ANALYTIC NILPOTENT CENTERS AS LIMITS OF NONDEGENERATE CENTERS REVISITED

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ABSTRACT. We prove that all the nilpotent centers of planar analytic differential systems are limit of centers with purely imaginary eigenvalues, and consequently the Poincaré–Liapunov method to detect centers with purely imaginary eigenvalues can be used to detect nilpotent centers.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

Consider the analytic family

$$\dot{x} = y + F_1(x, y; \lambda), \quad \dot{x} = F_2(x, y; \lambda) \tag{1}$$

with parameters $\lambda \in \mathbb{R}^p$ and having a nilpotent singularity at the origin.

In the papers [7–9] is stated a theorem which is slightly modified each time in order to correct the previous version but surprisingly it has never been properly written. Anyway we want to emphasize that the ideas presented in [7] have merit and in our opinion they are new and useful for understanding the nilpotent center problem mainly due to the computational algorithm that is derived from it. Here we present the correct statement and the right proof. We will present one counterexample for showing that the previous version of the mentioned theorem does not work. Also we end with an example for showing the analysis of a nilpotent center problem on a family using the right method provided here.

Here a *non-degenerate center* is a center with purely imaginary eigenvalues. See for example the book [11] for a modern treatment of the non-degenerate center problem. In this work we focus on the nilpotent center problem which has been studied by several authors [1, 2, 5, 10].

Given an analytic function f at a point p, we say that f has *order k* at p if the Taylor series of f at p starts with terms of degree k in x and y.

Theorem 1. *Suppose that the origin of system (1) with $\lambda = \lambda^*$ is a nilpotent center. Then there are two (non unique) functions $P(x, y)$ and $Q(x, y)$ analytic at the origin and of order at least two such that the 1-parameter family*

$$\dot{x} = y + F_1(x, y; \lambda^*) + \varepsilon P(x, y), \quad \dot{y} = -\varepsilon x + F_2(x, y; \lambda^*) + \varepsilon Q(x, y) \tag{2}$$

possesses a non-degenerate center at the origin for any $\varepsilon > 0$. Also there is an analytic function $f(x, y)$ at the origin of order at least two such that

$$(x - Q)\frac{\partial f}{\partial y} = P\left(1 + \frac{\partial f}{\partial x}\right). \tag{3}$$

Theorem 1 is proved in section 2.

The rest of the paper is organized as follows. In section 3 we do some remarks related to Theorem 1, and in section 4 using Theorem 1 we extend the algorithm of Poincaré-Liapunov for characterizing the nilpotent centers. Finally in sections 5 and 6 we provide the mentioned counterexample and example, respectively.

2. Proof of Theorem 1

Following [5] (see also [10]), if the analytic system (1) with $\lambda = \lambda^*$ has a center at the origin, then it is analytically orbitally equivalent to a time-reversible system. More precisely there exists an analytic near identity change of variables $(x, y) \mapsto (u, v) = \Phi(x, y) = (x + f(x, y), y + g(x, y))$ and a time rescaling $t \mapsto \tau(u, v)$ with $d\tau/dt = U(u, v)$ such that $U(0, 0) = 1$ and the new differentiable system is invariant under the involution $(u, v, \tau) \mapsto (-u, v, -\tau)$. More precisely, in the new variables (u, v) the differential system (1) becomes

$$\dot{u} = (v + \hat{F}_1(u^2, v))U(u, v), \quad \dot{v} = (u\hat{F}_2(u^2, v))U(u, v)$$

and after the time recaling we get the time-reversible system

$$u' = v + \hat{F}_1(u^2, v), \quad v' = u\hat{F}_2(u^2, v),$$

where the prime denotes derivative with respect to τ. Clearly the origin $(u, v) = (0, 0)$ is a nilpotent center. We perturb the system introducing the real parameter $\varepsilon > 0$ modifying the linear part as

$$(4) \qquad u' = v + \hat{F}_1(u^2, v), \quad v' = -\varepsilon u + u\hat{F}_2(u^2, v).$$

