreasing) map.*

Proof. Let us assume that $\mathcal{P} = \mathcal{P}_p$, with p being a finite non-degenerate center. From Corollary 2.7 we know that the value of the period function, as we approach the point p, tends to $2\pi/|Im(f'(p)g(\bar{p})|)$. So, it is enough to show that, as we approach the point p, the derivative of the period function tends to 0. Similar to the proof of Theorem A it is easy to verify that (assuming γ_s negatively oriented)

$$T'(s) = -2\,\mathrm{Im}\left(\int_{\gamma_s} \frac{\bar{g}'(w)}{\bar{g}^2(w)g(\bar{w})}\,dw\right) = 4\int\int_{\mathrm{int}(\gamma_s)} \left|\frac{\bar{g}'(w)}{\bar{g}^2(w)}\right|^2 dx\,dy,$$

since the center is non-degenerate. If we take limit when $s \to 0$, the left-hand side tends to the derivative of the period function at p while, we claim, that the right-hand side tends to zero. To see the claim we observe that the function inside the integral tends to a number bounded below (since p is non-degenerate, $\bar{g}(p) \neq 0$).

In case of $p = \infty$ being a non-degenerate center we can transform it to the origin by the change $w = 1/z$, and the result follows similarly as before. □

4. Consequences of Theorem A and examples

It is well known that if the boundary of a period annulus has a finite singular point q, which is not a non-degenerate center, then the period function tends to infinity as we approach it. From Corollary 2.5, this general fact for Eq. (1), reads as follows: If the boundary of the period annulus has either a degenerate center or it has a finite singular point q such that $g(\bar{q}) = 0$ and $f(q) \neq 0$ (i.e., a finite generalized saddle), then the period function tends to infinity as the periodic orbits approach to it. However, when there is no (generalized) saddle in the boundary of the period annulus, it is not possible to know, by using the above remark, Theorem A and Corollary 2.7, the behavior of the period function near it. Several examples showing different shapes are given along this section and in the appendix.

We start with an example in which the period functions in all the period annuli would be determined without explicitly computing them, i.e., using only the phase portrait and

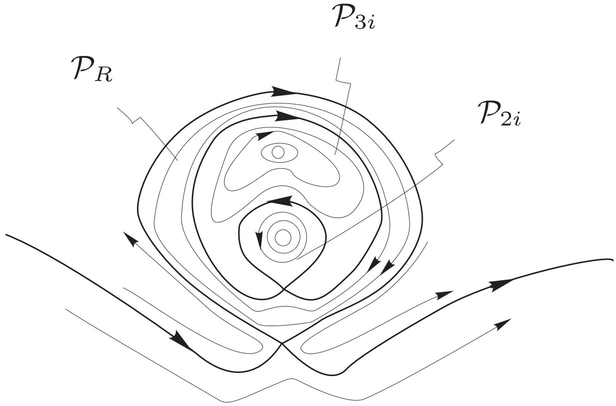

Fig. 2. Sketch of the phase portrait of $\dot{z} = (z - 2i)(z - 3i)\bar{z}(\bar{z} + i)$. Note that this differential equation has three period annuli.

the above-mentioned results. Let us consider the equation

$$\dot{z} = (z - 2i)(z - 3i)\bar{z}(\bar{z} + i). \tag{10}$$

In Fig. 2 we sketch its phase portrait. It has three different period annuli. On the one hand we have $\mathcal{P}_{2i}$ and $\mathcal{P}_{3i}$, associated to its non-degenerate finite centers located at $z = 2i$ and $z = 3i$, and, on the other hand, we have a ring period annulus $\mathcal{P}_R$. Moreover, there are two saddle points, located at $z = 0$ and i. Applying Corollary 3.3 we show that the period function associated to the period annuli $\mathcal{P}_{2i}$ and $\mathcal{P}_{3i}$ is a strictly increasing map tending to infinity since there is a finite saddle in their common outer boundary. Similar arguments and Theorem A show that the period function associated to $\mathcal{P}_R$ has a unique critical period (a minimum) in its interior and tends to infinity when we approach the inner/outer boundaries. So, the shape of the period functions of all annuli of Eq. (10) are qualitatively determined without being computed.

