EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. We provide sufficient conditions for the existence of a periodic solution for a class of second order differential equations of the form $\ddot{x} + g(x) = \varepsilon f(t, x, \dot{x}, \varepsilon)$, where ε is a small parameter.

1. Introduction and statement of the results

The second order differential equations of the form

$$\ddot{x} + g(x) = \varepsilon f(t, x, \dot{x}, \varepsilon),$$

have been studied by many authors because they have many applications, see for instance [2, 3, 7, 8, 9, 12, 15]. Two of the main families studied are the Duffing equations see [5, 6], ... or the forced pendulum see the nice survey [11] and the references quoted therein.

The aim of this work is to study periodic solutions of the second order differential equation

$$\ddot{x} + g(x) = \mu^{2n+1} p(t) + \mu^{4n+1} q(t, x, y, \mu), \tag{1}$$

where n is a positive integer, μ is a small parameter, and the functions

$$g(x) = x + x^{2n+1}\left(b + x h(x)\right),$$

and $h(x)$ are smooth, $b \neq 0$, $p(t)$ and $q(t, x, y, \mu)$ are smooth and periodic with period 2π in the variable t.

Let $\Gamma(x)$ the Gamma function, see for more details [1], and let α and β the first Fourier coefficients of the periodic function $p(t)$, i.e.

$$\alpha = \frac{1}{\pi}\int_0^{2\pi} p(t)\cos t\, dt, \qquad \beta = \frac{1}{\pi}\int_0^{2\pi} p(t)\sin t\, dt.$$

Then our main result is the following.

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Theorem 1. *If $\alpha\beta \neq 0$ then for $\mu \neq 0$ sufficiently small the differential equation* (1) *has a 2π–periodic solution $\mathbf{x}(t,\mu)$ such that*

$$\mathbf{x}(0,\mu) = \pi^{\frac{1}{4n+2}} \left(\frac{\Gamma(n+2)}{2b\Gamma\left(n+\frac{3}{2}\right)} \right)^{\frac{1}{2n+1}} \alpha \left(\frac{\beta^2}{\alpha^2} + 1 \right)^{-n} + O(\mu^{2n}).$$

Theorem 1 is proved in section 3. Its proof uses the averaging theory for computing periodic solutions, see section 2 for a summary of the results on this theory that we shall need.

2. The averaging theory

We want to study the T–periodic solutions of the periodic differential systems of the form

$$\mathbf{x}' = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{2}$$

with $\varepsilon > 0$ sufficiently small, where $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of the differential system (2) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. Suppose that the unperturbed system

$$\mathbf{x}' = F_0(t,\mathbf{x}), \tag{3}$$

has an open set V with $\overline{V} \subset \Omega$ such that for each $\mathbf{z} \in \overline{V}$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic.

Let $\mathbf{y}$ be an $n \times n$ matrix, and consider the first order variational equation

$$\mathbf{y}' = D_{\mathbf{x}} F_0(t, \mathbf{x}(t,\mathbf{z},0))\mathbf{y}, \tag{4}$$

of the unperturbed system (3) on the periodic solution $\mathbf{x}(t,\mathbf{z},0)$. Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of the linear differential system (4) with periodic coefficients such that $M_{\mathbf{z}}(0)$ is the $n \times n$ identity matrix.

Theorem 2. *Consider the function $F : \overline{V} \to \mathbb{R}^n$*

$$f(\mathbf{z}) = \int_0^T M_{\mathbf{z}}^{-1}(t) F_1(t, \mathbf{x}(t,\mathbf{z},0)) dt. \tag{5}$$

If there exists $\alpha \in V$ with $f(\alpha) = 0$ and

$$\det\left((d\,f/d\mathbf{z})\,(\alpha) \right) \neq 0, \tag{6}$$

then there exists a T–periodic solution $\mathbf{x}(t,\varepsilon)$ of system (2) *such that $\mathbf{x}(0,\varepsilon) = \alpha + O(\varepsilon)$.*

The existence of the periodic solution of Theorem 2 is due to Malkin [10] and Roseau [13], for a shorter and easier proof see [4]. The proof for the stability follows in a similar way to the proof of Theorem 11.6 of [14].

