Joining polynomial and exponential combinatorics for some entire maps

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Abstract

We consider families of entire transcendental maps given by $F_{\lambda,m}(z) = \lambda z^m \exp(z)$ where $m \geq 2$. All these maps have a superattracting fixed point at $z = 0$ and a critical point at $z = -m$. In parameter planes we focus on the capture zones, i.e., we consider λ values for which the critical point belongs to the basin of attraction of $z = 0$. We first show that the Julia set is either a Cantor bouquet or a Cantor bouquet with pinchings depending on the parameter λ. Second we investigate the connection between the dynamics near zero and the dynamics near infinity at the boundary of the immediate basin of attraction of the origin. So we glue the exponential an polynomial behaviour in the same dynamical plane

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1 Introduction

In this paper, we combine symbolic dynamics with polynomial–like theory to investigate the combinatorics of the Julia set of the families of trascendental entire functions

$$F_{\lambda,m}(z) = \lambda z^m e^z, \tag{1.1}$$

with $m \geq 2$ and $\lambda \in \mathbb{C} \setminus \{0\}$. The Julia set of an entire map f, $J(f)$, is the set of points where the family of iterates $\{f^n\}$ fails to be a normal family. Its complement in $\mathbb{C}$ is an open set of the plane known as Fatou set, where the dynamics is tame.

For all functions in (1.1), 0 is a critical and asymptotic value and $F_{\lambda,m}(-m) = \lambda(-m)^m \exp(-m)$ is a critical value. Therefore, these maps belong to a general class of entire transcendental maps with only finitely many critical and asymptotic values also known as *critically finite*. The interest on critically finite maps resides in that they resemble rational maps, as their Fatou set contains neither wandering nor Baker domains (see [7], [13, 14] and [16]). In contrast, the point at infinity plays a crucial role. For instance, the little Picard Theorem says that an entire function in any neighborhood of infinity assumes infinitely many times each value in the complex plane with at most one exception.

The families $F_{\lambda,m}$ have been previously considered in the literature. In [3], Bergweiler considers functions related to $F_{\lambda,m}$ to provide examples of a Baker domain at a positive distance from any singular orbit. In [15], Fagella and Garijo present a thorough analysis of the topology of capture zones in parameter plane and of Julia sets in dynamical plane.

As the exponential family $E_\lambda(z) = \lambda e^z$ is the transcendental version of the quadratic family $Q_c(z) = z^2 + c$, the family $F_{\lambda,2}$ is the transcendental version of the one parameter slice of the cubic family $z \mapsto z^3 - 3a^2 z + b$, given by

$$M_a(z) \;=\; z^3 - \frac{3}{2}\, a z^2.$$

We review some of its properties here, for further details see [18]. It is easy to see that M_a possesses a superattracting fixed point at $z = 0$ and a free critical point at $z = a$. When a belongs to the basin of attraction of the origin, we say that this critical point has been *captured*. The connected components of the parameter space for which this phenomenon occurs are thus called *capture zones*. Define the *main capture zone* as the set of parameter values a for which the free critical point belongs to the immediate basin of the origin. Roughly speaking, with the use of internal and external rays, Milnor showed how the dynamics near zero (conjugated to $\theta \mapsto 2\theta$) meets the dynamics near infinity (conjugated to $\theta \mapsto 3\theta$) at the boundary of the immediate basin of attraction of the origin. In [20] Roesch extended Milnor's results to the family

$$M_{a,m}(z) = z^{m+1} - \frac{m+1}{m}\, a z^m \;.$$

We consider (1.1) as the transcendental version of Roesch's family in the following sense: for all nonzero values of λ, $F_{\lambda,m}$ has a superattracting fixed point at $z = 0$, and a free simple critical point at $z = -m$.

The dynamical behavior of the free critical point will be crucial to describe the structure of the dynamical plane. Basically there are three possibilities. Either $z = -m$ is captured or it is related to a Fatou component different to the basin of $z = 0$, or it is not related to any Fatou component (landing on a periodic repelling point, escaping to infinity, etc.).

In this paper we deal only with parameters in capture zones. The problem of describing the topology of the Julia set for these parameters has been previously solved in a more general setting in [2, 19] where it has been shown that the Julia set can be described as a *Cantor bouquet* with or without *pinchings* (see Section 2 for definitions and results). Nevertheless, this topological description does not give a full understanding of the combinatorics in the Julia set, and this is precisely the main goal of the present work. To do so, we follow Milnor's approach and study how the dynamics near zero meets the dynamics of the *tails* of the Cantor bouquet near infinity. More specifically, in Theorem A we show how the pinchings occur at the boundary of the immediate basin of attraction of the origin. In order to present a clear exposition we restrict its formulation to a particular case and discuss its generalization in Theorem C. Moreover, we completely describe accessible points in the Julia set from the Fatou set in Theorem B.

The outline of the paper is as follows: in Section 2 we give precise definitions of the Cantor bouquet with or without pinchings and describe previous results concerning the topology of $J(F_{\lambda,m})$ and the family $F_{\lambda,m}$. We finish this section with the statements of the two main results of the paper. In Section 4 we prove Theorem A using a polynomial–like construction around the origin and symbolic dynamics. The polynomial–like construction rigourously explains how $F_{\lambda,2}$ acts as $z \mapsto z^2$ around the origin. The proof of Theorem B is found in Section 5. Finally in Section 6 we provide a partial generalization of Theorem A by showing how $F_{\lambda,m}$ acts as z^m around the origin for $m > 2$.

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2 Preliminaries and statement of the results

2.1 The notion of Cantor bouquets with or without pinchings

Nowadays there exist a comprehensive study related to dynamics and topology of Julia sets for entire transcendental maps. Some of the early prime works dealt mainly with complex exponential family $E_\lambda(z) = \lambda e^z$, [11, 10, 6, 21], while some generalizations to other entire transcendental maps were also found in [11] and more recently in [2, 19]. Among several topological results, it is shown that under certain assumptions, the Julia set of an entire transcendental map is a *Cantor bouquet*. Roughly speaking, a Cantor bouquet is a Cantor set of curves extending to infinity in a specific asymptotic direction, each one of them having a distinguished landing point, called the *endpoint*. All points in these curves, except perhaps the endpoints, have forward orbits that tend to infinity along the curves.

The concept of a Cantor bouquet was first introduced in [11] and described in a dynamical setting. Let f be a (critically finite) entire transcendental map and denote by $J(f)$ its Julia set. For a fixed $N \in \mathbb{N}$, denote by

$$\Sigma_N = \{s = (s_0 s_1 s_2 \cdots) \mid s_j \in \{0, 1, 2, \cdots, N-1\} \text{ for each } j\},$$

the space of one-sided infinite sequences of N symbols and by σ the *right shift map* acting on Σ_N.

Definition 2.1. An invariant set C_N of $J(f)$ is an N-*Cantor bouquet* if there is a homeomorphism $h : \Sigma_N \times [1, \infty) \to C_N$ that satisfies the following conditions:

(a) For the projection $\pi : \Sigma_N \times [1, \infty) \to \Sigma_N$,

$$\pi \circ h^{-1} \circ f \circ h(s, t) = \sigma(s).$$

(b) For each $s \in \Sigma_N$,

$$\lim_{t \to \infty} h(s, t) = +\infty.$$

(c) If $t > 1$, then for each $s \in \Sigma_N$,

$$\lim_{j \to \infty} f^j(h(s, t)) = +\infty.$$

Fixing $s \in \Sigma_N$, the curve $\{h(s, t) \mid t > 1\}$ will be called a *tail* and $h(s, 1) = z_s$ is its *endpoint*. The union of the tail with its endpoint is known as a *hair* associated to s.

In order to exemplify this definition, we sketch the construction of an N-Cantor bouquet for E_λ, with $\lambda \in (0, 1/e)$. See [11] for further details. For these parameters there exists a unique attracting fixed point that traps the orbit of the unique asymptotic value $z = 0$, therefore E_λ is *hyperbolic*. The horizontal lines $I_k = \{z \in \mathbb{C} \mid \text{Im}(z) = (2k+1)\pi\}$, $k \in \mathbb{Z}$, are the infinitely many components of the preimage of the negative real line $\mathbb{R}^-$ under E_λ. Denote by M_k the open strip bounded by I_k and I_{k+1}. A point $z \in J(E_\lambda)$ has a well defined *itinerary* $s = (s_0, s_1, \ldots)$ if and only if $E_\lambda^k(z) \in M_{s_k}$, $k \geq 0$. Fix N and take constants ξ and η such that the image under E_λ of each rectangle

$$R_k = \{z \in \mathbb{C} \mid \xi < \text{Re}(z) < \eta,\ (2k+1)\pi < \text{Im}(z) < (2k+3)\pi\}$$

contains $R = \cup R_k$ for $k = 0, 1, \ldots, N-1$. Since the exponential map is expansive on R one can show that each point, with an orbit never escaping from R, has a unique itinerary in Σ_N

and belongs to $J(E_\lambda)$. The set of non-escaping points is shown to be homeomorphic to Σ_N and constitute the endpoints of the bouquet.