Now the origin is a non-degenerate reversible center for any $\varepsilon > 0$. We go back to the initial time variable t and we obtain that the differential system

$$(5) \qquad \dot{u} = (v + \hat{F}_1(u^2, v))U(u, v), \quad \dot{v} = (-\varepsilon u + u\hat{F}_2(u^2, v))U(u, v)$$

also has a non-degenerate center at the origin for any $\varepsilon > 0$ because $U(0, 0) = 1$. Using the chain rule we have

$$\dot{u} = \dot{x} + \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial y}\dot{y}, \quad \dot{v} = \dot{y} + \frac{\partial g}{\partial x}\dot{x} + \frac{\partial g}{\partial y}\dot{y}$$

and inverting we get

$$(6) \quad \dot{x} = \frac{1}{\Delta(x, y)}\left[\left(1 + \frac{\partial g}{\partial y}\right)\dot{u} - \frac{\partial f}{\partial y}\dot{v}\right], \quad \dot{y} = \frac{1}{\Delta(x, y)}\left[\left(1 + \frac{\partial f}{\partial x}\right)\dot{v} - \frac{\partial g}{\partial x}\dot{u}\right],$$

where

$$\Delta(x, y) = 1 + \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

From here it is easy to pull back (5) to the original variables (x, y) obtaining system (2) with

$$(7) \qquad \begin{aligned} P(x, y) &= \frac{U \circ \Phi(x, y)}{\Delta(x, y)}(x + f(x, y))\frac{\partial f}{\partial y}, \\ Q(x, y) &= x - \frac{U \circ \Phi(x, y)}{\Delta(x, y)}(x + f(x, y))\left(1 + \frac{\partial f}{\partial x}\right). \end{aligned}$$

Therefore it is evident that $P(x, y)$ and $Q(x, y)$ are analytic and have order at least two at $(x, y) = (0, 0)$ and also that (3) holds. This completes the proof of Theorem 1.

3. Remarks

Remark 2. We emphasize that the expressions for the functions P and Q in the analogous to Theorem 1 given in the references [7–9] are wrong. More specifically in [7] it is stated that $P(0, y) = Q(0, y) \equiv 0$, and in [8] that P and Q only depend on f in the specific way

$$(8) \qquad P(x, y) = (x + f(x, y))\frac{\partial f}{\partial y}, \quad Q(x, y) = -(x + f(x, y))\frac{\partial f}{\partial x} - f(x, y).$$

The error in the proof was the assumption that nilpotent centers are conjugated to a time-reversible system, hence without performing the time rescaling $t \mapsto \tau$. Later on in [9] it is taking into account that actually nilpotent centers are orbitally conjugated to time-reversible systems but in the proof also appear mistakes yielding again that P and Q only depend on f as in (8). Our proof shows that P and Q depend on f, g and U. In particular the computations involved with the method associated to Theorem 1 to detect nilpotent centers become harder since now we have more freedom in choosing P and Q. Anyway all the center conditions founded in all the examples studied in [7,9] are correct either because we have checked them using Theorem 1, or because they have been studied by other authors.

Remark 3. Instead of perturbing as in (4) we can perturb in the following form

$$u' = v + \hat{F}_1(u^2, v) - \varepsilon\hat{G}_1(u^2, v), \quad v' = -\varepsilon u + u\hat{F}_2(u^2, v) - \varepsilon u\hat{G}_2(u^2, v),$$

with any pair of analytic functions $\hat{G}_1(u^2, v)$ and $\hat{G}_2(u^2, v)$ starting in at least second and first order respectively. In this case we obtain system (2) with

$$\begin{aligned} P(x, y) &= \frac{U \circ \Phi}{\Delta}\left[(1 + \hat{G}_2 \circ \Phi)(x + f)\frac{\partial f}{\partial y} - \left(1 + \frac{\partial g}{\partial y}\right)(\hat{G}_1 \circ \Phi)\right], \\ Q(x, y) &= x - \frac{U \circ \Phi}{\Delta}\left[(1 + \hat{G}_2 \circ \Phi)(x + f)\left(1 + \frac{\partial f}{\partial x}\right) - (\hat{G}_1 \circ \Phi)\frac{\partial g}{\partial x}\right]. \end{aligned}$$

This proves that P and Q are not unique due to the arbitrariness of the functions $\hat{G}_1$ and $\hat{G}_2$. In the particular case when $\hat{G}_1 = \hat{G}_2 \equiv 0$ we recover the former analysis in the proof of Theorem 1.

It is worth to emphasize that, in general, we cannot choose $\hat{G}_2$ satisfying the functional equation $(U \circ \Phi)/\Delta = 1 + \hat{G}_2 \circ \Phi$. The reason is that if it was possible, then adding $\hat{G}_1 \equiv 0$ we recover the expressions (8). But we know (see the section where we analyze system (10)) that this is not possible.

Remark 4. Using (3) we see that x must factor out the homogeneous polynomial of minimal degree in the Taylor expansion of P at the origin. This means that the function P has a Taylor expansion of the form $P(x, y) = \sum_{i+j \geq 2} p_{ij} x^i y^j$ with the coefficient $p_{02} = 0$.