3. Proof of Theorem 1

The differential equation of second order (1) can be written as the first order differential system

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x - x^{2n+1}\left(b + xh(x)\right) + \mu^{2n+1} p(t) + \mu^{4n+1} q(t,x,y,\mu). \end{aligned} \tag{7}$$

In order to apply the averaging theory described in section 2 to this differential system we do the scaling $x \to \mu x$ and $y \to \mu y$. Hence the differential system (7) becomes

$$\begin{aligned} &\dot{x} = y, \\ &\dot{y} = -x + \mu^{2n}\left(-bx^{2n+1} + p(t)\right) + \mu^{4n}\, q^*(t,x,y,\mu). \end{aligned} \tag{8}$$

This system is written into the normal form (2) for applying the averaging theory described in section 2, where

$$\begin{aligned} &\mathbf{x} = (x,y), \\ &\varepsilon = \mu^{2n}, \\ &\mathbf{F}_0(\mathbf{x}) = (y,-x), \\ &\mathbf{F}_1(\mathbf{x},t) = (0, -bx^{2n+1} + p(t)), \\ &\mathbf{F}_2(\mathbf{x},t,\varepsilon) = \left(0, \bar{q}(t,x,y,\varepsilon)\right). \end{aligned} \tag{9}$$

From section 2 the solution $\mathbf{x}(t,\mathbf{z},0) = (x(t,\mathbf{z},0), y(t,\mathbf{z},0))$ of system (8) with $\varepsilon = 0$ satisfies $\mathbf{x}(0,\mathbf{z},0) = \mathbf{z} = (x_0,y_0)$, and consequently

$$\begin{aligned} x(t,\mathbf{z},0) &= x_0 \cos t + y_0 \sin t, \\ y(t,\mathbf{z},0) &= -x_0 \sin t + y_0 \cos t. \end{aligned}$$

The fundamental matrix $M_{\mathbf{z}}(t) = M(t)$ of the the first order variational equation (4) satisfying (9) is

$$M(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}.$$

According with Theorem 2 in order to compute the 2π–periodic solutions of the differential system (8) we must compute the integral

$$
\begin{aligned}
f(\mathbf{z}) = & \begin{pmatrix} f_1(x_0,y_0)\\ f_2(x_0,y_0)\end{pmatrix}\\
= & \int_0^{2\pi} M^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))dt\\
= & \begin{pmatrix} b\displaystyle\int_0^{2\pi}\sin t(x_0\cos t+y_0\sin t)^{2n+1}dt-\int_0^{2\pi}p(t)\sin t dt\\ -b\displaystyle\int_0^{2\pi}\cos t(x_0\cos t+y_0\sin t)^{2n+1}dt+\int_0^{2\pi}p(t)\cos t dt\end{pmatrix}.
\end{aligned}
$$

Doing induction with respect to n it is not difficult to show that

$$
\begin{aligned}
\int_0^{2\pi}\sin t\,(x_0\cos t+y_0\sin t)^{2n+1}\,dt =& \frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)}y_0\left(x_0^2+y_0^2\right)^n,\\
\int_0^{2\pi}\cos t\,(x_0\cos t+y_0\sin t)^{2n+1}\,dt =& \frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)}x_0\left(x_0^2+y_0^2\right)^n.
\end{aligned}
$$

Therefore we must solve the system

$$
\begin{pmatrix} f_1(x_0,y_0)\\ f_2(x_0,y_0)\end{pmatrix} = \begin{pmatrix} \dfrac{2\sqrt{\pi}\,b\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)}y_0\left(x_0^2+y_0^2\right)^n-\pi\beta_1\\ \dfrac{-2\sqrt{\pi}\,b\,\Gamma\left(\frac{3}{2}+n\right)}{\Gamma(2+n)}x_0\left(x_0^2+y_0^2\right)^n+\pi\alpha_1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.
$$

This system has a unique solution

$$
\begin{pmatrix} x_0^*\\ y_0^*\end{pmatrix} = \pi^{\frac{1}{4n+2}}\left(\frac{\Gamma(n+2)}{2b\Gamma\left(n+\frac{3}{2}\right)}\right)^{\frac{1}{2n+1}}\begin{pmatrix}\alpha\left(\frac{\beta^2}{\alpha^2}+1\right)^{-n}\\ \beta\left(\frac{\alpha^2}{\beta^2}+1\right)^{-n}\end{pmatrix}.
$$

The determinant (6) of the Jacobian matrix $Df(x_0^*,y_0^*)$ is

$$
\begin{aligned}
\det(Df(x_0^*,y_0^*)) = & 4^{\frac{1}{2n+1}}(2n+1)\pi^{\frac{2n}{2n+1}+1}\left(\frac{\Gamma(n+2)}{b\Gamma\left(n+\frac{3}{2}\right)}\right)^{-\frac{2}{2n+1}}\\
& \left(\left(\beta\left(\frac{\alpha^2}{\beta^2}+1\right)^{-n}\right)^{\frac{2}{2n+1}}+\left(\alpha\left(\frac{\beta^2}{\alpha^2}+1\right)^{-n}\right)^{\frac{2}{2n+1}}\right)^{2n},
\end{aligned}
$$

and by assumptions it is positive because $\alpha\beta b\neq 0$.

In summary all the assumptions of Theorem 2 hold and consequently from Theorem 2 it follows Theorem 1.

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