Now we illustrate the case when the outer boundary does not have any (generalized) finite saddle. We compute the period function and show that different shapes are possible. We consider the following family of equations:

$$\dot{z} = izF(z)\overline{F(z)} = iz|F(z)|^2, \tag{11}$$

with $F(z) = 0$ either never vanishing or vanishing only when $z = 0$. Note that these equations have a global center. We write them in polar coordinates $z = re^{i\theta}$. We get

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= |F(re^{i\theta})|^2. \end{aligned} \tag{12}$$

Note that for any $\rho \geqslant 0$ their solutions are contained in the circles $r = \rho$. Hence integrating the second equation we can have an exact expression of the period function parameterized by ρ:

$$T(\rho) = \int_0^{2\pi} \frac{d\theta}{|F(\rho e^{i\theta})|^2}.$$

Taking $F(z) = z^k e^{az}$, for $a \geqslant 0$ and $k \geqslant 0$ we have

$$T(\rho) = \frac{1}{\rho^{2k}} \int_0^{2\pi} e^{-2a\rho\cos\theta} d\theta.$$

If $a = 0$ we obtain that $T(\rho) = 2\pi/\rho^{2k}$ which is a strict decreasing function tending to 0 when $\rho \to \infty$ for $k > 0$, and it is constant for $k = 0$.

If $a > 0$ it follows that

$$T(\rho) > \tilde{T}(\rho) := \int_{3\pi/4}^{5\pi/4} \frac{e^{-2a\rho\cos\theta}}{\rho^{2k}} d\theta > \frac{\pi e^{-2a\rho/\sqrt{2}}}{2\rho^{2k}}.$$

So, $T(\rho) \to \infty$ when $\rho \to \infty$. We remark that following the same steps as in the Appendix, we can obtain an explicit expression of the period function in terms of the Bessel functions. More concretely,

$$T(\rho) = \frac{2\pi}{\rho^{2k}} \sum_{j=0}^{\infty} \frac{(a\rho)^{2j}}{(j!)^2} = \frac{2\pi}{\rho^{2k}} J_0(2a\rho i),$$

being $y = J_0(t)$ the usual Bessel function, solution of $t^2 y''(t) + t y'(t) + t^2 y(t) = 0$. So, when $k = 0$ the period function is strictly increasing, when $k > 0$ it has a unique critical period and tends to infinity as we approach the inner as well as the outer boundary.

4.1. The Hamiltonian case

We study the planar Hamiltonian systems lying inside the family given by Eq. (1). As we already mentioned in the Introduction there is a re-parametrization (not defined everywhere, although) which transforms Eq. (1) into Eq. (3), which induces a planar Hamiltonian system. In other words we can say that $|f(z)|^{-2}$ is a singular integrating factor for Eq. (1). The presence of singularities is natural because the topological picture of the phase portraits near these points has elliptic or parabolic sectors and thus the flow cannot preserve the area in a whole neighborhood of them.

However, inside the family of Eq. (1) there are pure Hamiltonian systems. Since the period function of Hamiltonian systems is broadly studied and, as far as we know our general results particularized on this subfamily of Eq. (1) are new, we detail them in the sequel. It is easy to see that, when Eq. (1) does not depend on $\bar{z}$ (that is, $g(\bar{z}) \equiv 1$), the only Hamiltonian case is $\dot z = ikz,\ k \in \mathbb{R}$. Next lemma extends the above remark to a general g.

Lemma 4.1. *Eq.* (1) *is Hamiltonian if and only if there exist* $c \in \mathbb{R}$ *such that* $f'(z) = ic\bar g(z)$.