Redefining constants ξ and η, it is possible to obtain for each $s \in \Sigma_N$ a sequence of rectangles $R_{s_k} \subset M_{s_k}$ with increasing real part and $R_{s_k} \subset E_\lambda(R_{s_{k-1}})$. Similar arguments show the existance of a single point in R_{s_0} with an orbit escaping to infinity from the right hand direction following s. A continuity argument shows that in fact there exists a continuous curve (or tail) inside M_{s_0} with same dynamics. Appropriate pullbacks of the tail can be done to see how the tail limits to the left at the endpoint associated to s, giving thus a full hair.

As Σ_{N+1} naturally contains Σ_N, it follows that $C_N \subset C_{N+1}$ for each N, and the set

$$\mathcal{C} = \overline{\bigcup_{N \geq 0} C_N}$$

is now called a *Cantor bouquet.* In Figure 1(a) the Cantor bouquet of an exponential map is shown, where all hairs extend to infinity to the right.

Devaney and Tangerman generalized ([11]) above result showing that the Julia set for critically finite transcendental entire maps contains a Cantor bouquet. Let f be an entire transcendental map with finitely many singular values. Let D be an open disk in the plane which contains all of the critical and asymptotic values of f. Let Γ be the complement of D. We call a component of T of $f^{-1}(\Gamma)$ an *exponential tract.*

We fix a ray $\zeta = \zeta(r) = re^{i\theta}$ which is disjoint from T and defined for $r \geq \rho$, and we use the components of the preimage of this ray in T to set up the fundamental domains for the Cantor bouquet. More precisely, let $\gamma_i = \gamma_i(r)$ for $i \in \mathbb{Z}$ denote the pre-images of ζ in T. That is, $\gamma_i(r) = f^{-1}(\zeta(r))$ for an appropriate branch of f^{-1}. We choose the index i in the natural way so that γ_i and γ_{i+1} are adjacent for each i. The curves γ_i and γ_{i+1} bound a strip which serves as a fundamental domain for $f|T$, we also denote this strip by T_i. Let $W_N = \bigcup_{i=-N}^{N} T_i$ and

$$\Lambda_N = \{z \in W_N \mid f^{\circ j}(z) \in W_N \text{ for all } j \geq 0\}.$$

Definition 2.2. $f|T$ has asymptotic direction θ^* if $\gamma_i(r)$ is a C^1-asymptotic to a straight line with direction θ^* for each curve γ_i, defining the fundamental domains.

Definition 2.3. T is a hyperbolic exponential tract if there exist positive constants R_1, α, C such that, if z and $f(z)$ lie in W_N, with $|z| = r \geq R_1$, then

1. $|f(z)| > C\exp(r^\alpha)$.
2. $|f'(z)| > C\exp(r^\alpha)$.
3. $|\mathrm{Arg}(f'(z))| < C\exp(-r^\alpha)$.

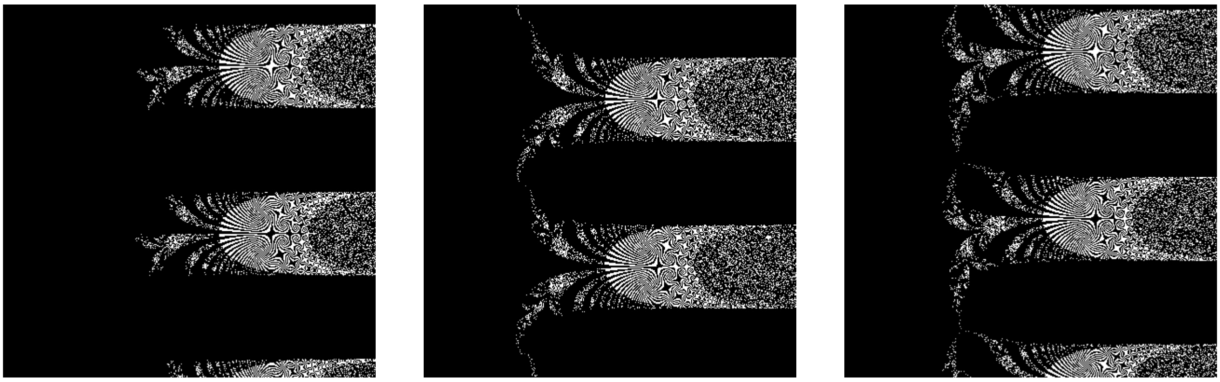
In the next theorem it is shown that the set of points whose orbits remain in T contains a Cantor bouquet ([11]).

Theorem 2.4. *Let $f \in \mathcal{S}$. Let T be a hyperbolic exponential tract on which f has assymptotic direction θ^*. Then, for each N, Λ_N is a Cantor bouquet. Consequently,*

$$J_T(f) = \{z \mid f^{\circ j}(z) \in T \text{ for all } j \geq 0\}$$

contains a Cantor bouquet.

Surprisingly, it was shown in [1] that every Cantor bouquet associated to the exponential family with a completely invariant basin of attraction (and other entire transcendental maps) are in fact homeomorphic to a unique topological abstract model called *straight brush.*



(a) An attracting fixed point for $\lambda = -0.05 + 1.14i$.

(b) An attracting two–cycle for $\lambda = -4 + 1.14i$.

(c) An attracting three–cycle for $\lambda = -1.06 + 1.89i$.

Figure 1: The Julia set for E_λ is shown in white.

Definition 2.5. A *straight brush* $\mathcal{B}$ is a subset of $[1, \infty) \times \mathcal{D}$, with $\mathcal{D}$ a dense subset of the irrational numbers, that satisfies the following properties.

(a) *Hairiness:* If $(y, \alpha) \in \mathcal{B}$ then there exists $y_\alpha \leq y$ such that $(t, \alpha) \in \mathcal{B}$ if and only if $y_\alpha \leq t$. The point (y_α, α) is called the *endpoint* of the *hair* $[t, \infty) \times \{\alpha\}$ at α.

(b) *Density:* the set $\{\alpha \mid (y, \alpha) \in \mathcal{B} \text{ for some } y\}$ is dense in $\mathcal{D}$. Moreover, given any $(y, \alpha) \in \mathcal{B}$ there are sequences $\beta_n \nearrow \alpha$, $\gamma_n \searrow \alpha$ so that the sequences of endpoints (y_{β_n}, β_n) and (y_{γ_n}, γ_n) converge to (y, α) in $\mathcal{B}$.

(c) *Compact sections:* $\mathcal{B}$ is a closed subset of $\mathbb{R}^2$.

In [1] it is shown that any two straight brushes are ambiently homeomorphic, that is there exists a homeomorphism of $\mathbb{R}^2$ taking one straight brush onto the other. This leads to a formal definition of a Cantor bouquet.

Definition 2.6. A *Cantor bouquet* is a compact subset of $\overline{\mathbb{C}}$ that is homeomorphic to a straight brush (with ∞ mapped to ∞).

The construction of the N–Cantor bouquet sketched above and the existence and the abstract model was heavily based on the existence of an attracting fixed point, so the Fatou set concides with its immediate basin of attraction. Recently Barański studied the structure of the Julia set for a large class of maps. More precisely,

Theorem 2.7. *Let f be an entire transcendental function of finite order so that all critical and asymptotic values are contained in a compact subset of a completely invariant attracting basin of a fixed point. Then $J(f)$ consists of disjoint hairs homeomorphic to the half-line* $[0, \infty)$.

Definitions 2.1 and 2.5 describe how each endpoint of a Cantor bouquet has a unique tail attached to it. However, Julia sets of entire transcendental functions may resemble a Cantor bouquet far to the right and with some endpoints becoming landing points of more than one hair. In [4] it is shown that, for the exponential family, the existence of an attracting cycle of period two or higher provides sufficient conditions for multiple landings. See Figure 1(b)–(c). In order to understand this new structure, [5] provided a topological abstract model known as *modified straight brush*. This model is the quotient of the straight brush with a dynamical equivalence relation defined among a subset of its endpoints. In the dynamical setting, we say

Definition 2.8. A Julia set homeomorphic to a modified straight brush is called a Cantor bouquet *with pinchings*.

Examples of Cantor bouquets with pinchings were mostly known among complex exponential and sine families with an attracting periodic cycle. However, in a recent paper, Rempe [19] have shown that these pinchings will occur in a large number of dynamical planes of entire transcendental maps. To do this he study the escaping set, defined as the of points whose orbits tends to infinity. It is well known that the Julia set of an entire transcendental map consist of the boundary of the escaping set.

