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First integrals and Darboux polynomials of natural polynomial Hamiltonian systems ☆

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ABSTRACT

In this Letter we study some aspects of the relationship between the existence of Darboux polynomials and additional polynomial first integrals in natural polynomial Hamiltonian systems with a finite number of degrees of freedom. More precisely, first we improve results of the paper of Maciejewski and Przybylska [A.J. Maciejewski, M. Przybylska, Phys. Lett. A 326 (2004) 219]; and after we answer two open questions presented in that paper.

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1. Introduction and main results

Probably the most natural class of Hamiltonian systems which appears in Mechanics are the ones whose Hamiltonian is a sum of the kinetic and potential energy.

In this Letter we consider the polynomial Hamiltonians of the form

$$H(q,p) = T(p) + V(q) = \frac{1}{2}\sum_{i=1}^{m} \mu_i p_i^2 + V(q), \tag{1}$$

where $V(q)$ is a polynomial and $\mu_i \in \mathbb{C}$ for $i = 1, \dots, m$. So the Hamiltonian system defined in $\mathbb{C}^{2m}$ with m degrees of freedom and Hamiltonian (1) is

$$\frac{dq_i}{dt} = \mu_i p_i, \qquad \frac{dp_i}{dt} = -\frac{\partial V}{\partial q_i}, \quad \text{for } i = 1, \dots, m,$$

with positions $q = (q_1, \dots, q_m) \in \mathbb{C}^m$, momenta $p = (p_1, \dots, p_m) \in \mathbb{C}^m$ and $t \in \mathbb{R}$. To avoid the easy linear differential systems we assume that $r = \deg(V) > 2$.

We denote by $\mathcal{X}_H$ the associated Hamiltonian vector field in $\mathbb{C}^{2m}$, i.e.

$$\mathcal{X}_H = \sum_{i=1}^{m} \mu_i p_i \frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \frac{\partial V(q)}{\partial q_i}\frac{\partial}{\partial p_i}.$$

A non-constant polynomial $F \in \mathbb{C}[q,p]$ is a *Darboux polynomial* of the polynomial Hamiltonian vector field $\mathcal{X}_H$ if there exists a polynomial $K \in \mathbb{C}[q,p]$, called the *cofactor* of F such that $\mathcal{X}_H F = KF$. We say that F is a *proper* Darboux polynomial if its cofactor is not identically zero, i.e. if F is not a first integral of $\mathcal{X}_H$.

Given an *involution* $\tau : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ (i.e. a diffeomorphism such that τ^2 is the identity) and a function $F : \mathbb{C}^{2m} \to \mathbb{C}$, we define $F^\tau : \mathbb{C}^{2m} \to \mathbb{C}$ as $F^\tau = \tau_* F = F \circ \tau$. In what follows we denote by σ the involution defined by $\sigma(q,p) = (q,-p)$.

A first integral $I(q,p)$ of the Hamiltonian vector field $\mathcal{X}_H$ is called an *additional first integral* when H and I are functionally independent, i.e. when the gradient vectors of $H(q,p)$ and $I(q,p)$ are linearly independent in $\mathbb{C}^{2m}$ except perhaps in a zero Lebesgue measure set.

Our work is a natural continuation of the interesting paper [5] of Maciejewski and Przybylska, see also [6]. There, the authors stated the following two main results.

Theorem 1. *(See [5].) Assume that in Hamiltonian (1) the potential $V(q)$ is of odd degree, then every Darboux polynomial of its Hamiltonian system is a first integral.*

Theorem 2. *(See [5].) Assume that in Hamiltonian (1) at least two μ_i are not zero and that the potential $V(q)$ is of even degree. If its Hamiltonian system possesses a proper Darboux polynomial, then it admits an additional polynomial first integral.*

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