CLASSICAL PLANAR ALGEBRAIC CURVES REALIZABLE BY QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. In this paper we show examples of planar quadratic differential systems having some famous planar invariant algebraic curves. We carry out a non exhaustive classification taking into account the degree of the invariant algebraic curve. Also we pay particular attention to the Darboux integrability of the systems.

1. Introduction

Throughout this work we will consider quadratic polynomial differential systems

$$\dot{x} = P(x,y) \ , \qquad \dot{y} = Q(x,y) \ , \tag{1}$$

defined on $\mathbb{R}^2$ or *quadratic systems* for short. This name comes from the restriction that $P, Q \in \mathbb{R}[x,y]$ are polynomials with real coefficients such that $2 = \max\{\deg P, \deg Q\}$. Here the dot denotes, as usual, differentiation with respect to the time t. We also denote $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ the vector field associated to system (1).

Quadratic systems appear very often in several branches of science, as biology, physics, chemistry, mechanics, etc. See for instance [1, 2] for a summary of several properties of these systems.

From the mathematical point of view quadratic systems are perhaps the most simple nonlinear differential systems. Despite its simplicity there are important open questions around them. May be the most important open problem involving quadratic systems is the famous *Hilbert 16th problem* restricted to them. The problem was posed by David Hilbert at the Paris conference of the International Congress of Mathematicians in 1900, together with the other 22 problems. Actually the problem consists of two parts: (i) an investigation of the relative positions of the branches of real algebraic curves of fixed degree; (ii) the determination of the upper bound for the

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number of limit cycles in polynomial differential systems of fixed degree an investigation of their relative positions. We recall that a *limit cycle* is an isolated periodic orbit inside the set of all periodic orbits of system (1). This question appears to be one of the most persistent problems in the Hilbert problem list (see [5]), with the only exception of the Riemann conjecture. Even the simplest case, which is just the restriction of Hilbert 16th problem to the quadratic systems, remains unsolved. Recently it appeared again on Smale's list of problems for the 21st century [15]. See for instance [7] for an account of Hilbert's problems and their sequels and also the excellent survey [9].

In some sense the invariant algebraic curves of system (1) are a connection between the two parts of Hilbert 16th problem. An algebraic curve $f(x, y) = 0$ with $f \in \mathbb{R}[x, y]$ is called an *invariant algebraic curve* of system (1) if it is an invariant set for the flow associated to system (1) in the phase plane. Thus, invariant algebraic curves are formed by the union of some orbits of (1). This implies that $\mathcal{X}$ and the gradient vector ∇f are orthogonal at the points of the curve $f = 0$. In other words $f(x, y) = 0$ is an invariant algebraic curve of system (1) if and only if there is a polynomial $K \in \mathbb{R}[x, y]$ called the *cofactor* such that f is a solution of the linear partial differential equation $\mathcal{X}(f) = Kf$. Moreover analyzing the degrees of the polynomials in the previous equation it is clear that $\deg K \leq 1$.

Let $\mathcal{U}$ be an open subset of $\mathbb{R}^2$. A $\mathcal{C}^k$ function $H : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ such that it is constant on each trajectory of (1) and it is not locally constant is called a $\mathcal{C}^k$ *first integral* of system (1) on $\mathcal{U}$. The equation $H(x, y) = h$ for a fixed $h \in \mathbb{R}$ gives a set of trajectories of the system defined in an implicit way. When $k \geq 1$, H is a first integral if and only if $\mathcal{X}(H) \equiv 0$ in $\mathcal{U}$ and H is not locally constant. The problem of finding such a first integral is what is called the *integrability problem* for system (1). Moreover writing the differential equation of the orbits of system (1) in the Pfaffian form $\omega := P(x, y)\, dy - Q(x, y)\, dx = 0$, an *integrating factor* is defined as a $\mathcal{C}^1$ function $\mu : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$ such that the differential 1-form $\mu\omega$ is closed, i.e. $d(\mu\omega) = 0$, in $\mathcal{U}$. Then in the case in which $\mathcal{U}$ is simply connected, the 1-form $\mu\omega$ is exact, that is $\mu\omega = dH$, and therefore a $\mathcal{C}^2(\mathcal{U})$ first integral $H(x, y)$ of system (1) is constructed. As a consequence the vector field $\mathcal{X}$ is topologically equivalent in $\mathcal{U}$ to the Hamiltonian vector field $\mu\mathcal{X}$. For a survey about integrating factors and their properties you can see the paper [4].

Invariant algebraic curves also play an important role in the integrability theory of polynomial differential systems. In 1878 Darboux in [3] presented a simple method to construct first integrals and integrating factors for planar polynomial differential systems having a sufficient number of invariant algebraic curves. We refer to the survey [10] and references therein about this theory. In its simplest form the *Darboux integrability theory* works as follows. Assume that a planar polynomial vector field $\mathcal{X}_m$ of degree m possesses q irreducible invariant algebraic curves $f_i(x, y) = 0$ with associated cofactors $K_i(x,y)$ for $i = 1,\ldots,q$ and consider the Darbouxian function

$$\prod_{i=1}^{q} f_i^{\lambda_i}(x,y) \,, \tag{2}$$

for suitable $\lambda_i \in \mathbb{R}$. Then the following holds:

- If $q \geq m(m+1)/2 + 1$, then (2) is a first integral of $\mathcal{X}_m$.
- If $q = m(m+1)/2$ then (2) is a first integral or an integrating factor of $\mathcal{X}_m$.
- If $q < m(m+1)/2$ but either $\sum_{i=1}^{q} \lambda_i K_i = 0$ or $\sum_{i=1}^{q} \lambda_i K_i = -\mathrm{div}\mathcal{X}_m$ then (2) is a first integral or an integrating factor of $\mathcal{X}_m$, respectively.