Proof. A necessary condition for a general planar vector field $\dot z = F(z,\bar z)$ to be Hamiltonian is that $\partial(\operatorname{Re} F(z,\bar z))/\partial x + \partial(\operatorname{Im} F(z,\bar z))/\partial y \equiv 0$. It is not difficult to see that

$$\operatorname{Re}\left(\frac{\partial F(z,\bar z)}{\partial z}\right) = \frac{1}{2}\left(\frac{\partial \operatorname{Re}(F(z,\bar z))}{\partial x} + \frac{\partial \operatorname{Im}(F(z,\bar z))}{\partial y}\right).$$

So the Hamiltonian condition gives $\operatorname{Re}\left(\frac{\partial F(z,\bar z)}{\partial z}\right) \equiv 0$. In order to apply above condition to Eq. (1), we observe that in our case $F(z,\bar z) = f(z)g(\bar z)$. Hence, Eq. (1) will be Hamiltonian when

$$\operatorname{Re}(f'(z)\, g(\bar z)) \equiv 0,$$

or equivalently, when $f'(z) = ic\bar g(z)$, where $c \in \mathbb{R}$, as desired.

For the converse, we claim that

$$H(z,\bar z) = -\frac{c}{2}\int_{z_0}^{z} \bar g(s)\, ds \int_{w_0}^{\bar z} g(s)\, ds \tag{13}$$

is a Hamiltonian function for Eq. (1) with $f'(z) = ic\bar g(z)$. To see the claim, we observe that the Hamiltonian system generated by a general map $H(z,\bar z)$,

$$\dot x = H_y,$$
$$\dot y = -H_x$$

writes, in the $(z,\bar z)$-variables, as $\dot z = -2i\, H_{\bar z}$. Hence, the Hamiltonian system generated by (13) is $\dot z = f(z)g(\bar z)$, with $f'(z) = ic\bar g(z)$. □

Corollary 4.2. *The following statements hold.*

(a) *The equation* $\dot z = g(\bar z)$ *is always Hamiltonian.*
(b) *If Eq.* (1) *is polynomial Hamiltonian and* $f'(z) \not\equiv 0$, *then*
  (b1) *it has an odd degree*, $2n+1$, *and is written as*

$$\dot z = ic\left(a_0 + a_1 z + \frac{a_2}{2}z^2 + \cdots + \frac{a_{n+1}}{n+1}z^{n+1}\right)\left(\overline{a_1} + \overline{a_2}\bar z + \cdots + \overline{a_{n+1}}\bar z^{n}\right),$$

  with $c \in \mathbb{R},\ a_j \in \mathbb{C},\ j = 0,1,\dots,n+1,$ *and*
  (b2) *there exists a neighborhood of infinity, where all orbits are periodic. In other words we always have a period annulus of the form* $\mathcal{P}_\infty$.

Proof. Statements (a) and (b1) are straightforward. From Proposition 2.8, all we need to see to prove the remainder statement (b2) is that $\mathrm{Re}(a_n(\overline{b_m})^{-1}) = 0$. If we denote by b_m the coefficient of z^m for $g(\bar{z})$ (hence $\overline{b_m}$ is the coefficient of z^m for $\bar{g}(z)$) and by a_n the coefficient of z^n for $f(z)$ (so, $n = m + 1$), we easily get $a_n = ic(m+1)\overline{b_m}$. Thus $\mathrm{Re}\big(a_n(\overline{b_m})^{-1}\big) = 0$. □

As a corollary of the previous result we may characterize the Hamiltonian equation given by (1) having a finite center. Here we present the list up to degree 5.

Corollary 4.3. *Suppose $\dot{z} = f(z)g(\bar{z})$ is a planar polynomial Hamiltonian equation of degree $m \leqslant 5$ having a finite center. Then there is an affine change of variables that is written as*

(a) $\dot{z} = icz$,
(b) $\dot{z} = ic\,(az + z^2/2)\,(\bar{a} + \bar{z})$,
(a) $\dot{z} = ic\,(az + bz^2/2 + z^3/3)\,(\bar{a} + \bar{b}\bar{z} + \bar{z}^2)$,

where $c \in \mathbb{R},\ a, b \in \mathbb{C}$.

