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Generalized Nonlinear Superposition Principles for Polynomial Planar Vector Fields*

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Abstract. In this paper we study some aspects of the integrability problem for polynomial vector fields $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. We analyze the possible existence of first integrals of the form $I(x,y) = (y - g_1(x))^{\alpha_1}(y - g_2(x))^{\alpha_2} \cdots (y - g_\ell(x))^{\alpha_\ell} h(x)$, where $g_1(x), \ldots, g_\ell(x)$ are unknown particular solutions of $dy/dx = Q(x,y)/P(x,y)$, α_i are unknown constants and $h(x)$ is an unknown function. We show that for certain systems some of the particular solutions remain arbitrary and the other ones are explicitly determined or are functionally related to the arbitrary particular solutions. We obtain in this way a nonlinear superposition principle that generalize the classical nonlinear superposition principle of the Lie theory. In general, the first integral contains some arbitrary solutions of the system but also quadratures of these solutions and an explicit dependence on the independent variable. In the case when all the particular solutions are determined, they are algebraic functions and our algorithm gives an alternative method for determining such type of solutions.

Keywords: nonlinear differential equations, polynomial planar vector fields, nonlinear superposition principle, Darboux first integral, Liouvillian first integral.
AMS classification: Primary 34C05; Secondary 34C14, 22E05.

1. Introduction

We consider in this paper two–dimensional systems

$$\frac{dx}{dt} = \dot{x} = P(x,y)\ , \qquad \frac{dy}{dt} = \dot{y} = Q(x,y)\ , \tag{1}$$

in which $P, Q \in \mathbb{R}[x,y]$ are polynomials in the real variables x and y and the independent variable (the time) t is real. Throughout this paper we will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1). Obviously, we can also express system (1) as the differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}\ . \tag{2}$$

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