We remark that if $\lambda_i \in \mathbb{Z}$ for all $i = 1,2,\ldots,q$, then (2) is a rational function and system (1) is called *rationally integrable* in case that (2) is a first integral. In [11] Poincaré stated the problem of determining when $\mathcal{X}_m$ has a rational first integral. In this sense Jouanoulou [6] showed that if at least $m(m+1)/2 + 2$ different irreducible invariant algebraic curves are known, then there exists a rational first integral. On the other hand, the works [12] and [14] give a characterization of when $\mathcal{X}_m$ has a first integral which is an elementary or Liouvillian function.

Darboux integrability theory has also been useful for studying different relevant problems of planar polynomial differential systems such as problems related to centers, limit cycles, and bifurcation problems, see [13].

These notes are devoted to show some examples of quadratic systems having "classical" algebraic curves as invariant curves. We use the word classical in the sense of famous. See for instance a famous curves index in the web page `http://www-history.mcs.st-andrews.ac.uk/Curves/Curves.html` or in the book [8].

We say that a given algebraic curve is *realized* by a quadratic system when there is a system (1) possessing it as invariant algebraic curve. We have opted to carry out a classification of classical algebraic curves realizable by quadratic systems, which is not exhaustive, taking into account the degree of the invariant algebraic curve.

The structure of the paper is the following. In Section 2 we present some classical algebraic curves realizable by quadratic systems. In all the studied cases we give the quadratic differential system (1), the realized invariant algebraic curve $f(x,y) = 0$ and its associate cofactor $K(x,y)$. Also we pay attention to the integrability of the systems distinguishing two integrability types: rationally integrable systems and Hamiltonian systems. In Section 3 we organize in tables the classification made according to the degree of the curves. In these tables we denote by (**RI**), (**DI**) and (**DIF**) the rationally integrable systems, and the systems having either a Darboux first integral or a Darboux integrating factor, respectively. All the planar algebraic curves in the forthcoming tables are given mainly in the implicit form $f(x,y) = 0$ although some of them are given in parametric form $x = F(\tau)$, $y = G(\tau)$, or in polar coordinates form $r = F(\theta)$. In the final section we plot several pictures of some realizable classical curves by quadratic systems studied in section 2.

2. Realizable classical algebraic curves for quadratic systems

To check whether an algebraic curve is realized by a quadratic system is just a matter of linear algebra. This direct method consists in to use arbitrary real coefficients a_{ij}, b_{ij} and k_{ij} for both the quadratic vector field $\mathcal{X}$ and the cofactor K. Thus we take $P(x,y) = \sum_{i+j=0}^{2} a_{ij}x^iy^j$, $Q(x,y) = \sum_{i+j=0}^{2} b_{ij}x^iy^j$ and $K(x,y) = \sum_{i+j=0}^{1} k_{ij}x^iy^j$, and we impose that $\mathcal{X}(f) = Kf$ to be satisfied. This procedure leads to a system of algebraic equations for the unknowns a_{ij}, b_{ij} and k_{ij}. In case that this algebraic system becomes compatible we get the realizable invariant algebraic curve as well as the quadratic system. Otherwise the curve is non realizable, see Tables 2 and 3.

Along this section we present some classical algebraic curves realizable for quadratic systems. In all cases we give the quadratic differential system (1), the realized invariant algebraic curve $f(x,y) = 0$ and its associate cofactor $K(x,y)$.

2.1. **The trivial case.** First of all we remark that any cubic algebraic curve $f(x,y) = 0$ is always realized by the quadratic Hamiltonian system

$$\dot{x} = \frac{\partial f}{\partial y}\ ,\ \dot{y} = -\frac{\partial f}{\partial x}\ .$$

For example, this is the case of the cubic curves in Table 1.

Name	Curve $f(x,y) = 0$
Slüse's Concoid	$f(x,y) = a(x-a)(x^2+y^2) + k^2x^2$
Cramer's Curve	$f(x,y) = x(x^2+y^2) + (r+l)x^2 - (r-\ell)y^2.\ r > \ell > 0$
Oblique Estrofoide	$f(x,y) = x(x^2+y^2) - a[(x^2-y^2)\sin t + 2xy\cos t]$
Folium of Descartes	$f(x,y) = x^3 + y^3 - 3axy$
Ofiúrida	$f(x,y) = x(x^2+y^2) - y(cy - bx).\ b > 0,\ c > 0$
Panestrofoide	$f(x,y) = x(x^2+y^2) + g(x^2-y^2) + k(x+g)$
Visiera	$f(x,y) = (2y-a)(x^2+y^2) - ay^2.\ a > 0$
Oblique Versiera	$f(x,y) = (x^2+y^2)(x\cos\alpha + y\sin\alpha - 4r\cos^2\alpha) + 2ry^2$

Table 1. Some third degree classical algebraic curves realizable by quadratic Hamiltonian systems.