Precisely he dealt with functions in the set $\mathcal{B}$, that is functions for which the set of critical and asymptotic values is bounded (notice that all functions in the hypothesis of Theorem 2.7 belong to the set $\mathcal{B}$). The main result in [19] is that if two entire transcendental maps in $\mathcal{B}$ are quasiconformally equivalent near infinity, then their escaping set are quasiconformally conjugate near infinity (notice that, fixed m, all functions in family $F_{\lambda,m}$ are quasiconformally equivalent near infinity).

Using this strong rigidity theorem for the escaping set of entire maps, the author is able to prove the following result for *hyperbolic* maps, that is, maps for which all its singular values belongs to an attracting basin of a periodic point (stated as Theorem 1.4 and 5.2 in [19]).

Theorem 2.9. *Let $f, g \in \mathcal{B}$ be two hyperbolic maps of finite order and quasiconformally equivalent near infinity. Assume that g also verity the hypothesis of Theorem 2.7. Then f and g are conjugate on their set of escaping points and the conjugacy extends to a continuous surjective map from $J(g)$ onto $J(f)$.*

As a consequence of all results stated in this section, we will show in Section 3 that the Julia of$F_{\lambda,m}$ is a Cantor bouquet or a Cantor Bouquet with pinchings accordingly with the parameter λ.

2.2 The family $F_{\lambda,m}$

We state without proofs some results on the parameter and dynamical plane of $F_{\lambda,m}$ that we will strongly use in subsequent sections. Recall that $F_{\lambda,m}(z) = \lambda z^m e^z$. Clearly, there exists a superattracting fixed point at $z = 0$ for all choices of λ and $m \geq 2$. Denote its basin of attraction by

$$A(0) = A_{\lambda,m}(0) = \{z \in \mathbb{C} |\, F^n_{\lambda,m}(z) \to 0 \text{ as } n \to \infty\}, \tag{2.1}$$

and its immediate basin by $A^*(0) = A^*_{\lambda,m}(0)$. In [15] it is proved that $A^*(0)$ contains the disc $D_\varepsilon = \{z \in \mathbb{C} \mid |z| < \varepsilon\}$ where ε depends on λ and m. Moreover, $F_{\lambda,m}$ has also a (free) simple critical point at $z = -m$. With this in mind, define a *capture zone* as a connected component in parameter plane given by

$$\mathcal{C}^k_m = \{\lambda \in \mathbb{C} \,|\, F^k_{\lambda,m}(-m) \in A^*(0)\,,\ F^j_{\lambda,m}(-m) \notin A^*(0),\ 0 \leq j \leq k-1\}, \tag{2.2}$$

for each $m \geq 2$ and $k \geq 0$. Any $\lambda \in \mathcal{C}^k_m$ is called a *capture parameter.* In particular

$$\mathcal{C}^0_m = \{\lambda \in \mathbb{C} \,|\, -m \in A^*(0)\}, \tag{2.3}$$

is known as *main capture zone.* The next theorem gathers some of the most important results related to the family $F_{\lambda,m}$ and found in [15].

Theorem 2.10. *The following statements hold for all $m \geq 2$.*

(a) The main capture zone is a bounded component. If $\lambda \in \mathcal{C}^0_m$, then $A(0) = A^*(0)$.

(b) $\mathcal{C}^1_m = \emptyset$. In other words, $-m \in A^*(0)$ if and only if $F_{\lambda,m}(-m) \in A^*(0)$.

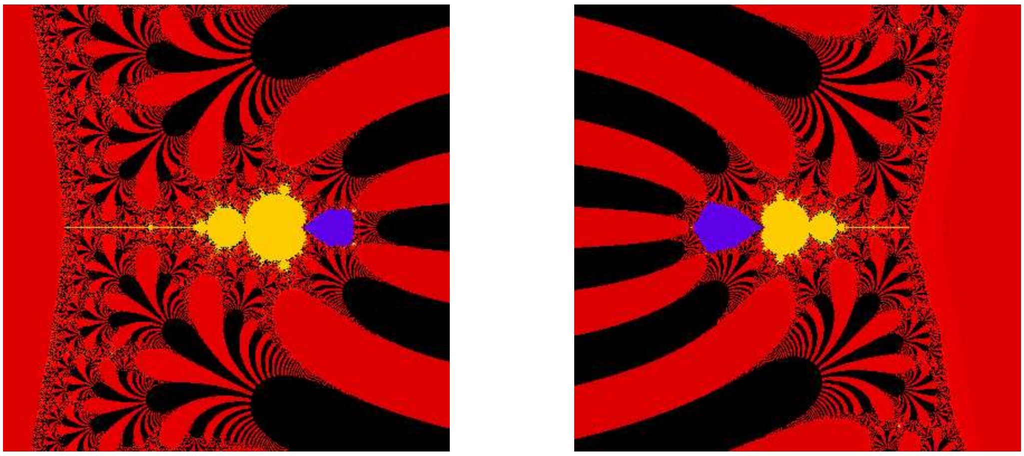
(c) $\mathcal{C}^2_m$ is an unbounded component extending into the left or right hand plane depending on m.

(d) For $k > 2$, $\mathcal{C}^k_m$ has infinitely many connected components extending to infinity in an asymptotic direction.

(e) For $k \geq 2$, if $\lambda \in \mathcal{C}^k_m$ then $A(0)$ has infinitely many connected components. All these components, except $A^*(0)$, are unbounded.

In Figure 2 we illustrate the parameter plane of $F_{\lambda,m}$ for $m = 2$ and 3. The main capture zone is drawn in blue[1] while other capture zones are shown in red. The parameter values for which the orbit of the free critical point does not converge to zero (so, it is *not captured*) but it is bounded are drawn in orange. The parameter values for which the orbit of the free critical point is unbounded are drawn in black. If m is even, capture zones extend to $+\infty$ as the real part of λ tends to $+\infty$, whereas if m is odd these strips extend to $-\infty$ as the real part of λ tends to $-\infty$.

[1]Color plots are available in the online version of this paper. Otherwise, blue is darker than red and orange is light.

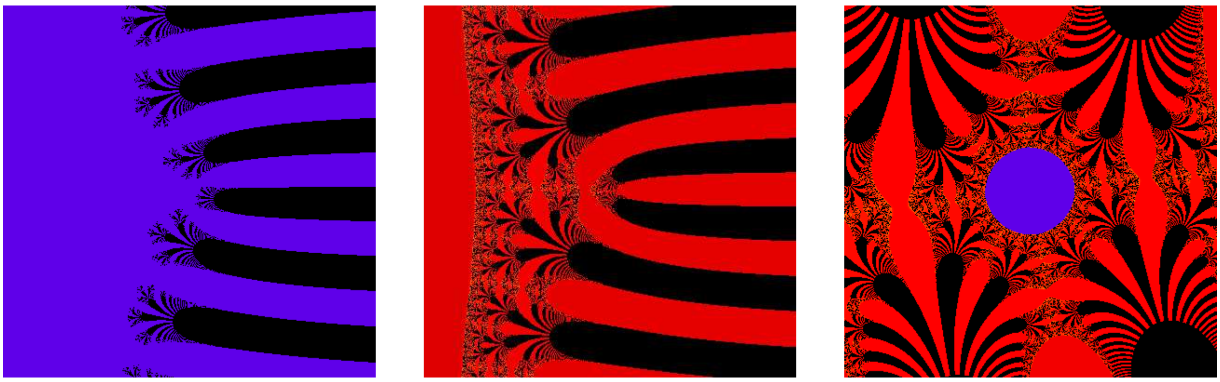


(a) Range $(-24, 8) \times (-16, 16)$.

(b) Range $(-5, 11) \times (-8, 8)$.

Figure 2: Parameter plane for (a) $F_{\lambda,2}$ and (b) $F_{\lambda,3}$. Color codes are explained in the text.

In Figure 3(a)–(c), we display the Julia set of $F_{\lambda,2}$ for two different values of λ (Figure 3(c) is a magnification of Figure 3(b)). The immediate basin of attraction of $z = 0$ is shown in blue (although it is not visible in Figure 3(b)). The connected components $A(0) \setminus A^*(0)$ are shown in red and the Julia set is in black. In Figure 3(a) λ is drawn from $\mathcal{C}_2^0$ and the Julia set is thus a Cantor bouquet. In Figures 3(b) and (c) the λ value belongs to $\mathcal{C}_2^2$ and the Julia set is a pinched Cantor bouquet with pinchings located at the boundary of the immediate basin of attraction of $z = 0$ (and all its preimages).