Remark 5. Clearly Theorem 1 provides necessary center conditions but only when it is applied to a family (1) having a monodromic nilpotent singularity at the origin. For example if the origin of system (1) with $\lambda = \lambda^\dagger$ is not monodromic but it is time-reversible with respect to the involution $(x, y, t) \mapsto (-x, y, -t)$, then the perturbation $\dot{x} = y + F_1(x, y; \lambda^\dagger)$, $\dot{y} = -\varepsilon x + F_2(x, y; \lambda^\dagger)$ has a time-reversible non-degenerate center at the origin for any $\varepsilon > 0$ showing that (2) holds with $P = Q \equiv 0$.

In summary, since we will only apply Theorem 1 to monodromic nilpotent families (1) we recall the following theorem of Andreev that characterizes that property.

Theorem 6 ([3]). *For an analytic system of the form* (1) *with* $\lambda = \hat{\lambda}$ *and having an isolated singularity at the origin let* $y = F(x)$ *be the unique solution of* $y + F_1(x, y; \hat{\lambda}) = 0$ *such that* $F(0) = F'(0) = 0$, *and let*

$$f(x) = F_1(x, F(x); \hat{\lambda}) \qquad and \qquad \xi(x) = (\partial F_1/\partial x + \partial F_2/\partial y)(x, F(x)).$$

Let $a \neq 0$ *and* $\alpha \geq 2$ *be such that* $f(x) = ax^\alpha + \cdots$. *When* ξ *is not identically zero let* $b \neq 0$ *and* $\beta \geq 1$ *be such that* $\xi(x) = bx^\beta + \cdots$. *Then the origin of* (1) *with* $\lambda = \hat{\lambda}$ *is monodromic if and only if* $\alpha = 2n - 1$ *is an odd integer,* $a < 0$, *and one of the following conditions holds:*
  (i) $\xi(x) \equiv 0$,
  (ii) $\beta \geq n$,
 (iii) $\beta = n - 1$ *and* $b^2 + 4an < 0$.

4. The Poincaré-Liapunov algorithm

Let (1) be a family of differential systems having a monodromic nilpotent singularity at the origin. Then from Theorem 1 we derive an algorithm to determine necessary conditions on the parameters of the family for having a nilpotent center at the origin. Of course this algorithm is just the well known Poincaré-Liapunov method applied to the larger perturbed family (2) as it is explained in [7]. More specifically, since (2) has an analytic first integral for any $\varepsilon > 0$ there are *focus quantities* $\eta_i(\lambda, \varepsilon)$ and a formal series $H(x, y; \lambda, \varepsilon) = \varepsilon x^2 + y^2 + \cdots$ such that

$$(9) \qquad \qquad \mathcal{X}_\varepsilon(H) = \sum_{i \geq 2} \eta_i(\lambda, \varepsilon)(x^2 + y^2)^i,$$

where $\mathcal{X}_\varepsilon = [y + F_1(x, y; \lambda) + \varepsilon P(x, y)]\partial_x + [-\varepsilon x + F_2(x, y; \lambda) + \varepsilon Q(x, y)]\partial_y$ is the vector field associated to family (2) with arbitrary λ. Then system (2) with $\lambda = \lambda^*$ has a center at the origin for any $\varepsilon > 0$ if and only if $\eta_i(\lambda^*, \varepsilon) \equiv 0$ for all $i \geq 2$.

In practice and using a computer algebra system such as Mathematica we find the first terms of a formal series $H(x, y; \lambda, \varepsilon) = \varepsilon x^2 + y^2 + \sum_{j+k \geq 3} h_{jk}(\lambda, \varepsilon)x^j y^k$ satisfying (9). Equating the terms of homogeneous degree d we get the expressions of $h_{jk}(\lambda, \varepsilon)$ for $j + k = d$ when d is odd, and the functions $h_{jk}(\lambda, \varepsilon)$ for $j + k = d$ and $j \neq 0$, together with the focal value $\eta_{d/2}(\lambda, \varepsilon)$ when d is even.

5. A Counterexample

We will show that the method proposed in [8, 9] fails to detect all nilpotent centers in a nilpotent family. We shall take a family of differential systems having a monodromic nilpotent singularity at the origin and we will prove that if we use the old wrong version of Theorem 1 with P and Q given by (8) with an arbitrary analytic function f having at least second order, then we obtain more restrictions than the necessary and sufficient ones. This will show that the method of [8,9] does not work.