Anyway we emphasize the possibility of having particular cases in Table 1 realized by a non Hamiltonian quadratic system in the following example.

Oblique Versiera: $f(x,y) = (x^2+y^2)(x\cos\alpha + y\sin\alpha - 4r\cos^2\alpha) + 2ry^2 = 0$ with $r > 0$ and $0 < \alpha < \pi/2$ has a multiple point at the origin. In short the origin is a node if $\alpha > \pi/4$, an isolated point if $\alpha < \pi/4$ and a cusp if $\alpha = \pi/4$. Taking the particular case $\alpha = \pi/4$ and renaming the new parameter $r = \sqrt{2}R$ we obtain that $f(x,y) = 0$ is realized by the following quadratic system

$$
\begin{array}{rcl}
\dot{x} & = & 3R(k_2 - 3k_1)x + (2k_1 - k_2)x^2 + (k_1 - k - 2)xy - k_2y^2 \ , \\
\dot{y} & = & R(k_1 - 3k_2 9x + 2R(k_2 - 3k_1)y + k_2x^2 + 2k_1xy + k_1y^2 \ ,
\end{array} \tag{3}
$$

with associated cofactor $K(x,y) = (k_2 - 3k_1)(5R - 2x - y)$.

2.2. **Some fourth and higher degree classical algebraic curves non realizable by non Hamiltonian quadratic systems.** In this section we do one explicit computation to show the meaning of Tables 2 and 3. For example, we take the Ampersand Curve $f(x,y) = (y^2 - x^2)(x-1)(2x-3) - 4(x^2 + y^2 - 2x)^2 = 0$, the vector field $\mathcal{X} = \left(\sum_{i+j=0}^{2} a_{ij}x^iy^j\right)\partial_x + \left(\sum_{i+j=0}^{2} b_{ij}x^iy^j\right)\partial_y$ and the cofactor $K(x,y) = \sum_{i+j=0}^{1} k_{ij}x^iy^j$, with arbitrary real coefficients a_{ij}, b_{ij} and k_{ij}, and after equating the coefficients in the monomials x^iy^j of the equation $\mathcal{X}(f) = Kf$ we get

- From the monomial x: $a_{00} = 0$;
- From the monomial y: $b_{00} = 0$;
- From the monomial x^2: $k_{00} = 2a_{10}$;
- From the monomial y^2: $b_{01} = a_{10}$;
- From the monomial xy: $b_{10} = 19a_{01}/3$;
- From the monomial x^3: $k_{10} = (-21a_{10} + 38a_{20})/19$;
- From the monomial x^2y: $k_{01} = (-607a_{01} + 114a_{11} - 18b_{20})/57$;
- From the monomial xy^2: $b_{11} = (361a_{02} - 136a_{10} + 57a_{20})/57$;
- From the monomial y^3: $b_{20} = (-136a_{01} + 19a_{11} - 19b_{02})/3$;
- From the monomial x^4: $a_{20} = -71a_{10}/133$;
- From the monomial x^3y: $b_{02} = (-3523a_{01} + 607a_{11})/544$;
- From the monomial x^2y^2: $a_{10} = 80731a_{02}/23224$;
- From the monomial xy^3: $a_{11} = 461a_{01}/33$;
- From the monomial y^4: $a_{02} = 0$;
- From the monomial x^4y: $a_{01} = 0$.

With these parameter restrictions the vector field reduce to the constant field $\mathcal{X} = 0$, a particular case of Hamiltonian field.

2.3. **Non Hamiltonian quadratic systems having a classical cubic invariant curve.** In this section we do the explicit computations in order to get Table 4.

Witch of Agnesi: The quadratic system

$$
\dot{x} = a^2k_1 + \frac{ak_2}{2}x + k_1x^2 \ , \quad \dot{y} = -ak_2y - 2k_1xy + k_2y^2 \ , \tag{4}
$$

has the Witch of Agnesi or “versiera” $f(x,y) = (a^2 + x^2)y - a^3 = 0$ as invariant algebraic curve with associated cofactor $K(x,y) = k_2y$. The curve has, see Figure 1, points of inflection at $y = 3a/4$ and the invariant straight line $y = 0$ is an asymptote to the curve. A rational parametrization of Agnesi's curve is given by

$$x(\tau) = a\tau \ , \quad y(\tau) = \frac{a}{1+\tau^2} \ .$$

Making the reparametrization $\tau \mapsto t$ defined by

$$\tau(t) = \frac{1}{4k_1}\left[-k_2 + \Delta \tan\left(\frac{a\Delta t}{4}\right)\right] \ , \quad \text{where} \ \ \Delta := \sqrt{16k_1^2 - k_2^2} \ ,$$

we obtain the flow into Agnesis's curve induced by (4).

Cubic Duplicatriz: The algebraic curve $f(x,y) = x^3 - \ell(x^2+y^2) = 0$ is realized by the quadratic system

$$\begin{array}{lrcl} & \dot{x} & = & -k_1\ell x - \dfrac{(k_2-2k_3)\ell}{3}y + k_1x^2 + k_2xy \ , \\ (5) & & & \\ & \dot{y} & = & \dfrac{(k_2-2k_3)\ell}{3}x - k_1\ell y + k_3x^2 + \dfrac{3k_1}{2}xy + \dfrac{3k_2}{2}y^2 \ , \end{array}$$

with cofactor $K(x,y) = -2k_1\ell + 3k_1x + 3k_2y$. The origin is an isolated point and the x-axis is a symmetry axis for the curve. Moreover, $(4\ell/3, \pm 4\ell/[3\sqrt{3}])$ are inflection points, see Figure 2.