(a) $\lambda = 0.32 + 0.82i$. Range $(-15, 15) \times (-15, 15)$.

(b) $\lambda = -21+3i$. Range $(-15, 15) \times (-15, 15)$.

(c) $\lambda = -21 + 3i$. Range $(-.2, .2) \times (-.2, .2)$.

Figure 3: The Julia set for $F_{\lambda,2}$. Color codes are explained in the text.

2.3 Statement of the results

Throughout this work, we only deal with parameter values belonging to a capture zone. Under this assumption, it is straightforward to argue that the Fatou set coincides with the basin of attraction of $z = 0$, and consequently, its complement is the Julia set. It is then natural to ask about the topology of this set.

Theorem A Let $\lambda \in \mathcal{C}_m^k,\ m \geq 2,\ k \geq 0$. Then, if $\lambda \in \mathcal{C}_m^0$ then $J(F_{\lambda,m})$ is a Cantor bouquet homeomorphic to a straight brush, and if $\lambda \in \mathcal{C}_m^k,\ k > 1$ then $J(F_{\lambda,m})$ is a pinched Cantor bouquet.

In the light of the above result we know that the Julia set for capture parameters in $\mathcal{C}_m^k$ is either a Cantor bouquet ($k = 0$) or a Cantor bouquet with pinchings ($k \geq 2$). Our main goal in this paper is to explain how these pinchings occur by matching the dynamics of the polynomial map $z \mapsto z^m$ with the dynamics of $z \mapsto \lambda e^z$. The bridge between these dynamical behaviors is build by means of symbolic dynamics and polynomial–like constructions, showing thus how tails of $F_{\lambda,m}$ land, in particular, at the boundary of the basin of attraction of $z = 0$. See Figure 3(c).

For simplicity and clarity in the exposition, we state and prove Theorem A for particular values of m and k. We have chosen them in order to capture the main arguments of the construction without introducing overwhelming notation. In Section 6 we state the general case for arbitrary values of m and k and discuss the refinement of the previous arguments that will constitute its proof.

Theorem B *Let $F_{\lambda,2}(z) = \lambda z^2 e^z$ and assume that $\lambda \in \mathcal{C}_2^3$. Then, the boundary of $A^*(0)$ is a quasi-circle where $F_{\lambda,2}$ on $\partial A^*(0)$ is conjugate to $\theta \mapsto 2\theta$ on the unit circle. Each point in the boundary of $A^*(0)$ is an endpoint. Moreover there exists a domain $\Gamma \subset \mathbb{C}$ such that*

(i) If $F_{\lambda,2}(-2) \notin \Gamma$, then each point in $\partial A^*(0)$ is a landing point of a unique hair, except the fixed point at $\partial A^*(0)$ and all its preimages, which are endpoints of exactly two hairs.

(ii) If $F_{\lambda,2}(-2) \in \Gamma$, then each point in $\partial A^*(0)$ is a landing point of a unique hair, except the two periodic points of period two on $\partial A^*(0)$ and all their preimages, which are endpoints of exactly two hairs.

The third result of the paper is about the accessible points from the Fatou set. Let f be an entire transcendental map. A point z_0 in $J(f)$ is *accessible (from the Fatou set)* if there is a continuous curve $\gamma : [0, 1) \to \mathbb{C}$ for which $\gamma(t)$ lies in the Fatou set for all t and $\lim_{t \to 1^-} \gamma(t) = z_0$. Notice that such a curve must therefore lie in a single component of the Fatou set. The existence and characterization of non–accesible points is an interesting problem by itself that arises not only in the entire transcendental setting (see for example [8, 5]) but also in rational dynamics, [17]. For instance, in the exponential family E_λ, for $\lambda \in (0, 1/e)$, the only accessible points are the endpoints. We can draw a result concerning the set of accessible points of the Julia set from the basin of attraction of 0.

Theorem C *Let $m \geq 2$ and $k \geq 2$. The set of points in $J(F_{\lambda,m})$ that are accessible can be completely characterized as follows.*

(a) If $\lambda \in \mathcal{C}_m^0$, this set coincides with the set of all endpoints.

(b) If $\lambda \in \mathcal{C}_m^k$, this set coincides with the set of endpoints lying in $\partial A^*(0)$ and all its preimages. However, for any natural number N there exists an N–Cantor bouquet that contains only non–accesible points.

3 Proof of Theorem A

The proof of Theorem A follows from the next two propositions. Firstly, if the parameter λ belongs to $\mathcal{C}_m^0$ then the corresponding Fatou set consists in a completely invariant basin of attraction and the Julia set is its complement. The situation is quite similar to the case of the exponential map with an attracting fixed point that we explained in the Section 2.1. Roughly speaking our map acts in a similar way than the exponential map. On one hand in both cases points escape to infinity when their real parts tend to infinity. On the other hand the partition used to label points tending to infinity is defined using the components of the preimage of the negative real line. In Proposition 3.1 we prove that the Julia set is a Cantor bouquet, i.e., the Julia set is homeomorphic to a straight brush. Secondly, if the parameter λ belongs to any other capture zone then the Julia set is a pinched Cantor bouquet, in this case the situation is similar to the case of the exponential map with an attracting k-cycle with $k > 1$.

Proposition 3.1. If $\lambda \in \mathcal{C}_m^0$ then $J(F_{\lambda,m})$ is Cantor bouquet homeomorphic to a straight brush.

Proof. First we find out a hyperbolic tract T in the dynamical plane, following Devaney-Tangerman constructon (see [11], pp. 490-491), in order to prove that the set of points with forward orbit in T contains a Cantor bouquet.

Let Δ_r denote the open disk of radius $r > 0$ centered at the origin. Select r small enough so neither $-m$ nor $F_{\lambda,m}(-m)$ lies in Δ_r. Let n_0 denote the smallest positive integer needed so the bounded component of $F_{\lambda,m}^{-n_0}(\Delta_r)$ contains both the origin and the critical value $z = F_{\lambda,m}(-m)$, but not $\overline{z = -m}$. Call this component D. Let Γ be the complement of D. Notice that for each $k \leq n_0$, $\overline{F_{\lambda,m}^{-k}(\Delta_r)}$ consists of two connected components, while

$$T := \overline{F_{\lambda,m}^{-n_0-1}(\Delta_r)} = \overline{F_{\lambda,m}^{-1}(\Gamma)}.$$

has only one connected component, and $D \cap T = \emptyset$. Clearly T is an exponential tract for $F_{\lambda,m}$ and other exponential tract intersect T. We consider the connected components of the preimage of $\mathbb{R}^-$ outside D. These countable number of curves, denoted by $\sigma_i, i \in \mathbb{Z}$, define a partition of T

(for a precise description and parametrization of these curves see [15] or Section 4 in this paper). Moreover all these curves extend to infinity to the right and have the asymptotic direction $\theta^\star = 0$.

Finally we claim that T is a hyperbolic exponential tract. Since the family behaves like the exponential family far to the rigth we left the details of checking the conditions to the reader.

Thus, from Theorem 3.3 in [11], the Julia set lies on T and *contains* a Cantor bouquet (in the sense of Devaney-Tangerman). Moreover we observe that if $\lambda \in \mathcal{C}_m^0$, the Fatou set is the completely invariant attracting basin of $z = 0$ (a particular case of Baranski's approach). So all points in the Julia set must belong to a unique hair or endpoint (in the tract T). This two results together imply that the Julia set is precisely the Cantor bouquet given in [11].

The rest of the proof shows that the Cantor bouquet is homeomorphic to a straight brush. Since the basic ideas are closed to the construction of the straight brush for the exponential family in [1], we only sckech here the key steps (see also [5] for a straight brush construction) in [1] in our setting.

Fixed $m \geq 2$ and $f \in F_{\lambda,m}$ with $\lambda \in \mathcal{C}_m^0$. Because of the expansivity of f in the tract T, if we take a preimage of T inside T we get infinitely many connected components, denoted in what follows by $H_i, \in \mathbb{Z}$ (one in each of the strips given by the curves σ_i). Precisely it is easy to check that if $z \in H_k$ then

$$(2k+1)\pi - \mathrm{Arg}(\lambda) - \frac{\pi}{2} \leq \mathrm{Im}(z) \leq (2k+1)\pi - \mathrm{Arg}(\lambda) + \frac{\pi}{2}$$

We observe that T is not contained in any half plane (as it happens with the complex exponential family with an invariant basin of attraction) so that the H_i extend arbitrarily far to the left (inside T) as i increases in absolute value. So we define, for each $x \in \mathbb{C}$ and $n \in \mathbb{Z}$, the *squares*:

$$S(x,n) = H_n \bigcup \{z \in \mathbb{C} \mid x \leq \mathrm{Re}(z) \leq x + \pi\}.$$

We notice that either $S(x,n) = \emptyset$ or $S(x,n) \cap J(f) \neq \emptyset$. When $S(x,n) \neq \emptyset$, its image by f is a piece of an annulus cutting across the sets H_i.

