THE HOPF CYCLICITY OF THE CENTERS OF THE POLYNOMIAL VECTOR FIELDS

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Abstract. We consider families of planar polynomial vector fields having a singularity with purely imaginary eigenvalues for which a basis of its Bautin ideal $\mathcal{B}$ is known. We provide an algorithm for computing an upper bound of the Hopf cyclicity less than or equal to the Bautin depth of $\mathcal{B}$. We also present a method for studying the cyclicity problem for the Hamiltonian and the time-reversible centers without the necessity of solving previously the Dulac complex center problem associated to the larger complexified family. As application we analyze the Hopf cyclicity of the quintic polynomial family written in complex notation as $\dot{z} = iz + z\bar{z}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3)$.

1. Introduction and statement of the main results

We consider a family of planar polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y + P(x, y, \lambda), \\ \dot{y} &= x + \lambda_1 y + Q(x, y, \lambda), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y, \lambda]$ are the polynomial nonlinearities of system (1) and $(\lambda_1, \lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda \subset \mathbb{R}^n$ are the parameters of the family. We assume that for some $(\lambda_1, \lambda) = (0, \lambda^*) \in \Lambda$ system (1) has a *center* at the origin. Of course the origin is always a *monodromic* singularity of family (1), i.e., it is a center or a *focus* and clearly when $\lambda_1 \neq 0$ it is a focus.

Using a transversal section $\Sigma = [0, \hat{h})$ with endpoint at the origin of coordinates and parameterized by h where $\hat{h} = \hat{h}(\lambda)$, we have the *displacement map* $d : \Sigma \times \Lambda \to \Sigma \times \Lambda$ defined by $d(h; \lambda) = \Pi(h; \lambda) - h$, where $\Pi : \Sigma \times \Lambda \to \Sigma \times \Lambda$ is the *Poincaré* or *return map*. We note that $\hat{h} > 0$ can be finite or infinite.

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Since the differential system (1) is analytic the displacement map $d(h;\lambda)$ is analytic in the variables $h \in [0,\hat{h})$ and λ. Hence we can expand the displacement function $d(h;\lambda) = \sum_{i\geq 1} a_i(\lambda)h^i$ in Taylor series at $h=0$. For $\lambda_1 = 0$ the *Bautin ideal* $\mathcal{B}$ at the origin of system (1) is defined as the ideal generated by all the polynomials $a_i(\lambda)$ in the ring of all polynomials in the variables λ. This ideal $\mathcal{B}$ is Noetherian and then by the Hilbert's basis Theorem it is generated by a finite number of polynomials. So we know that

$$\mathcal{B} = \langle v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda)\rangle, \tag{2}$$

where the generators $v_{i_j}(\lambda)$ for $j=1,\ldots,m$, of the ideal $\mathcal{B}$ are called *Poincaré–Liapunov constants.*

The relation between $a_i(\lambda)$ and $v_i(\lambda)$ is that $v_i(\lambda) = a_i(\lambda) \mod \mathcal{B}_{i-1}$ where $\mathcal{B}_{i-1} = \langle v_1(\lambda),\ldots,v_{i-1}(\lambda)\rangle$. In other words there are polynomials $p_{i,j} \in \mathbb{R}[\lambda]$ such that $v_i(\lambda) = a_i(\lambda) + \sum_{j=1}^{i-1} p_{i,j}(\lambda)a_j(\lambda)$.

Definition 1. Given the Bautin ideal $\mathcal{B} = \langle a_i(\lambda) : i \in \mathbb{N}\rangle$, we say that the basis $B = \{a_{j_1}(\lambda),\ldots,a_{j_m}(\lambda)\}$ of $\mathcal{B}$ with order $j_1 < \cdots < j_m$ is *minimal* if it satisfies the following properties:

(i) $a_i(\lambda) \equiv 0$ for $1 \leq i \leq j_1 - 1$ and $a_{j_1}(\lambda) \not\equiv 0$;

(ii) For $i \geq j_1 + 1$, if $a_i(\lambda) \notin \langle a_1(\lambda),\ldots,a_{i-1}(\lambda)\rangle$, then $a_i(\lambda) \in B$.