We consider the following polynomial differential systems of degree 7:

$$
\begin{aligned}
\dot{x} &= y + F_1(x, y; \lambda) \\
&= y + Ax^6y + Bx^5y^2 + Cx^4y^3 + Dx^3y^4 + Fx^2y^5 + Gxy^6 + Hy^7, \\
(10) \\
\dot{y} &= F_2(x, y; \lambda) \\
&= -x^7 + Vx^6y + Kx^5y^2 + Lx^4y^3 + Mx^3y^4 + Nx^2y^5 + Pxy^6 + Qy^7,
\end{aligned}
$$

with parameters $\lambda = (A, B, C, D, F, G, H, V, K, L, M, N, P, Q) \in \mathbb{R}^{14}$. In [4] it is proved that the origin is a center if and only if one of the following conditions is satisfied:

(i) Hamiltonian: $V = 3A + K = 5B + 3L = C + M = 3D + 5N = F + 3P = G + 7Q = 0$;

(ii) Time-reversible: $V = B = D = G = L = N = Q = 0$;

(iii) $V = 3A + K = 5B + 3L = 3D + 5N = F + 3P - 2A(C + M) = 5(G + 7Q) + 4L(C+M) = 2KL + 5N = 25P + 4L^2 - 25A(2A^2 + M) = 5Q + L(2A^2 + M) = 0$.

We will see that there are centers in (iii) which are not detected by the method of [8,9] when it is applied with $P(x, y)$ and $Q(x, y)$ given by the misleading expression (8). Performing the Poincaré-Liapunov algorithm on family (10) with $P(x, y)$ and $Q(x, y)$ given by (8) for some $f(x, y; \lambda) = \sum_{j+k \geq 2} f_{jk}(\lambda)x^j y^k$ we obtain $\eta_2(\lambda, \varepsilon) = \eta_3(\lambda, \varepsilon) \equiv 0$ but

$$
\eta_4(\lambda, \varepsilon) = \frac{2\varepsilon[5V + (5B + 3L)\varepsilon + (3D + 5N)\varepsilon^2 + 5(G + 7Q)\varepsilon^3]}{35 + 20\varepsilon + 18\varepsilon^2 + 20\varepsilon^3 + 35\varepsilon^4}.
$$

Imposing $\eta_4(\lambda, \varepsilon) \equiv 0$ for any $\varepsilon > 0$ produces the parameter conditions $V = 5B + 3L = 3D + 5N = G + 7Q = 0$. Clearly the last condition, $G + 7Q = 0$, is a wrong center condition as one can see from the conditions (iii) of one component of the center variety.

6. EXAMPLE

We give the right proof of Proposition 6 of [7].

Proposition 7. *System $\dot{x} = y + x^2 + k_2 xy$, $\dot{y} = -x^3 + k_1 x^2$ has a nilpotent center at the origin if and only if $k_1 = k_2 = 0$.*

Proof. Let $\mathcal{X}_\varepsilon$ be the vector field associated to family (2) with $F_1(x, y; \lambda) = x^2 + k_2 xy$, $F_2(x, y; \lambda) = -x^3 + k_1 x^2$ and parameters $\lambda = (k_1, k_2) \in \mathbb{R}^2$. We define $P(x, y) = \sum_{i+j \geq 2} p_{ij} x^i y^j$ and $Q(x, y) = \sum_{i+j \geq 2} q_{ij} x^i y^j$ two analytic functions of order greater or equal than 2 at the origin with coefficient $p_{02} = 0$. We start the Poincaré-Liapunov algorithm for this family obtaining

$$
\eta_2(\lambda, \varepsilon) = \frac{2}{3 + 2\varepsilon + 3\varepsilon^2}(2k_1 + O(\varepsilon)).
$$

The vanishing of η_2 for any $\varepsilon > 0$ implies the parameter condition $k_1 = 0$. Also, for instance, we can obtain from $\eta_2 \equiv 0$ the values of q_{20}, q_{21} and p_{12}. Next we obtain

$$
\eta_3(\lambda, \varepsilon) = \frac{-1}{6(1 + \varepsilon)(5 - 2\varepsilon + 5\varepsilon^2)}(36k_2 + O(\varepsilon)).
$$

From here we deduce the center condition $k_2 = 0$.

To see that actually system $\dot{x} = y + x^2$, $\dot{y} = -x^3$ has a nilpotent center at the origin it is sufficient to note that the origin is monodromic (just apply to it

Andreev's Theorem 6) and that the system is time-reversible with respect to the involution $(x, y, t) \mapsto (-x, y, -t)$. It is worth to emphasize that system $\dot{x} = y + x^2$, $\dot{y} = -x^3$ has no formal first integral, see [6]. However our method also detect this type of nonintegrable centers. $\qquad\square$

Acknowledgements

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