(FALTA ACABAR)

□

The following proposition, stated for our family $F_{\lambda,m}$, is a particular case of Corollary 5.3 in [19]. We state and prove it here for completeness.

Proposition 3.2. If $\lambda \in \mathcal{C}_m^k,\ k > 1$ then $J(F_{\lambda,m})$ is a pinched Cantor bouquet.

Proof. Using Rempe's result (see Section 2.1) we can relate the escaping set and the Julia set of two maps as long as these maps are quasiconformally equivalent near infinity. In our case we take two values λ_1, λ_2 with the first value in $\mathcal{C}_m^0$ and the second value in any other different capture zone. Using $\phi_1(z) = \frac{\lambda_2}{\lambda_1} z$ and $\phi_2(z) = z$ we have that $\phi_1 \circ F_{\lambda_1,m} = F_{\lambda_2,m} \circ \phi_2$, proving thus that both maps are quasiconformally equivalents. So we apply Theorem 2.9 to conclude that, if $I(F_{\lambda_1,m})$ and

$I(F_{\lambda_2,m})$ denote the escaping set of $F_{\lambda_1,m}$ and $F_{\lambda_2,m}$ respectively, there exists a homeomorphism $\psi : I(F_{\lambda_1,m}) \mapsto I(F_{\lambda_2,m})$, which extends as a surjective continuous map

$$\bar{\psi} : J(F_{\lambda_1,m}) \mapsto J(F_{\lambda_2,m}).$$

Since by the previous proposition $J(F_{\lambda_1,m})$ is a Cantor bouquet homeomorphic to a straight brush, we conclude that $J(F_{\lambda_2,m})$ is a Cantor bouquet with pinchings (or equivalently, a modified stragiht brush with identifications in some of the non escaping endpoints). □

4 Proof of Theorem B

We begin by showing that the boundary of $A^*_{\lambda,2}(0)$ is a quasi–circle using a polynomial–like construction (see [12] for an excellent exposition on polynomial–like mappings). After that, we use symbolic dynamics to show how the hairs land on $\partial A^*_{\lambda,2}(0)$ and characterize the pinchings.

Before we start the proof, we describe a partition of the dynamical plane derived from the components of the preimage of $\mathbb{R}^-$, for all $m \geq 2$ and any capture parameter λ. Hereafter, denote by $\text{Arg}(\cdot) \in (-\pi, \pi]$ the principal argument. From the expression $F_{\lambda,m}(z) = \lambda z^m e^z$, it is easy to see that

$$\text{Arg}(F_{\lambda,m}(z)) = \text{Arg}(\lambda) + m\text{Arg}(z) + \text{Im}(z) \qquad (\text{ mod } 2\pi),$$

so that, finding components of the preimage of $\mathbb{R}^-$ is equivalent to solve $\text{Arg}(F_{\lambda,m}(z)) = \pi$. Denote $r = |z|$ and $\alpha = \text{Arg}(z)$ so the above equation becomes

$$r = \rho_k(\alpha) = \frac{(2k+1)\pi - m\alpha - \text{Arg}(\lambda)}{\sin(\alpha)},$$

where $\alpha \in (-\pi, \pi)$ and $k \in \mathbb{Z}$. Thus, for fixed λ and m, the component of the preimage of $\mathbb{R}^-$ are given by

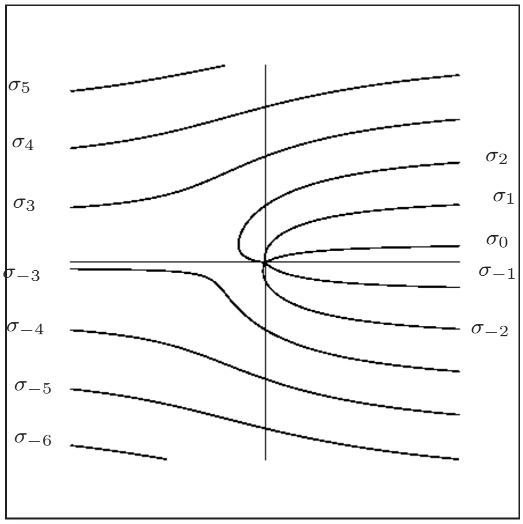
$$\sigma_k = \rho_k(\alpha) e^{i\alpha}. \tag{4.1}$$

In Figure 4 we show some of these curves for $m = 5$. As their real parts tend to $+\infty$, all the σ_k are asymptotic to the horizontal lines $\text{Im}(z) = (2k+1)\pi - \text{Arg}(\lambda)$. There are m of these curves that *start* at the origin (namely, $\sigma_{-j}, \ldots, \sigma_{j-1}$ when $m = 2j$, or $\sigma_{-j}, \ldots, \sigma_j$ when $m = 2j+1$), while all others *start* at $-\infty$.

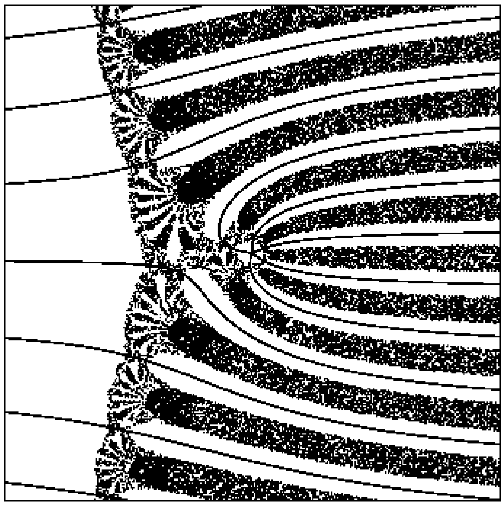
The family of curves σ_k, $k \in \mathbb{Z}$, divides the plane into infinitely many regions or strips. One of these regions, denoted in what follows by W, contains $\mathbb{R}^-$ and is bounded by four of these curves, namely, $\sigma_j, \sigma_{j-1}, \sigma_{-j}$ and $\sigma_{-(j+1)}$ when $m = 2j$, and by $\sigma_{j+1}, \sigma_j, \sigma_{-j}$ and $\sigma_{-(j+1)}$, when $m = 2j+1$. We say that W has two *arms* in the far right–hand side plane and refer to the *upper* and *lower arm* of W in the natural way. All other regions, denoted by M_k, are bounded by σ_k and σ_{k+1} with $k \neq j-1, -j-1$ if $m = 2j$, and $k \neq j, -j-1$ if $m = 2j+1$.

From the above construction, it is clear that

$$F_{\lambda,m} : M_k \to \mathbb{C} \setminus \mathbb{R}^-$$



(a) Graphs of σ_k for $m = 5$.



(b) The Julia set of $F_{\lambda,5}$ lying over the graphs of σ_k.

Figure 4: Strips in the dynamical plane.

is a bijective map for each k. Denote by

$$L_k : \mathbb{C} \setminus \mathbb{R}^- \to M_k \tag{4.2}$$

the inverse of $F_{\lambda,m}$ taking values in each M_k. In contrast, the map

$$F_{\lambda,m} : W \to \mathbb{C} \setminus \mathbb{R}^-$$

is a covering map of degree 2, since contains the critical point $z = -m$ of multiplicity one.

We are now ready to prove the polynomial–like construction.

Proposition 4.1. Let λ be a parameter in $\mathcal{C}_2^3$ and $r > 0$ large enough. Then, there exist U_r and V_r open, bounded and simply connected domains of $\mathbb{C}$, such that $0 \in \overline{U_r} \subset V_r$ and satisfying that $(F_{\lambda,2}, U_r, V_r)$ is a polynomial–like mapping of degree 2. Moreover, the filled Julia set of $(F_{\lambda,2}, U_r, V_r)$ is a quasi–disk and coincides with $\overline{A^*_{\lambda,2}(0)}$.

Proof. Since $\lambda \in \mathcal{C}_2^3$, we have that $F^3_{\lambda,2}(-2)$ lies in the immediate basin of 0. Consider a simple (non closed) curve, completely contained in $A^*_{\lambda,2}(0)$, joining 0 and $F^3_{\lambda,2}(-2)$ such that it is an straight line inside D_ε (a small disc inside $A^*_{\lambda,2}(0)$). Pulling back this curve twice, it defines a simple curve γ in the Fatou set joining $+\infty$ and $F_{\lambda,2}(-2)$ such that it cuts across each straight line $\mathrm{Re}(z) = c$ once, for sufficiently big $c \in \mathbb{R}$. The rest of the proof is divided in three steps, and the third step in two cases.