Parabolic Folium: The parabolic folium $f(x,y) = x^3 - bxy - a(x^2-y^2) = 0$ is realized with cofactor $K(x,y) = -2(4a^2+b^2)(3ak_1+bk_2) + 12a^2(3k_1x + 2k_2y)$ by the quadratic system

$$\begin{array}{lrcl} & \dot{x} & = & \delta x + 12a^2(k_3y + k_1x^2) + 8a^2k_2xy \ , \\ (6) & \dot{y} & = & 12a^2k_3x + (\delta + 12abk_3)y - (3abk_1 + 4a^2k_2 + b^2k_2 + 18ak_3)x^2 \\ & & & +2a(9ak_1 - bk_2)xy + 12a^2k_2y^2 \ , \end{array}$$

where $\delta := -12a^3k_1 - 3ab^2k_1 - 4a^2bk_2 - b^3k_2 - 6abk_3$. The parabolic folium has a singular point of node type at the origin, see Figure 3. Moreover, this curve admits the following polynomial parametrization

$$x(\tau) = (1-\tau)^2a + b\tau \ , \quad y(\tau) = \tau(1-\tau^2)a + b\tau^2 \ .$$

In the particular case $b = 0$, the curve is known as right parabolic folium. In order to simplify computations, if we assume moreover $k_1 = k_2 = 0$ and $a = 1 = k_3 = 1$, system (6) reduce to the Hamiltonian system $\dot{x} = 12y$, $\dot{y} = 6(2-3x)x$. In this case, using the reparametrization

$$\tau(t) = \frac{1}{\exp(6t) - 1} \ ,$$

we obtain the flow into the right parabolic folium induced by the Hamiltonian system.

Newton's Serpentine: The quadratic system

$$\begin{aligned} \dot{x} &= -a^2(k_1+k_2)+k_1x^2 \ , \\ \dot{y} &= \frac{1}{c}\left[-ac^2(k_1+k_2)+ck_2xy+4ak_1y^2+2ak_2y^2\right] \ , \end{aligned} \tag{7}$$

with $c \neq 0$ has the Newton's serpentine $f(x,y) = y(x^2+a^2) - acx = 0$ as invariant curve with cofactor $K(x,y) = \frac{1}{c}\left[(2k_1+k_2)(cx+2ay)\right]$. Newton's serpentine has the origin as a symmetry center and the straight line $y = 0$ is an asymptote to the curve, see Figure 4. Moreover,

$$\mu(x,y) = \frac{[-a^2(k_1+k_2)+k_1x^2]^{1+k_2/(2k_1)}}{[-acx+(a^2+x^2)y]^2} \ ,$$

is an integrating factor for system (7).

Pseudoversiera: The cubic curve $f(x,y) = (a^2+x^2)y - 2a^3 = 0$ is realized by system

$$\dot{x} = a^2k_1 + ak_2x + k_1x^2 \ , \quad \dot{y} = y(-2ak_2 - 2k_1x + k_2y) \ , \tag{8}$$

with associate cofactor $K(x,y) = k_2y$. Pseudoversiera has the straight lines $y = 0$ and $x = 0$ as asymptote and symmetry axis respectively and the points $(\pm a/\sqrt{3}, 3a/2)$ are infection points, see Figure 5. Moreover, system (8) has the following Darboux first integral

$$H(x,y) = \left[\frac{a}{2}(k_2+\sqrt{\Delta})+k_1x\right]^{\Delta-k_2\sqrt{\Delta}} \left[\frac{a}{2}(k_2-\sqrt{\Delta})+k_1x\right]^{\Delta+k_2\sqrt{\Delta}} y^{\Delta} f^{-\Delta}(x,y) \ ,$$

where $\Delta := k_2^2 - 4k_1^2$.

2.4. **Quadratic systems having a classical quartic invariant curve.** In this section we do the explicit computations summarized in Table 5.

Oblique Bifolium: The quadratic system

$$\begin{aligned} \dot{x} &= 3b^3x + 6abx^2 - 8(3a^2+2b^2)xy - 2aby^2 \ , \\ \dot{y} &= -ab^2x + 2b^3y + 2(3a^2+2b^2)x^2 + 8abxy - 6(3a^2+2b^2)y^2 \ , \end{aligned} \tag{9}$$

has the invariant oblique folium $f(x,y) = -x^2(ax+by) + (x^2+y^2)^2 = 0$ with cofactor $K(x,y) = 8[b^3 + 3abx - 3(3a^2+2b^2)y]$.

Right Bifolium: The curve $f(x,y) = -ax^3 + (x^2+y^2)^2 = 0$ is realized by system

$$\begin{aligned} \dot{x} &= -\frac{3}{4}akx + \frac{3}{4}kx^2 - 4\ell xy - \frac{1}{4}ky^2 \ , \\ \dot{y} &= -\frac{9}{16}aky + \ell x^2 + kxy - 3\ell y^2 \ , \end{aligned} \tag{10}$$

with cofactor $K(x,y) = \dfrac{3}{4}(-3ak + 4kx - 16\ell y)$.