Although, apparently, from all we have been done, it seems easy to determine the phase portrait of any of those Hamiltonian equations (even for an arbitrary given degree), we have to be precise about the existence (or not) of possible separatrix connections changing the relative position of the centers could be a deep obstruction to do so. However, the existence of a first integral is certainly helpful in many cases. In contrast, once we know the phase portrait (and the degenericity of the centers), the qualitative behavior of the period function is trivial.

Precisely, the relevant information about the equation given in Corollary 4.3(b), can be summarized as follows.

- When $a = 0$ the period annulus is global and $p = 0$ is a degenerate center. According to the computations in the example given by (11), the period function is a strictly decreasing map.
- When $a \neq 0$ the origin has a period annulus and its phase portrait has two further period annuli, $\mathcal{P}_{-2a}$ and $\mathcal{P}_\infty$; see Fig. 3(a). Moreover, the period function is
- increasing from a constant value to infinity in $\mathcal{P}_0$ and $\mathcal{P}_{-2a}$,
- decreasing from infinity to a constant value in $\mathcal{P}_\infty$.

A similar approach to the equation given in Corollary 4.3(c) gives rise to several different phase portraits, all of them having infinity as a non-degenerate center. To avoid all the casuistic, we assume that all the zeroes of f and g are simple, and there are no common zeros of f and g. So we have three non-degenerate centers (the zeroes of $az + bz^2/2 + z^3/3$) and two hyperbolic saddles (the zeroes of $\bar{a} + \bar{b}\bar{z} + \bar{z}^2$). In this framework there are 4 different configurations depending on the number and relative position of the heteroclinic connections between the separatrices of the two distinct saddles (2 phase portraits with no connection between them, 1 with two connections and one with 4 connections). As an example we give the phase portrait when $a = 3$ and $b = 5$; see Fig. 3(b). There are five period annuli denoted by $\mathcal{P}_0, \mathcal{P}_{-6}, \mathcal{P}_{-3/2}, \mathcal{P}_R$ and $\mathcal{P}_\infty$.

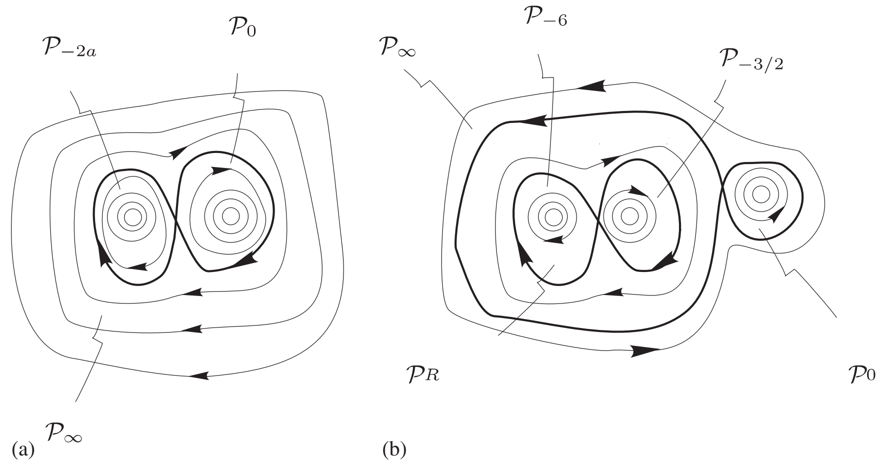

Fig. 3. Sketch of the phase portrait of some polynomial Hamiltonian equation. (a) corresponds to degree 3 while (b) corresponds to degree 5.

Notice that since we can compute the Hamiltonian associated to this equation it is easy to determine the phase portrait. As before, the relevant information about the period function for these parameter values can be summarized as follows.