The cardinality m of B is called the *Bautin depth* of $\mathcal{B}$ in [9] and it is associated to the chain of ideals $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}$ where $\mathcal{B}_s = \langle v_1(\lambda),\ldots,v_s(\lambda)\rangle$ for certain integer $s \geq 1$.

Following Bautin's seminal work [1] in Chapter 4 of [15] and in Chapter 6 of [14] it is proved that when (2) is a minimal basis of the ideal $\mathcal{B}$ then the displacement map $d(h;\lambda)$ can be written in the form

$$d(h;\lambda) = \sum_{j=1}^{m} v_{i_j}(\lambda)h^{i_j}q_j(h;\lambda), \tag{3}$$

where $q_j(h;\lambda)$ are analytic functions in the variables h and λ near $(h,\lambda) = (0,\lambda^*)$ such that $q_j(0;\lambda^*) = 1$. Clearly $v_{i_j}(\lambda^*) = 0$ for all $j=1,\ldots,m$ when the differential system (1) has a center at the origin for $\lambda = \lambda^*$.

The maximum number of small amplitude limit cycles that can bifurcate from a center at the origin of family (1) with $\lambda = \lambda^*$ under arbitrarily small perturbations inside family (1), that is for $\|\lambda - \lambda^*\| \ll 1$, is called the *cyclicity* of the center with parameters λ^*. See [4] for an interesting point of view. It is well known that the cyclicity of any center at the origin of (1) is at most the Bautin depth m of $\mathcal{B}$, see Theorem 7.

Given a ground field $\mathbb{K}$ and a polynomial ideal $\mathcal{J}$ in $\mathbb{K}[\mathbf{x}]$ with $\mathbf{x} \in \mathbb{K}^d$ we define by $\mathbf{V}(\mathcal{J})$ the affine variety in $\mathbb{K}^d$ determined by $\mathcal{J}$. If the ideal $\mathcal{J} = \langle p_1(\mathbf{x}), \ldots, p_s(\mathbf{x})\rangle$, then $\mathbf{V}(\mathcal{J}) = \{\mathbf{x}^* \in \mathbb{K}^d : p_j(\mathbf{x}^*) = 0 \text{ for } 1 \leq j \leq s\}$.

In our forthcoming Theorem 2 we give an upper bound j^* with $1 \leq j^* \leq m$ for the maximum number of small amplitude limit cycles that can bifurcate from the center at the origin of family (1), hence improving sometimes the known bound m given by the Bautin depth.

To present the result we will specify an *arbitrary* analytic curve $\varepsilon \mapsto \gamma(\varepsilon) = (\lambda_1(\varepsilon), \lambda(\varepsilon)) \subset \Lambda \subset \mathbb{R}^n$ in the parameter space passing through a point $(\lambda_1(0), \lambda(0)) = (0, \lambda^*)$ with $\lambda^* \in \mathbf{V}(\mathcal{B})$. More specifically we consider *any* analytic perturbation of the center of system (1) with $\lambda(0) = \lambda^*$ of the form

$$
\begin{aligned}
\dot{x} &= -y + \lambda_1(\varepsilon)x + P(x, y, \lambda(\varepsilon)),\\
\dot{y} &= x + \lambda_1(\varepsilon)y + Q(x, y, \lambda(\varepsilon)).
\end{aligned}
\tag{4}
$$

Let $\mathcal{X}_0$ be the vector field defined by the unperturbed family (1) having a center at the origin, i.e., with parameters $(\lambda_1, \lambda) = (0, \lambda^*)$; and let $\mathcal{X}_\varepsilon$ be the vector field defined by the perturbed system (4). We are interested in the maximum number of periodic orbits that can bifurcate from the origin of $\mathcal{X}_0$ under the perturbation $\mathcal{X}_\varepsilon$. In short we want to find for the family of centers (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ its *Hopf cyclicity*, $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$, under perturbations $\mathcal{X}_\varepsilon$; that is, the sharp upper bound of the maximum number of small amplitude limit cycles of $\mathcal{X}_\varepsilon$ that can bifurcate from the origin when $|\varepsilon|$ is sufficiently small.