BOW: The quadratic system

$$\dot{x} = x(2 - 9y) \ , \quad \dot{y} = 2(x^2 + y - 6y^2) \ , \tag{11}$$

possesses the invariant Bow $f(x, y) = x^4 - x^2y + y^3 = 0$ with cofactor $K(x, y) = 6(1 - 6y)$. Moreover, system (11) has the rational first integral

$$H(x, y) = \frac{x^4(8x^2 + 27x^4 - 54x^2y - 9y^2 + 54y^3)}{f^2(x, y)} \ .$$

CARDIOID: The curve $f(x, y) = (x^2+y^2-ax)^2-a^2(x^2+y^2) = 0$, see Figure 6, is invariant for the quadratic system

$$\dot{x} = -2akx+a\ell y+kx^2+4\ell xy-3ky^2 \ , \quad \dot{y} = -3\ell x^2-3aky+4kxy+\ell y^2 \ , \tag{12}$$

and $K(x, y) = 2(-3ak + 2kx + 2\ell y)$ is the associated cofactor.

CAMPILA: The algebraic curve $f(x, y) = (x^2 + y^2) - a^2x^4 = 0$ is realized by the system

$$\dot{x} = xy \ , \quad \dot{y} = x^2 + 2y^2 \ , \tag{13}$$

with associate cofactor $K(x, y) = 4y$. The Campila curve has an isolated point at the origin and the straight lines $y = 0$ and $x = 0$ are symmetry axis, see Figure 7. A rational first integral for system (13) is given by

$$H(x, y) = \frac{x^4}{x^2 + y^2} \ .$$

On the other hand, Campila curve can be parameterized by means of

$$x(\tau) = \frac{1}{a}\frac{1}{\cos\tau} \ , \quad y(\tau) = \frac{1}{a}\frac{\sin\tau}{\cos^2\tau} \ .$$

KÜLP'S CONCOID: The quadratic system

$$\dot{x} = -xy \ , \quad \dot{y} = a^2 + y^2 \ , \tag{14}$$

admits the polynomial Külp's Concoid first integral

$$H(x, y) = -a^2(a^2 - x^2) + x^2y^2 \ .$$

CONCHAL: The quadratic system

$$\dot{x} = (a + x)y \ , \quad \dot{y} = 2ax - 2x^2 - y^2 \ , \tag{15}$$

possesses the polynomial Conchal first integral

$$H(x, y) = (a + x)^2[(x - a)^2 + y^2] - a^2k^2 \ .$$

Conchal curve $H(x, y) = 0$ possesses, depending on the value of their parameters, a very different topology. In short, if $k = a$ the origin is a node and see Figures 9 and 10 for $k \neq a$.

CURVE ANTIVERSIERA: The algebraic curve $f(x, y) = -2rx^3+x^4+4r^2y^2 = 0$ is realized by the system

(16)

$$\dot{x} = -2k_1rx+4k_2ry+k_1x^2-8k_2xy \ , \quad \dot{y} = -3k_1ry+3k_2x^2+2k_1xy-16k_2y^2 \ ,$$

with associated cofactor $K(x,y) = -6k_1r + 4k_1x - 32k_2y$.

STEINER'S CURVE: The quadratic system

$$\begin{aligned} \dot{x} &= 9k_1r^2 + 6r(k_1x - k_2y) - 3k_1x^2 - 4k_2xy + k_1y^2 \ , \\ \dot{y} &= 9k_2r^2 - 6r(k_2x + k_1y) + k_2x^2 - 4k_1xy - 3k_2y^2 \ , \end{aligned} \tag{17}$$

has the invariant Steiner's Curve $f(x,y) = -27r^4 + 18r^2(x^2+y^2) + (x^2+y^2)^2 + 8rx(3y^2 - x^2) = 0$. Moreover, $\mu(x,y) = f^{-5/6}(x,y)$ is a Darboux integrating factor for system (17).

SIMPLE FOLIUM: The quadratic system

$$\dot{x} = 3k_1(3+r)x - 3k_1x^2 - 4k_2xy + k_1y^2 \ , \quad \dot{y} = k_2x^2 + 9k_1y - 4k_1xy - 3k_2y^2 \ , \tag{18}$$

possesses the invariant Simple Folium $f(x,y) = -4rx^3 + (x^2+y^2)^2$ with cofactor $K(x,y) = 12(3k_1 - k_1x - k_2y)$.

MONTFERRIER'S LEMNISCATE: The curve $f(x,y) = x^2(x^2-a^2) + b^2y^2 = 0$ is realized by system

$$\dot{x} = b^2xy \ , \quad \dot{y} = -a^2x^2 + 2b^2y^2 \ , \tag{19}$$

with associated cofactor $K(x,y) = 4b^2y$. The straight lines $y = 0$ and $x = 0$ are symmetry axis and the origin is a node with tangents $ax \pm by = 0$, see Figure 14. A rational first integral for system (19) is given by $H(x,y) = x^4/f(x,y)$.