- It is increasing from a constant value to infinity in $\mathcal{P}_0, \mathcal{P}_{-6}$ and $\mathcal{P}_{-3/2}$.
- It has a unique critical period in $\mathcal{P}_R$, and tends to infinity as we approach the inner as well as the outer boundary.
- It is decreasing from infinity to a constant value in $\mathcal{P}_\infty$.

Appendix. An alternative approach to the proof of Theorem A

In this section, we give a different approach to the proof of Theorem A, as long as the period annulus is not of the form $\mathcal{P}_R$. As it will become clear in the sequel, the main difficulty in extending this different point of view to all the cases, is that it is strongly based on the existence of a holomorphic linearizing change of variables for non-degenerate holomorphic centers of the equation $\dot{z} = icz + O(z^2)$. Note that period annulus of the form $\mathcal{P}_R$ does not have a center in its boundary.

Assume that Eq. (1) has a singular point of center type (degenerate or not) at the origin, and as usual denote by $\mathcal{P}_0$ its period annulus. The differential equation can be written as

$$\frac{dz}{dt} = f(z)g(\bar{z})\frac{\bar{g}(z)}{\bar{g}(z)} = \frac{f(z)}{\bar{g}(z)}|\bar{g}(z)|^2. \tag{14}$$

Since the origin is a center (degenerate or not), by Proposition 2.6 we know that $f(z)/\bar{g}(z) = icz + O(z^2)$, $c \in \mathbb{R}$, is an holomorphic function on $\mathcal{P}_0$ and $\bar{g}(z) = az^k + 0(z^{k+1})$, for some $k \in \mathbb{N}$ and $a \in \mathbb{C} \setminus \{0\}$ (of course, $k = 0$ corresponds to the non-degenerate case). Applying Theorem 2.4(b) we also know that there is a holomorphic linearizing map $w = \phi(z) = z + 0(z^2)$ for $\dot{z} = f(z)/\bar{g}(z)$, i.e., equation $\dot{z} = f(z)/\bar{g}(z)$, in this new variable writes as $\dot{w} = icw$. In [8] it is shown that

$$w = \phi(z) = z \exp\left(\int_0^z \left(\frac{ic\bar{g}(s)}{f(s)} - \frac{1}{s}\right) ds\right),$$

so the conformal conjugacy extends to the whole period annulus $\mathcal{P}_0$. Notice that ϕ transforms $\mathcal{P}_0$ in a disc $D(0, R)$, where R geometrically corresponds to the supremum of the radius of the images by ϕ of the periodic orbits in $\mathcal{P}_0$.

Thus in the new variable w, Eq. (14) is written as

$$\frac{dw}{dt} = icw\left|\bar{g}\left(\phi^{-1}(w)\right)\right|^2. \tag{15}$$

From above, we can write $\bar{g}(\phi^{-1}(w))$ in the form $w^k/H(w)$, where $H(w) = w^k/\bar{g}(\phi^{-1}(w))$ is holomorphic at least in $D(0, R)$. Because of that we notice that, in $D(0, R)$, $H(w)$ can be written as a power series, $H(w) = \sum_{j=0}^{\infty} h_j w^j$. Thus the radius of convergence of this series is $\bar{R} \geqslant R$.

Hence, writing Eq. (15) in polar coordinates we get

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= c\left|\frac{w^k}{H(w)}\right|^2_{w=re^{i\theta}}. \end{aligned} \tag{16}$$

The circles $r = \rho$ are invariant in the above equation, and for $\rho < R$, they correspond to the periodic orbits $\phi^{-1}(\{w : |w| = \rho\})$ of Eq. (14). The period of these orbits is given by

$$T(\rho) = |c| \int_0^{2\pi} \frac{|H(w)|^2_{w=\rho e^{i\theta}}}{\rho^{2k}}\, d\theta.$$