Theorem 2. *Assume that the unperturbed system* (4) *with* $\varepsilon = 0$ *has a center at the origin. Assume that* $\{v_{i_1}(\lambda(\varepsilon)), v_{i_2}(\lambda(\varepsilon)), \ldots, v_{i_m}(\lambda(\varepsilon))\}$ *is a minimal basis of the Bautin ideal* $\mathcal{B}$ *associated to the perturbed system* (4). *Consider the Taylor expansions* $v_{i_j}(\lambda(\varepsilon)) = \sum_{r\geq 1} \bar{v}_{i_j,r}\, \varepsilon^r$ *of the Poincaré–Liapunov constants for* $j = 1, \ldots, m$. *Let first* $k \geq 1$ *and last* j^* *with* $1 \leq j^* \leq m$ *be the smallest integer numbers such that* $\bar{v}_{i_{j^*},k} \neq 0$. *Then the cyclicity* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ *of the origin of* (4) *is bounded by* j^*.

We can compute the upper bound j^* applying Theorem 2 with an arbitrary perturbation (4) to a family of centers for which we know a basis of its Bautin ideal, and after we can study if it is a sharp upper bound, i.e. if j^* is reached. By definition this sharp upper bound is $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$.

Unfortunately there are few families of polynomial vector fields for which the basis of their Bautin ideal $\mathcal{B}$ is known. Hence $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$ of few families $\mathcal{X}_0$ is known. To known a finite set of generators of $\mathcal{B}$ is in general much harder than to known its associated variety $\mathbf{V}(\mathcal{B})$. This is the main reason by which the *center problem* consisting in describing the *center variety* $\mathbf{V}(\mathcal{B})$ in the parameter space is easier than the *cyclicity problem* consisting in obtaining $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0)$.

If the perturbation is such that we can choose adequately some Poincaré–Liapunov constants of the perturbed field (see the next corollary), then the above upper bound j^* computed via Theorem 2 can be reached. More precisely we have the following consequence from the proof of Theorem 2.

Corollary 3. *Consider that the unperturbed system* (4) *with* $\varepsilon = 0$ *has a center at the origin and assume* $\{v_{i_1}(\lambda(\varepsilon)), v_{i_2}(\lambda(\varepsilon)), \ldots, v_{i_m}(\lambda(\varepsilon))\}$ *is a minimal basis of the Bautin ideal* $\mathcal{B}$ *associated to the perturbed system* (4). *Let* j^* *with* $1 \leq j^* \leq m$ *be defined as in Theorem 2. Assume that we can perturb this system in such a way that*

$$|v_{i_1}(\lambda(\varepsilon))| \ll |v_{i_2}(\lambda(\varepsilon))| \ll \cdots \ll |v_{i_{j^*}}(\lambda(\varepsilon))| \ll 1,$$

and $v_{i_j}(\lambda(\varepsilon))v_{i_{j+1}}(\lambda(\varepsilon)) < 0$ *for* $j = 1, \ldots, j^* - 1$. *Then we have* $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = j^*$.

In the celebrated paper [1] Bautin proved that the Hopf cyclicity of a center of a quadratic polynomial vector field is at most 3. Bautin's result is improved by Żołądek in Theorem 3 at page 237 of [17] where the Hopf cyclicity of the quadratic family having its parameters on different irreducible components of the center variety is computed. Next in [18] Żołądek found that there are centers in (1) with P and Q homogeneous cubic polynomials in x and y such that $\mathrm{Cycl}(\mathcal{X}_\varepsilon, 0) = 5$.