PEAR CURVE: The quadratic system

$$\dot{x} = (x-r)(y-r) \ , \quad \dot{y} = (r-2y)y \ , \tag{20}$$

possesses the Pear invariant curve $f(x,y) = r^4 - 2r^3y + (x-r)^2y^2 = 0$ with cofactor $K(x,y) = -2y$. The Pear curve has the straight line $x = r$ as symmetry axis and asymptote, see Figure 15. On the other hand, system (20) admits the rational first integral $H(x,y) = (r-2y)/[(x-r)^2y^2]$. A parametrization of the Pear $f(x,y) = 0$ is given by

$$x(\tau) = r(1+\cos\tau) \ , \quad y(\tau) = \frac{r}{1+\sin t} \ .$$

VIRTUAL PARABOLA: System

$$\begin{aligned} \dot{x} &= (d-b)(bc+ad) + [bc^2 - da^2 + 3ac(d-b)]x + 2ac(c-a)x^2 \\ &\quad +(bc-ad)y^2 \ , \\ \dot{y} &= y[bc^2 - da^2 + ac(c-a)x] \ , \end{aligned} \tag{21}$$

has the invariant virtual parabola $f(x,y) = y - [ax+b]^{1/2} - [cx+d]^{1/2} = 0$ with associate cofactor $K(x,y) = 2(c-a)(bc+ad+2acx)$. In addition, system (21) possesses the rational first integral

$$H(x,y) = \frac{y^2[(a+c)(bc+ad) + 2ac(a+c)x - acy^2]}{f(x,y)} \ .$$

3. The classification tables

Name	Curve $f(x,y)=0$
Ampersand Curve	$f(x,y)=(y^2-x^2)(x-1)(2x-3)-4(x^2+y^2-2x)^2$
Bicorn	$f(x,y)=(x^2+2ay-a^2)^2-y^2(a^2-x^2)$
Bicusp	$f(x,y)=(x^2-a^2)(x-a)^2+(y^2-a^2)^2$
Bifoliate	$f(x,y)=x^4+y^4-2axy^2$
Bullet Nose	$f(x,y)=1/x^2-1/y^2-1$
Durero's Concoid I	$f(x,y)=(xy+b^2-y^2)^2-(x+y-a)^2(b^2-y^2)$
Durero's Concoid II	$f(x,y)=x^2y^2-a^2(a^2-x^2)$
Leaf of Clover	$f(x,y)=a^2(y^2-b^2)^2-4b^2x^2(y`2+b^2)$, $a>0$, $b>0$
Oblique Concoid	$f(x,y)=[xy\cos t-(y^2+ay-\ell^2)\sin t]^2$ $-(x\sin t+y\cos t+a\cos t)^2(\ell^2-y^2)$
Double Heart	$f(x,y)=(y^2+x^2)^2-6axy^2-ax^3+a^2x^2$
Jerabek's Curve	$f(x,y)=r^2(x^2+y^2-ax)^2-a^2(x^2+y^2)(x-a)^2$
Mascheroni's Curve	$f(x,y)=(x^2+y^2)x^2-(\ell x+ay)^2$
Perseo's Curve or Espiric	$f(x,y)=(x^2+y^2+p^2+d^2-r^2)^2-4d^2(x^2+p^2)$
Bullet Tip Curve	$f(x,y)=a^2y^2-b^2x^2-x^2y^2$
Bernoulli's Lemniscata	$f(x,y)=(x^2+y^2)^2-2a^2(x^2-y^2)$
Booth's Lemniscata	$f(x,y)=(x^2+y^2)^2+(2m^2+n)x^2+(2m^2-n)y^2$
Ortoconcoide	$f(x,y)=(y^2+ay-\ell^2)^2-x^2(\ell^2-y^2)$
Cassini's Oval	$f(x,y)=(x^2+y^2)^2-2c^2(x^2-y^2)-a^4+c^4$
Poliode of a Straight Line	$f(x,y)=4m^4+4km^2y+[-4m^2+(k-y)^2](x^2+y^2)$
Clock of Sand	$f(x,y)=a^2(y^2-b^2)^2-4b^2x^2(y^2+b^2)$. $a>0$, $b>0$

Table 2. Fourth degree classical algebraic curves non realizable by non Hamiltonian quadratic systems.

Name	Curve $f(x,y)=0$
Radial Astroid	$f(x,y)=[(x-x_0)^2+(y-y_0)^2]^3-144a^2(x-x_0)^2(y-y_0)^2$
Butterfly Curve	$f(x,y)=y^6-(x^2-x^6)$
Generalized Campila	$f(x,y)=b^2x^2y^2-(x^2-a^2)(x^2-a^2+b^2)^2$
Scarab	$f(x,y)=(x^2+y^2)(x^2+y^2+cx)^2-a^2*(x^2-y^2)^2$
Mill of Wind	$f(x,y)=4x^2y^2(x^2+y^2)-a^2(x^2-y^2)^2$
Nephroid	$f(x,y)=(x^2+y^2)(x^2+y^2-a^2)^2-4a^2(x^2+y^2-ax)^2$
Poliode of a Circumference	$f(x,y)=4m^4(1-x^2-y^2)+[2mx(x^2+y^2)]/a+(x^2+y^2)$ $\times[-x+(x^2+y^2)/(2a)]^2$. $a>0$, $m>0$, $m<2a$
Cornoid	$x(\tau)=r\cos\tau(1-2\sin^2\tau)$, $y(\tau)=r\sin\tau(1+2\cos^2\tau)$
Cayley's Sextic	$r(\theta)=\ell\cos\theta/4$ with $\ell>0$
Münger's Oval	$f(x,y)=(d^2-r^2)x^{4n}-2dx^{1+2n}(x^2+y^2)^n+(x^2+y^2)^{1+2n}$

Table 3. Higher degree non realizable classical algebraic curves.