The first step is to consider the preimage of γ in W. We claim that $\gamma \cap \mathbb{R}^- = \emptyset$. (FALTA PRUEBA)

Since $z = -2$ is a simple critical point, the preimage of γ consists of two simple curves that meet only at $z = -2$. Denote these curves by α and β and note that they extend to $+\infty$ in a different arm of W. To fix notation, let α be the curve extending along the lower arm and β the one extending in the upper arm.

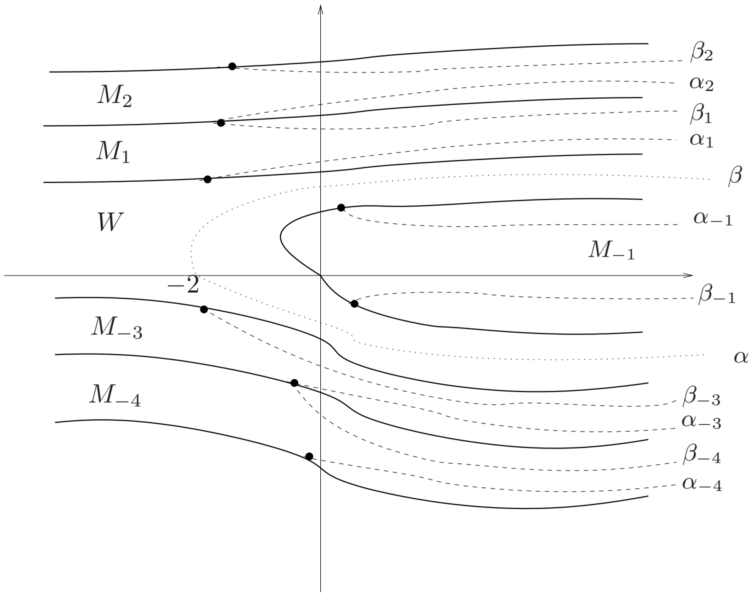


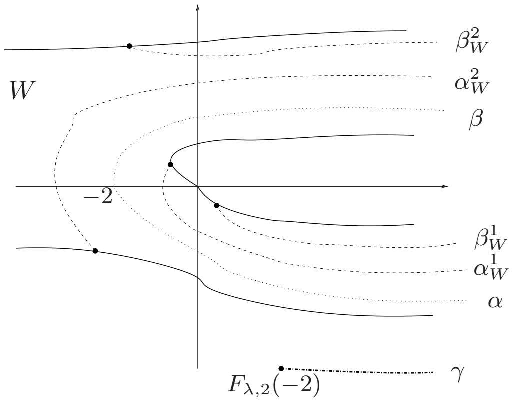
Figure 5: Dynamical plane of $F_{\lambda,2}$ divided into fundamental domains. The curves α and β are drawn in dotted lines. The components of the preimages of α and β in each M_k, denoted by α_k and β_k, are drawn in dashed lines.

The second step is to take the components of the preimage of $\alpha \cup \beta$ in W and on each fundamental domain M_k. Clearly, $F_{\lambda,2}^{-1}(-2)$ consists of infinitely many points, denoted by q_k, and each one lying in a unique σ_k curve. On the other hand, using the inverse branches L_k (4.2), we get exactly one preimage of α and one preimage of β in each domain M_k. We denote them by α_k and β_k, respectively. By continuity, α_k and β_k are joined at the corresponding point q_k. See Figure 5. The components of the preimages of α and β in W are obtained as follows. Since $F_{\lambda,2}$ is a covering map of degree 2 in W, components of the preimages consist of four curves denoted by α_W^i and β_W^i, $i = 1, 2$, extending to $+\infty$ through the lower ($i = 1$) and upper ($i = 2$) arms of W.

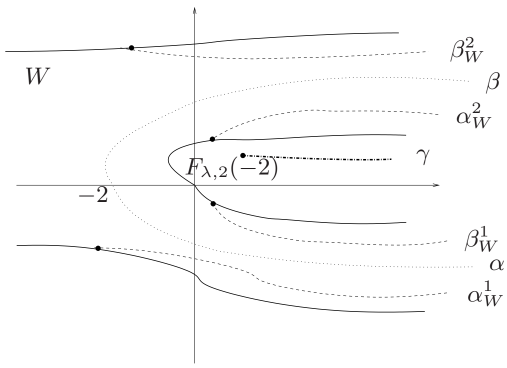
The third step in the polynomial–like mapping construction is to define the sets U_r and V_r of the statement with the desired properties. We denote by Γ the connected component of the complement of $\alpha \cup \beta$ containing the origin. The proof splits in two possible scenarios depending on the location of $F_{\lambda,2}(-2)$ with respect of Γ.

Case 1. $F_{\lambda,2}(-2) \notin \Gamma$. Far to the right, the relative position of α and β with respect to $\{\alpha_W^1, \beta_W^1, \alpha_W^2, \beta_W^2\}$ when going from bottom to top of Γ is either $\{\alpha, \alpha_W^1, \beta_W^1, \beta, \alpha_W^2, \beta_W^2\}$ if $F_{\lambda,2}(-2)$ lies *below* α, or $\{\alpha_W^1, \beta_W^1, \alpha, \alpha_W^2, \beta_W^2, \beta\}$ if $F_{\lambda,2}(-2)$ lies *above* β. To see the claim suppose that

$F_{\lambda,2}(-2)$ lies *below* α (the second case is similar). Let L be a vertical segment far to the right and contained in the lower arm of W. By construction, its bottom and top endpoints are in σ_{-2} and σ_{-1} respectively, so $F_{\lambda,2}(L)$ is a simple curve that surrounds the origin starting and ending at $\mathbb{R}^-$. As we move in L from bottom to top, $F_{\lambda,2}(L)$ travels in a counterclockwise direction, cutting γ, α and β in this order. Their components of the preimage located in the lower arm of W are ordered as α, α_W^1 and β_W^1. For the upper arm of W the arguments are similar. See Figure 6(a).
For any $r > 0$ sufficiently large, define $V_r = \{z \in \Gamma \mid \text{Re } z < r\}$ and let U_r be the connected component of the preimage of V_r containing the origin. Then U_r is bounded by pieces of $\{\alpha_W^1, \beta_W^1, \alpha_{-1}, \beta_{-1}\}$ plus two (almost) vertical lines. Clearly $\overline{U}_r \subset V_r$ and $F_{\lambda,2}$ maps ∂U_r onto ∂V_r in a 2–to–1 fashion. Thus $(F_{\lambda,2}, U_r, V_r)$ is a desired polynomial–like map of degree 2. In Figure 7(a) we illustrate the sets U_r and V_r.



(a) Case 1 in the proof of Proposition 4.1.



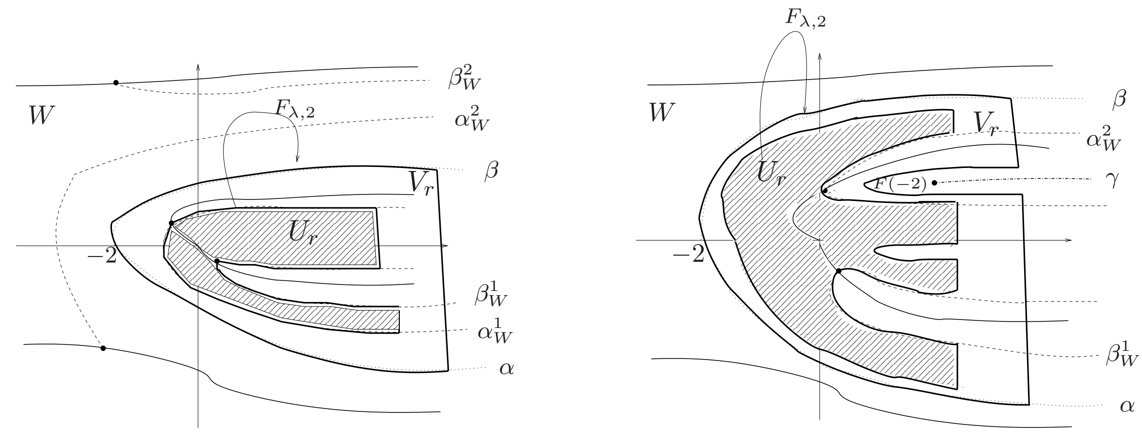
(b) Case 2 in the proof of Proposition 4.1.

Figure 6: Sketch of the relative positions of α and β with respect to their component of the preimage in W.