For $\rho < \bar{R}$, using the power series expansion of H, it is easy to verify that (see, for instance, [13])

$$T(\rho) = \frac{2\pi|c|}{\rho^{2k}} \sum_{j=0}^{\infty} |h_j|^2 \rho^{2j},$$

since $|H(w)|^2 = (h_0 + h_1 w + \cdots)(\bar{h}_0 + \bar{h_1}\bar{w} + \cdots)$, and the only terms of the product series not vanishing after integration are $|h_k|^2 w^k \bar{w}^k$ for all k. Clearly, the radius of convergence of $\sum_{j=0}^{\infty} |h_j|^2 \rho^{2j}$ is the same as the radius of convergence of $\sum_{j \geqslant 0} h_j w^j$, that is $\bar{R}$.

Finally, for $\rho < R$, we write down the period function as

$$T(\rho) = \frac{2\pi|c|}{\rho^{2k}} \sum_{j=0}^{\infty} |h_j|^2 \rho^{2j} = 2\pi|c| \left(\sum_{j=-k}^{0} |h_{j+k}|^2 \rho^{2j} + \sum_{j=1}^{\infty} |h_{j+k}|^2 \rho^{2j} \right).$$

So, under this parametrization by ρ, all derivatives of $\rho^{2k} T(\rho)$ are positive.

If $k = 0$, i.e. the origin is a non-degenerate center, the period function, parameterized by ρ, is an strictly increasing function defined in $[0, R)$. We remark that this does not imply that the period function tends to infinity, as we approach the outer boundary of the period annulus. As the following example illustrates, the radius of convergence of the above series can be infinite but the function defined by the series only coincide with the period function in $[0, R)$.

Let $\dot{z} = z(i + z)(i + z)(-i + \bar{z})$. The phase portrait has two centers at $z = 0$ (non-degenerate) and $z = -i$ (degenerate) with a common boundary given by $\mathrm{Im}(z) = -1/2$. We focus at $z = 0$. The linearizing map is given by $w = \frac{z}{i+z}$, so Eq. (15) becomes

$$\frac{dw}{dt} = iw \frac{1}{|1 - w|^2}, \tag{17}$$

and the period function can be explicitly computed, $T(\rho) = 2\pi(1+\rho^2)$, where $\rho = |w|$. Of course there are no problems of convergence and the function is well defined in all $\mathbb{R}$ and tends to infinity as ρ does. However, the function $T(\rho)$ is only the period function of the period annulus at $z = 0$ as long as the circle of radius ρ lies in the image by $\phi(z) = z/(i + z)$ of $\mathcal{P}_0$. It is easy to verify that the boundary of $\mathcal{P}_0$ given by $\mathrm{Im}(z) = -1/2$ goes to the circle of radius $\rho = 1$, so the period function is $T(\rho) = 2\pi(1 + \rho^2)$, $\rho < 1$, and it tends to 4π when $\rho \to 1$ and it tends to 2π when $\rho \to 0$.

In case of $k > 0$ the origin is a degenerate center. As expected the period function tends to infinity when we approach to the origin. To deduce the existence of at most one zero of $T'(\rho)$, we observe that a zero of $T'(\rho)$ implies the existence of a solution of the equation

$$-\sum_{j=-k}^{0} |h_{j+k}|^2 j \rho^{2j-1} = \sum_{j=1}^{\infty} |h_{j+k}|^2 j \rho^{2j-1}. \tag{18}$$

The left-hand side of this expression is a positive decreasing function tending to infinity when ρ tends to zero. The right-hand side is a positive increasing function. Hence, above equation has at most one zero, obtaining thus the existence of at most one critical period. The fact that this zero does not always exist is illustrated in example $\dot{z} = z(i + z)z\bar{z}$ for which the period function in $\mathcal{P}_0$ is $T(\rho) = 2\pi(1 + 1/\rho^2)$ being $\rho = |w| < 1$, as in the previous example. So, it is decreasing from infinity to 4π.

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