Name	Curve $f(x,y)=0$
Witch of Agnesi	$f(x,y)=(a^2+x^2)y-a^3.\ a>0$
Cubic Duplicatriz	$f(x,y)=x^3-\ell(x^2+y^2).\ \ell>0$
Parabolic Folium	$f(x,y)=x^3-bxy-a(x^2-y^2).\ a>0,\ b>0$
(**DIF**) Newton's Serpentine	$f(x,y)=y(x^2+a^2)-acx.\ a>0,\ c>0$
(**DI**) Pseudoversiera	$f(x,y)=(a^2+x^2)y-2a^3.\ a>0$

Table 4. Third degree classical algebraic curves realizable for non Hamiltonian quadratic systems.

Name	Curve
Oblique Bifolium	$f(x,y)=-x^2(ax+by)+(x^2+y^2)^2$
Right Bifolium	$f(x,y)=-ax^3+(x^2+y^2)^2$
(**RI**) Bow	$f(x,y)=x^4-x^2y+y^3$
Cardioid	$f(x,y)=(x^2+y^2-ax)^2-a^2(x^2+y^2)$
(**RI**) Campila	$f(x,y)=(x^2+y^2)-a^2x^4$
(**RI**) Külp's Concoid	$f(x,y)=-a^2(a^2-x^2)+x^2y^2$
(**RI**) Conchal	$f(x,y)=-a^2k^2+(a+x)^2[(x-a)^2+y^2]$
Curve Antiversiera	$f(x,y)=-2rx^3+x^4+4r^2y^2$
(**DI**) Steiner's Curve	$f(x,y)=-27r^4+18r^2(x^2+y^2)+(x^2+y^2)^2$ $+8rx(3y^2-x^2)$
Simple Folium	$f(x,y)=-4rx^3+(x^2+y^2)^2$
(**RI**) Montferrier's Lemniscate	$f(x,y)=x^2(x^2-a^2)+b^2y^2$
(**RI**) Pear Curve	$f(x,y)=r^4-2r^3y+(x-r)^2y^2$
(**RI**) Virtual Parabola	$f(x,y)=y-[ax+b]^{1/2}-[cx+d]^{1/2}$

Table 5. Fourth Degree Realizable Classical Algebraic Curves.

4. Some pictures

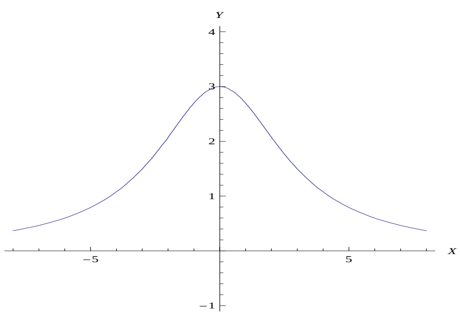

Figure 1. Witch of Agnesi or Versiera.

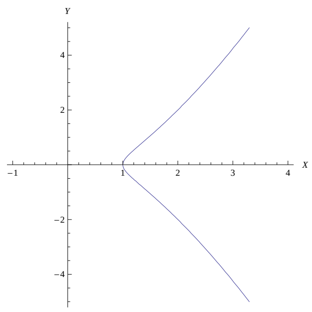

FIGURE 2. Cubic Duplicatriz.

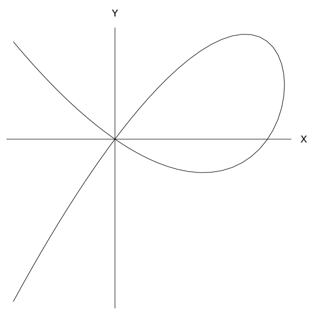

FIGURE 3. Parabolic Folium.

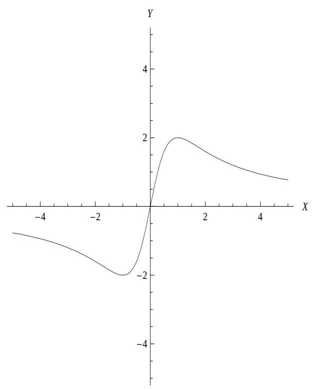

FIGURE 4. Newton's Serpentine.

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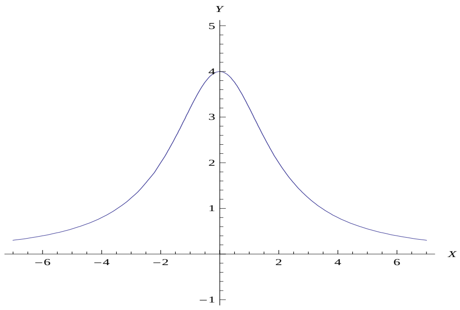

FIGURE 5. Pseudoversiera.

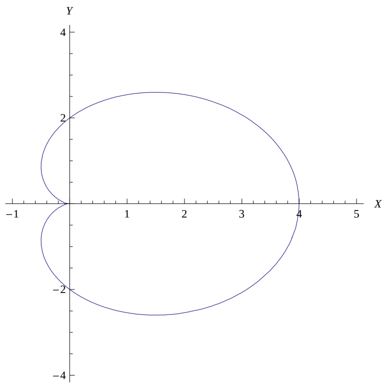

FIGURE 6. Cardioid.