Case 2. $F_{\lambda,2}(-2) \in \Gamma$. Since $F_{\lambda,2}(-2)$ belongs to the Fatou set and takes two iterates to be in $A^*(0)$, we have that $F_{\lambda,2}(-2)$ must belong to one of the two connected components of $F_{\lambda,2}^{-2}(A^*(0))$ in Γ. In general for each $k \geq 2$ and $m \geq 2$, $F_{\lambda,m}^{-k}(A^*(0))$ consists of finitely many unbounded and connected components extending towards ∞ when m is even, or $-\infty$ when m is odd. We name these type of components *fingers*.

Denote by $\mathcal{F}$ the finger where γ is located. Then, far to the right, the relative position of the curves α and β with respect to $\{\alpha_W^1, \beta_W^1, \alpha_W^2, \alpha_W^2\}$ when going from bottom to top of Γ is $\{\alpha_W^1, \alpha, \beta_W^1, \alpha_W^2, \beta, \beta_W^2\}$ (regardless of the finger where γ is located). See Figure 6(b). The above claim follows easily from similar arguments as in Case 1.
Let r be a positive real number such that $r > 1 + \text{Re }(F_{\lambda,2}(-2))$ and $\gamma \cap (\text{Re}(z) = r)$ is a unique point (see the definition of γ). We define $B_\varepsilon(\gamma)$ to be a closed ε−neighborhood of a piece of γ, completely contained in $\mathcal{F}$. More precisely,

$$B_\varepsilon(\gamma) = \{z \in \mathcal{F} \mid \text{Re }(z) \leq r \text{ and } \text{dist}(z, \gamma) \leq \varepsilon\}, \tag{4.3}$$



(a) Polynomial–like construction for $m = 2$ and $k = 3$ when $F_{\lambda,2}(-2) \notin \Gamma$.

(b) Polynomial–like construction for $m = 2$ and $k = 3$ when $F_{\lambda,2}(-2) \in \Gamma$.

Figure 7: Domains U_r and V_r in Proposition 4.1

.

where $\operatorname{dist}(z, \gamma)$ denotes the natural distance between compact sets. If ε is small enough, we have that $B_\varepsilon(\gamma)$ is simply connected.
Let now $V_r = \{z \in \Gamma \mid \operatorname{Re}(z) < r\} \setminus B_\varepsilon(\gamma)$ and define U_r as the connected component of the preimage of V_r containing the origin. We claim that, for $r > 0$ large enough, $(F_{\lambda,2}, U_r, V_r)$ is the desired polynomial–like mapping of degree 2. To see this, we study the preimage of the boundaries of V_r in Γ. The preimage of the arcs of α and β that bound V_r are arcs of the curves β_W^1, α_{-1}, β_{-1} and α_W^2. The preimage of $B_\varepsilon(\gamma)$ in Γ has two connected components: one is a connected component contained in M_{-1}, and the other is a connected component contained in the intersection of Γ with a small neighbourhood of a piece of $\alpha \cup \beta$. Finally, the preimage of $\Gamma \cap \{z \mid \operatorname{Re} z = r\}$ is the suitable union of (almost) vertical lines that bound U_r from the right. We illustrate this construction in Figure 7(b). Since $\overline{U_r} \subset V_r$ and the map $F_{\lambda,2} : U_r \to V_r$ sends ∂U_r to ∂V_r with degree 2, we conclude that for r large enough, the triple $(F_{\lambda,2}, U_r, V_r)$ is a polynomial–like mapping of degree 2.

Our final step is to show that the filled Julia set of $(F_{\lambda,2}, U_r, V_r)$ is a quasi–disk. By the Straightening Theorem, [12], there exists a quasi–conformal mapping φ that conjugates $F_{\lambda,2}$ to a polynomial Q of degree 2 on the set U_r. That is $(\varphi \circ F_{\lambda,2} \circ \varphi^{-1})(z) = Q(z)$ for all $z \in U_r$. Since $z = 0$ is a superattracting fixed point for $F_{\lambda,2}$ and φ is a conjugacy, we have that $z = 0$ is superattracting for Q. Hence, after perhaps a holomorphic change of variables, we may assume that $Q(z) = z^2$. Thus, the filled Julia set of $(F_{\lambda,2}, U_r, V_r)$ given by

$$K(F_{\lambda,2}) = \{z \in U_r \mid F_{\lambda,2}^n(z) \in U_r \text{ for all } n\} = \bigcap_{n \geq 0} F_{\lambda,2}^{-n}(U_r),$$

coincides with $\overline{A^*(0)}$ and is the image under the quasi–conformal map φ^{-1} of the closed unit disk.

So $\partial A^*(0)$ is a quasi–circle. Using φ we can parametrize $\partial A^*(0)$ so that any point $z \in \partial A^*(0)$ can be written as $z = \varphi^{-1}(\theta)$, or simply $z = z_\theta$, for some $\theta \in \mathbb{S}^1$. Since φ conjugates $F_{\lambda,2}$ on $\partial A^*(0)$ with the map $\theta \mapsto 2\theta$ on $\mathbb{S}^1$, we have that $F_{\lambda,2}(z_\theta) = z_{2\theta}$.

□

Remark 4.2. An important consequence of the previous proposition is that the only points in U_r that never escape under iteration are precisely $\overline{A^*_{\lambda,2}(0)}$, a quasi–disk. Since $J(F_{\lambda,2})$ is a pinched Cantor bouquet it follows that all points in $\partial A^*_{\lambda,2}(0)$ must be endpoints of at least one hair.

What remains to prove involves the use of symbolic dynamics that will allow us to show how pinchings occur at the boundary of $A^*_{\lambda,2}(0)$. As stated in the theorem, this depends on the relative position of $F_{\lambda,2}(-2)$ with respect to the boundaries of Γ.

Case 1. $F_{\lambda,2}(-2) \notin \Gamma$. Fix r large enough and let Ω be the connected component of the preimage of Γ containing 0. Observe that Ω is the union of U_r and two strips H_i, $i = 0, 1$ extending to infinity in the asymptotic direction. Also $\partial\Omega$ (as well as $\partial\Gamma$) belongs to the Fatou set. On the right hand side boundary of U_r (equivalently, the left hand side boundary of H_i) there are infinitely many hairs crossing it and landing at some endpoint inside U_r. This follows from the topological structure of the Julia set and the existence of infinitely many endpoints inside U_r.

Next step is to characterize hairs with endpoints in the boundary of $A^*(0)$. If one of those hairs has an endpoint in U_r but not in $\partial A^*(0)$, it must eventually escape from Ω since the only points in the Julia set with forward orbit inside U_r are the points in $\partial A^*(0)$ (see Remark 4.2). Consequently the hairs with endpoints in $\partial A^*(0)$ must remain under forward iteration in Ω and their tails must also remain in $H_0 \cup H_1$. Since each H_i maps one–to–one into $\Gamma \setminus V_r$ we can associated to each of those tails an itinerary $s \in \Sigma_2$ in the natural way (0 for H_0 and 1 for H_1) and viceversa, i.e., for each sequence $s \in \Sigma_2$ there is a unique tail in Ω with this itinerary. The existence and uniqueness of the tail with a prescribed itinerary follow from the fact that the image by $F_{\lambda,2}(-2)$ of the left hand side boundary of H_i cut across each H_i in an *almost* vertical line, for all r sufficiently large. See [11] (proof of Theorem 3.3), or Proposition 1.2 in [9] for details.

To finish the proof, note that each of the endpoints in $\partial A^*(0)$ has dynamics governed by $\theta \mapsto 2\theta$ and the tails have associated a unique sequence $s \in \Sigma_2$. This will determine how many tails land on each endpoint. Indeed, there are two fixed tails associated to $\bar{0}$ and $\bar{1}$, and a unique fixed point in $\partial A^*(0)$, so both fixed tails must land on it giving two pinched hairs. The preimage of these hairs in Ω contains two new hairs (with sequences $1\bar{0}$ and $0\bar{1}$) landing at the preimage point of the fixed point in $\partial A^*(0)$, thus again given a pinching. Clearly, this type of pinching occurs at each point in the backward orbit of the fixed point restricted to $\partial A^*(0)$.

Any other point not contained in the backward orbit of the fixed point has a unique tail landing on it. To see this, let $z \in \partial A^*(0)$ be one of those points and set $\theta = \varphi(z)$. Then, the itinerary for θ in $\mathbb{S}^1$ is given by its binary expansion whereas from the construction above, the tail landing on z must share the same itinerary with respect to H_0 and H_1. Since there is a unique tail associated to an itinerary $s \in \Sigma_2$, the result follows.