We will consider the quintic polynomial family written in complex form as

$$\dot{z} = (i + \lambda_1)z + z\bar{z}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3) \tag{5}$$

with $z = x + iy \in \mathbb{C}$ and parameters $\lambda_1 \in \mathbb{R}$ and $(A, B, C, D) \in \mathbb{C}^4$. The center problem for this family has been solved in [12], but the Hopf cyclicity is only stated for the easier case of having a focus at $z = 0$. Hence we will restrict our attention on the cyclicity problem of the center at $z = 0$ of (5) and our results are stated below.

Theorem 4. *The following statements hold.*

(a) *Any nonlinear center at the origin of family* (5) *has cyclicity at most* 6 *when we perturb it inside this family.*

(b) *There are perturbations of the linear center $\dot{z} = iz$ inside family* (5) *producing 6 limit cycles bifurcating from the origin.*

A center in family (1) is *time-reversible* if after a rotation it is invariant under the discrete symmetry $(x, y, t) \mapsto (x, -y, -t)$. We remark that it has been possible to prove Theorem 4 thanks to the use of a new procedure that allows to study the cyclicity problem for the centers which are either time-reversible, or for which we know an explicit formal first integral. This method does not need to solve previously the Dulac complex center problem associated to the larger complexified family, see the Approach I in Subsection 2.1. Also techniques for bounding the cyclicity in the harder case of non-radical Bautin ideal are used, see Subsection 2.2.

Our computations show strong evidences for stating the following conjecture.

Conjecture 5. *An upper bound for the cyclicity of the linear center at the origin in family* (5) *perturbing it within this family is seven.*

We end by emphasizing that similar techniques can be applied to get the cyclicity (not only a bound of it) inside certain subfamilies of the full family (5) fixing some relations between the parameters that give rise to a radical Bautin ideal.

Proposition 6. *The cyclicity of the center at the origin in the subfamilies of* (5) *obtained by fixing one of the real parameters* $\mathrm{Re}(C)$ *or* $\mathrm{Im}(C)$ *is* 3*, and is* 2 *when we fix* $D = 0$*.*

2. Background on the cyclicity problem

In this section we summarize several results concerning the cyclicity problem and the approach to that problem using methods from computational commutative algebra. Most of this background can be found in the excellent book [14], see also the paper [16].

Using the rearrangement (3) of the displacement map $d(h; \lambda)$ and applying Rolle's Theorem several times the following theorem is proved, see for example [1, 9, 14, 15].

Theorem 7. *Let* $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda)\} \subset \mathbb{R}[\lambda]$ *be a minimal basis of the Bautin ideal* $\mathcal{B}$ *associated to the origin of family* (1)*. Then the cyclicity of any center at the origin in* (1) *is at most* m*.*

The Poincaré-Liapunov constants are difficult to work with mainly because to compute them we must perform quadratures. Therefore instead of working with the Poincaré-Liapunov constants, from the computational point of view it is better to obtain other polynomials

$\eta_j(\lambda) \in \mathbb{R}[\lambda]$ that arise as the obstructions in order to get a formal first integral $H(x,y) = x^2 + y^2 + \cdots$ of family (1) with $\lambda_1 = 0$ which is another characterization of centers, see Poincaré [13] and Liapunov [11]. More precisely we seek for a formal series $H(x,y;\lambda) = x^2+y^2+\cdots$ in such a way that $\mathcal{X}_\lambda(H) = \sum_{j\geq 1} \eta_j(\lambda)(x^2+y^2)^j$ where $\mathcal{X}_\lambda = (-y + P(x,y,\lambda))\partial_x + (x + Q(x,y,\lambda))\partial_y$ is the associate vector field to family (1) with $\lambda_1 = 0$.

