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NON-ALGEBRAIC INVARIANT CURVES FOR POLYNOMIAL PLANAR VECTOR FIELDS

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Abstract. In this paper we give, as far as we know, the first method to detect non-algebraic invariant curves for polynomial planar vector fields. This approach is based on the existence of a generalized cofactor for such curves. As an application of this algorithmic method we give some Lotka-Volterra systems with non-algebraic invariant curves.

1. **Introduction.** Let us consider a *planar polynomial differential system* of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = P(x,y) = \sum_{k=0}^{m} P_k(x,y) \ , \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y} = Q(x,y) = \sum_{k=0}^{m} Q_k(x,y) \ , \qquad (1.1)$$

in which P, $Q \in \mathbb{R}[x,y]$ are relative prime polynomials in the variables x and y. Moreover P_k and Q_k are homogeneous polynomials of degree k. Throughout this paper we will denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of system (1.1).

There are several papers studying whether some invariant curves of system (1.1) are algebraic, i.e., it can be described implicitly by $f(x,y) = 0$ where f is a polynomial, see for instance [5] and references therein. An invariant curve $f(x,y) = 0$ is an *algebraic invariant curve* of system (1.1) when $f \in \mathbb{C}[x,y]$ and it is irreducible. Let $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$ be the vector field associated to (1.1). It is clear that the orbital derivative $\mathcal{X}f$ should vanish on the algebraic curve $f(x,y) = 0$. On the other hand, since the ideal $< f >$ is radical, then $\mathcal{X}f \in < f >$ and therefore there exists a polynomial $K(x,y) \in \mathbb{C}[x,y]$ of degree less than or equal to $m-1$, called *cofactor* associated to the algebraic invariant curve $f = 0$ such that $\mathcal{X}f = Kf$.

It is known that algebraic invariant curves and integrability have a narrow relationship for planar polynomial systems like it is clearly shown in the Darboux theory. Darboux in [8] showed how first integrals of polynomial systems possessing sufficiently many algebraic invariant curves can be constructed. In short, he proved

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