Case 2. $F_{\lambda,2}(-2) \in \Gamma$. Select $r > 0$ large enough and let Ω be the connected component of the preimage of Γ containing 0, minus an ε–neighbourhood of a piece of γ, as in 4.3. Observe that Ω is the union of U_r and four arms H_i, $i = 0, \ldots, 3$ extending to infinity in the asymptotic direction. We know that on the right hand side boundary of U_r there are infinitely hairs crossing it and landing at some endpoint inside U_r, since the boundary of Ω belongs to the Fatou set. As before, the hairs with an endpoint in U_r but not in $\partial A^*(0)$ must eventually escape under forward iteration from Ω, and hairs with endpoint in $\partial A^*(0)$ must remain in Ω, so their tails must remain in $\cup H_i$. However, in this case, the dynamics on those hairs are governed by a subshift of four symbols, since each strip fails to cover $\cup H_i$ under the action of $F_{\lambda,2}$.

Labeling the strips in an increasing order from bottom to top, the transition matrix of the subshift is given by

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

where $a_{ij} = 1$ (respectively, $a_{ij} = 0$) means the strip H_i covers (respectively, does not cover) H_j. For instance, there is a unique fixed sequence $\overline{1}$ and four period two sequences given by $\overline{03}$, $\overline{30}$, $\overline{23}$ and $\overline{32}$. Denote by Σ_A the space of allowed sequences generated by A.

As before, we can associate to any tail in $\cup H_i$ with endpoint in $\partial A^*(0)$ a sequence in Σ_A and viceversa. To finish the proof, note that each of the endpoints in $\partial A^*(0)$ has dynamics governed by $\theta \mapsto 2\theta$. We construct a partition in $\partial A^*(0)$ to determine how many tails land on each endpoint.

From the transition matrix its easy to see the existence of a unique fixed tail (with itinerary $\bar{1}$) that in turn, must land at the unique fixed point in $\partial A^*(0)$. Now, we can compute periodic points of period two in $\partial A^*(0)$. Under the angle doubling map restricted to $\mathbb{S}^1$, these point are $\theta = 1/3$ and $\theta = 2/3$. Using the conjugacy φ^{-1}, denote the corresponding 2–periodic points in $\partial A^*(0)$ by $z_{1/3}$ and $z_{2/3}$. As mentioned before, the four tails of period two have sequences $\overline{03}$, $\overline{30}$, $\overline{23}$ and $\overline{32}$. An easy combinatorial argument shows that the tails associated to $\overline{03}$ and $\overline{32}$ land at a single endpoint of period two whereas the tails associated to $\overline{30}$ and $\overline{23}$ land in the other periodic point. The set of periodic points of period two and their preimages, namely $z_{1/6}, z_{1/3}, z_{2/3}$ and $z_{5/6}$ defines the desired partition on $\partial A^*(0)$. To match the sequences in Σ_A with this partition, we label them in the following way: traveling along $\partial A^*(0)$ in a counterclockwise direction, associate the symbol 0 to the arc joining $z_{2/3}$ and $z_{5/6}$, the symbol 1 to the arc joining $z_{5/6}$ and $z_{1/6}$, 2 to the arc joining $z_{1/6}$ and $z_{1/3}$, and 3 to the arc joining $z_{1/3}$ and $z_{2/3}$. Is left to the reader to check that under $F_{\lambda,2}$ the arc with symbol 0 covers the arc with symbol 3 and so on. Hence, the transition matrix for this partition is exactly A.

As before, each point not in the backward orbits of $z_{1/3}$ and $z_{2/3}$ has a unique itinerary $s \in \Sigma_A$, and by uniqueness of the tails, this point is the landing point of a unique tail associated to s. This concludes the proof of Theorem A.

5 Proof of Theorem B

In this section we prove our result concerning accessibility of points in the Julia set. Part (a) follows directly from Theorem C in [2], since when λ belongs to $\mathcal{C}_m^0$, $F_{\lambda,m}$ has a completely invariant basin of attraction.

Part (b) is shown as follows. First, we will show that in out case a point z_0 in the Julia set is accessible if and only if it belongs to the boundary of some Fatou component. Assume first that z_0 belongs to the boundary of some connected component U in the Fatou set. From the polynomial–like construction (see Proposition 4.1) $\overline{U}$ is a quasi–disk, so there exists a quasi–conformal map $\varphi : \overline{U} \to \overline{\mathbb{D}}$. Setting $\theta_0 = \mathrm{Arg}(\varphi(z_0))$, the curve $\gamma(t) = \varphi^{-1}(t \cdot e^{i\theta_0})$, $t \in [0,1)$, is an accessible path for z_0. The reverse implication is straightforward.

There are points in $J(F_{\lambda,m})$ that are not accessible, as they do not belong to the boundary of any connected component of the Fatou set. In particular any point whose orbit escapes to infinity is non–accessible, and those points are dense in the Julia set. Moreover, any repelling periodic point that does not belong to $\partial A^*(0)$ must be non–accessible. In fact, we show that for any integer $N \geq 0$, there exists a non–accessible and forward invariant N-Cantor bouquet contained in $J(F_{\lambda,m})$.

We use the same notation as in Proposition 4.1. Consider the components of the preimage of $\alpha \cup \beta$ in all the fundamental domains outside $\Gamma \cup W$. These components of the preimage bound simply connected C–shaped regions, denoted by D_k, and induce a natural alphabet $\mathcal{A} = \{\pm 1, \pm 2, \ldots, \pm k, \ldots\}$. For each natural number N, the set C_N given by

$$C_N = \{z \in J(F_{\lambda,m}) \,|\, F_{\lambda,m}^n(z) \in \bigcup_{|k| \leq N} D_k \text{ for all } n \in \mathbb{Z} \text{ and } k \in \mathcal{A}\},$$

is an N-Cantor bouquet of non–accessible points. The fundamental domains D_k, with $k \in \mathcal{A}$, $|k| \leq N$ define in a natural way an itinerary $s = (s_0, s_1, \ldots)$, $s_i \in \mathcal{A}$ and $|s_i| \leq N$ for all points in C_N. For a given s, the only point with bounded orbit that follows s is the endpoint, while those points with unbounded orbit and itinerary s form the tail of the hair. The construction mimics the N–Cantor bouquet construction described in Section 2.

Since all accessible points must lie in the boundary of a Fatou component (and thus eventually enter the domain $\Gamma \cup W$ under iteration), is now straightforward to see that all points in C_N are non–accessible.

6 Generalizations

In the proof of Theorem A we used two main tools: polynomial–like construction and symbolic dynamics. In order to generalize our results for any $m \geq 2$ and $k \geq 2$, we start with the following proposition.

Proposition 6.1. Let $m \geq 2$, $k \geq 2$, $r > 0$ large enough and assume that $\lambda \in \mathcal{C}_m^k$. Then, there exist U_r and V_r open, bounded and simply connected domains of $\mathbb{C}$, such that $0 \in \overline{U_r} \subset V$ and satisfying that $(F_{\lambda,m}, U_r, V_r)$ is a polynomial–like mapping of degree m. Moreover, the filled Julia set of $(F_{\lambda,m}, U_r, V_r)$ is a quasi–disk and coincides with $\overline{A^*_{\lambda,m}(0)}$.

The proof of the above result splits into two cases depending once more on the relative position of the free critical value with respect to Γ. In the case when $F_{\lambda,m}(-m)$ does not lie in Γ, the same arguments as in Proposition 4.1 follow through. Thus, under the dynamics of $\theta \mapsto m\theta$, there exist $m-1$ fixed points in $\partial A^*_{\lambda,m}(0)$. At the same time, there are m fixed tails that must land at those fixed points. Following the same arguments as in Theorem A, case (i), we also have

Theorem C *Let $m \geq 2$, $k \geq 2$ and assume that $\lambda \in \mathcal{C}^k_m$ and $F_{\lambda,m}(-m) \notin \Gamma$. Then the Julia set $J(F_{\lambda,m})$ is a pinched Cantor bouquet. The boundary of $A^*(0)$ is a quasi-circle and each one of its points is an endpoint. $F_{\lambda,m}$ on $\partial A^*(0)$ is conjugate to $\theta \mapsto m\theta$ on the unit circle. Moreover, each point in $\partial A^*(0)$ is a landing point of a unique hair, except for a single fixed point in $\partial A^*(0)$ and all its preimages, which are endpoints of exactly two hairs.*

In the case when $F_{\lambda,m}(-m)$ lies in Γ, the arguments of the polynomial–like construction are again similar, nevertheless we must take into account the orbit of the critical value with respect to Γ. Let k' denote the number of iterates that takes $F_{\lambda,m}(-m)$ to leave Γ for the first time. Since λ is a capture parameter, $1 \leq k' \leq k-2$. Then, after removing k' ε–neighborhoods around $\gamma, F_{\lambda,m}(\gamma), \ldots, F^{k'-1}_{\lambda,m}(\gamma)$ in the definition of V_r, the argument now follows as in Proposition 4.1. The pinchings of the corresponding Cantor bouquet are determined by a transition matrix associated to this construction.

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