Using the complex coordinate $z = x + iy \in \mathbb{C}$ family (1) with $\lambda_1 = 0$ can be written into the form $\dot{z} = iz + F(z,\bar{z},\lambda)$ where $\bar{z} = x - iy$ and F is given by the polynomial $F(z,\bar{z},\lambda) = P\left(\frac{1}{2}(z+\bar{z}), \frac{i}{2}(\bar{z}-z), \lambda\right) + iQ\left(\frac{1}{2}(z+\bar{z}), \frac{i}{2}(\bar{z}-z), \lambda\right)$. We can adjoin to this complex polynomial differential equation its complex conjugate forming thus the complex system

$$
\begin{aligned}
\dot{z} &= iz + F(z,\bar{z},\lambda) = iz + \sum_{j+k=2}^{N} a_{j,k}(\lambda) z^j \bar{z}^k, \\
\dot{\bar{z}} &= -i\bar{z} + \bar{F}(z,\bar{z},\lambda) = -i\bar{z} + \sum_{j+k=2}^{N} \bar{a}_{j,k}(\lambda) \bar{z}^j z^k.
\end{aligned} \tag{6}
$$

Replacing the conjugates $\bar{z}$ and $\bar{a}_{j,k}$ by new independent complex state variable and complex parameters, say w and $b_{j,k}$ respectively, yields a larger complex family of systems

$$
\dot{z} = iz + \sum_{j+k=2}^{N} a_{j,k} z^j w^k, \quad \dot{w} = -iw + \sum_{j+k=2}^{N} b_{j,k} w^j z^k, \tag{7}
$$

defined in $\mathbb{C}^2$ with complex parameters $\mu = (a_{j,k}, b_{j,k})$. Family (7) is called the *complexification* of family (1) with $\lambda_1 = 0$.

Following Dulac [6] one can generalize the concept of center singularity of systems in $\mathbb{R}^2$ to systems in $\mathbb{C}^2$. To be specific we say that (7) has a (complex) center at the origin $(z,w) = (0,0)$ when $\mu = \mu^*$ if and only if it admits a formal (complex) first integral $\hat{H}(z,w;\mu^*) = zw + \cdots$. It is easy to check that system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a center at the origin if and only if (6) has a center at the origin for $\lambda = \lambda^*$.

We shall define the *focus quantities* $g_j(\mu) \in \mathbb{C}[\mu]$ with $\mu = (a_{j,k}, b_{j,k})$ of the complexification (7). Denote by $\hat{X}_\mu = (iz + \cdots)\partial_z + (-iw + \cdots)\partial_w$ the family of vector fields in $\mathbb{C}^2$ associated with (7). The focus quantities satisfy that when we look for a formal first integral $\hat{H}(z,w;\mu) = zw + \cdots$ of $\hat{X}_\mu$ then $\hat{X}_\mu(\hat{H}) = \sum_{j\geq 1} g_j(\mu)(zw)^{j+1}$.

Let $\mathcal{I}$ and $\mathcal{I}_k$ be the ideals in $\mathbb{C}[\lambda]$ given by $\mathcal{I} = \langle g_j(\mu) : j \in \mathbb{N} \rangle$ and $\mathcal{I}_k = \langle g_1(\mu), \ldots, g_k(\mu) \rangle$, respectively. We can also define $\tilde{g}_j \equiv g_j$

mod $\mathcal{I}_{j-1}$ so that $\mathcal{I}_k = \langle g_1(\lambda), \tilde{g}_2(\lambda), \ldots, \tilde{g}_k(\lambda)\rangle$. It is evident that (7) has a center at the origin when $\mu = \mu^*$ if and only if $\mu^* \in \mathbf{V}(\mathcal{I})$. In this work we refer to $\mathcal{I}$ and $\mathbf{V}(\mathcal{I})$ as the *complex Bautin ideal* and *complex center variety* respectively.