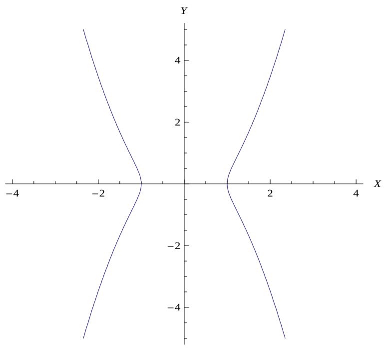

FIGURE 7. Campila.

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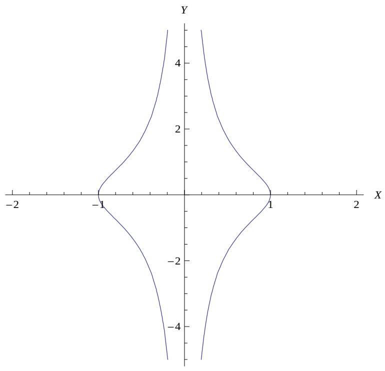

FIGURE 8. Külp's Concoid $H(x, y) = 0$.

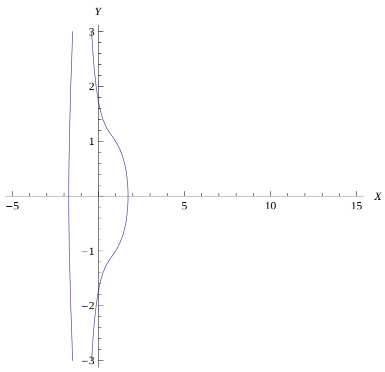

FIGURE 9. Conchal $H(x, y) = 0$ with $k > a$.

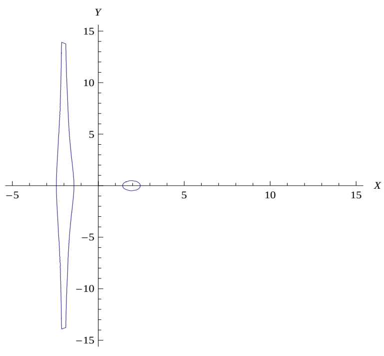

FIGURE 10. Conchal $H(x, y) = 0$ with $k < a$.

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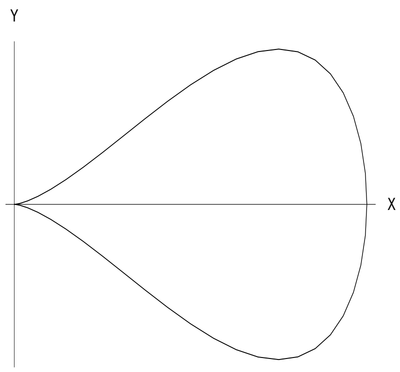

FIGURE 11. Curve Antiversiera.

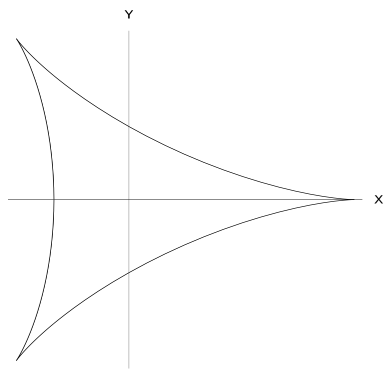

FIGURE 12. Steiner's curve.

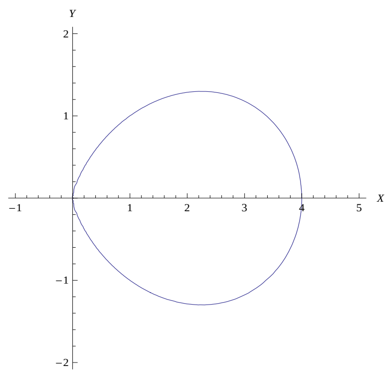

FIGURE 13. Simple Folium.

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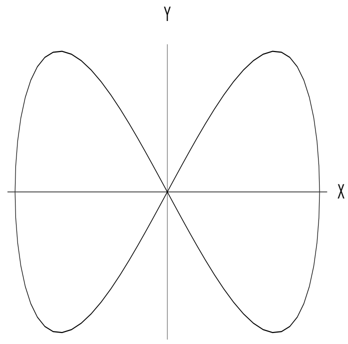

FIGURE 14. Montferrier's Lemniscate.

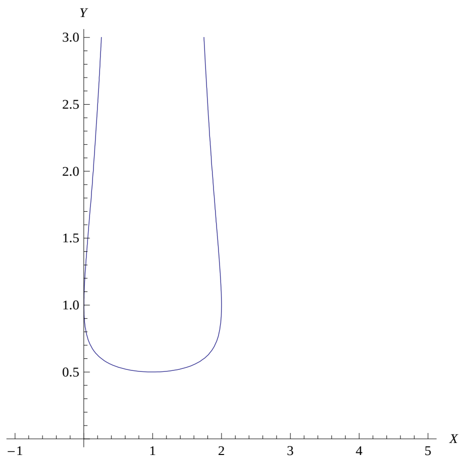

FIGURE 15. Pear curve.

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