We define the *real focus quantities* $f_j(\lambda)$ for family (1) as

$$f_j(\lambda) = g_j(a_{j,k}(\lambda), \bar{a}_{j,k}(\lambda) \in \mathbb{R}[\lambda]. \tag{8}$$

Theorem 6.2.3 of [14] describes the relationship between the Poincaré-Liapunov constants $v_j(\lambda)$ and the real focus quantities $f_j(\lambda)$ for family (1) when the following standard procedure is used to compute them. Taking polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ family (1) becomes $dr/d\theta = R(\theta, r; \lambda)$ where the function R is a 2π-periodic function of θ and is analytic for $|r|$ sufficiently small. Let $r(\theta; h, \lambda) = \sum_{j\geq 1} u_j(\theta; \lambda)h^j$ be the solution of that differential equation satisfying the initial condition $r(0; h, \lambda) = h$. Then $v_j(\lambda) = u_j(2\pi; \lambda)$. In [14] it is proved that the former procedure applied to family (1) with $\lambda_1 = 0$ gives $v_1(\lambda) = v_2(\lambda) \equiv 0$, $v_3(\lambda) = f_1(\lambda)$ up to a positive multiplicative constant and for any integer number $k \geq 2$ one has $v_{2k}(\lambda) \in \mathcal{B}_{2k-1}$ and $v_{2k+1}(\lambda) - f_k(\lambda) \in \mathcal{B}_{2k-1}$. In particular if $\{v_{k_1}, \ldots, v_{k_r}\}$ and $\{f_{j_1}, \ldots, f_{j_s}\}$ are two minimal bases for the Bautin ideal $\mathcal{B}$ formed by Poincaré-Liapunov constants and by real focus quantities respectively, then $r = s$ and $k_q = 2j_q + 1$. In this work we will use the notation $\tilde{f}_j \equiv f_j \mod \mathcal{B}_{j-1}$.

Remark 8. In summary we can finally obtain an upper bound of the cyclicity only in terms of the real focus quantities instead of Poincaré-Liapunov constants because Theorem 7 can be restated in terms of a minimal Basis of $\mathcal{B}$ formed by real focus quantities. The key point is that expression (3) of the displacement map can be rewritten as

$$d(h; \lambda) = \sum_{j=1}^{m} \tilde{f}_{k_j}(\lambda) h^{2k_j+1} \psi_j(h; \lambda), \tag{9}$$

where $\psi_j(h; \lambda)$ are analytic functions in the variables h and λ near $(h, \lambda) = (0, \lambda^*)$ such that $\psi_j(0; \lambda^*) = 1$. So we shall compute real focus quantities instead of the Poincaré-Liapunov constants due to their computational simplicity.

2.1. **Radical Bautin ideal.** Recall that the radical $\sqrt{\mathcal{J}}$ of an ideal $\mathcal{J}$ is the set of elements a power of which is in $\mathcal{J}$, that is $\sqrt{\mathcal{J}} = \{p \in \mathbb{K}[\mathbf{x}] : p^s \in \mathcal{J}$ for some $s \in \mathbb{N}\}$. Clearly $\mathcal{J} \subset \sqrt{\mathcal{J}}$ always. In case that $\mathcal{J} = \sqrt{\mathcal{J}}$ then $\mathcal{J}$ is called a *radical ideal.*

When the Bautin ideal $\mathcal{B}$ is radical, then in this very special case we can find a finite number of generator of $\mathcal{B}$ using two different approaches that we explain now. As starting point it is assumed that we have solved the center problem of the family in the sense that we have established the equality

$$\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{j_s}) \tag{10}$$

of varieties in $\mathbb{R}^{n-1}$ where $\mathcal{B}_{j_s} = \langle f_{j_1}(\lambda), \ldots, f_{j_s}(\lambda)\rangle$, or equivalently $\mathcal{B}_{j_s} = \langle v_{2j_1+1}(\lambda), \ldots, v_{2j_s+1}(\lambda)\rangle$ for certain integer $s \geq 1$. From the applicable point of view equality (10) is established in the following way. Compute the first real focal values $f_k(\lambda)$ satisfying that $f_k \notin \sqrt{B_{k-1}}$ for $k = 1, \ldots, j_s$ until we reach stabilization in the sense that $f_k \in \sqrt{B_{j_s}}$ for some consecutive values of k with $k \geq j_s$. This step is totally algorithmic and this computation leads to expect that (10) is true. One way to verify that actually (10) holds is performing the irreducible decomposition of the variety $\mathbf{V}(\mathcal{B}_{j_s}) = \bigcup_r V_r$ (also an algorithmic step) and check that for any $\lambda^* \in V_r$ its associated system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a center at the origin. This last part is not algorithmic and may be a difficult step which requires usually of some integrability or symmetry argument on system (1).

Approach I. It is motivated by the Strong Hilbert Nullstellensatz and also by the fact that it is possible for two ideals I and J in $\mathbb{R}[\lambda]$ that $\mathbf{V}(I) = \mathbf{V}(J)$ as real varieties included in $\mathbb{R}^k$, but $\mathbf{V}(I) \neq \mathbf{V}(J)$ when they are viewed as complex varieties in $\mathbb{C}^k$.

A key point in Approach I is to prove that (10) also holds in $\mathbb{C}^{n-1}$. This implies the equality $\sqrt{\mathcal{B}} = \sqrt{\mathcal{B}_{j_s}}$ of the radicals of ideals from the Strong Hilbert Nullstellensatz. Under the extra assumption (simple good fortune) that $\mathcal{B}_{j_s}$ is radical we get

$$\mathcal{B}_{j_s} \subset \mathcal{B} \subset \sqrt{\mathcal{B}} = \sqrt{\mathcal{B}_{j_s}} = \mathcal{B}_{j_s}$$

and therefore $\mathcal{B} = \mathcal{B}_{j_s}$ finishing Approach I. We can therefore state the following result.

Theorem 9 (First Radical Ideal Cyclicity Bound Theorem)**.** *Assume that $\{f_{j_1}(\lambda), \ldots, f_{j_m}(\lambda)\}$ is a minimal basis of the ideal $\mathcal{B}_{j_m} \subseteq \mathcal{B}$ where $\mathcal{B}$ is the Bautin ideal associated to family* (1)*. Suppose that $\mathcal{B}_{j_m}$ is radical and that the equality of varieties $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{j_m})$ holds in $\mathbb{C}^{n-1}$. Then $\mathcal{B} = \mathcal{B}_{j_m}$ and, in particular, the cyclicity of any center at the origin in* (1) *is at most m.*

Remark 10. Now we turn to the key point of how to prove that (10) holds in $\mathbb{C}^{n-1}$. Since $\mathcal{B}_{j_s} \subset \mathcal{B}$ it is clear that $\mathbf{V}(\mathcal{B}) \subset \mathbf{V}(\mathcal{B}_{j_s})$ holds in $\mathbb{C}^{n-1}$, therefore we only have to check that the reverse inclusion $\mathbf{V}(\mathcal{B}_{j_s}) \subset \mathbf{V}(\mathcal{B})$ holds in $\mathbb{C}^{n-1}$. To prove that we must check whether for any $\lambda^* \in \mathbb{C}^{n-1}$ satisfying $f_{j_1}(\lambda^*) = \cdots = f_{j_s}(\lambda^*) = 0$ this implies that $f_k(\lambda^*) = 0$ for all $k \in \mathbb{N}$ where $\mathcal{B}_{j_s} = \langle f_{j_1}(\lambda), \ldots, f_{j_s}(\lambda)\rangle$.

Now we will see that there is a different but equivalent way to prove the former condition. At this point we view family (1) as a system on $\mathbb{C}^2$ with complex parameters, i.e., we will study family (1) with $(x, y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}^{n-1}$. Now we do the linear complex change of coordinates $(x, y) \mapsto (X, Y) = (x + iy, x - iy)$. Notice that now $\bar{Y} \neq X$ but anyway (1) is transformed into

$$\dot{X} = iX + \mathcal{F}^+(X, Y; \lambda), \quad \dot{Y} = -iY + \mathcal{F}^-(X, Y; \lambda), \tag{11}$$

where $\mathcal{F}^\pm$ only contains nonlinear terms in X and Y because

$$\mathcal{F}^\pm(X, Y; \lambda) = P\left(\tfrac{1}{2}(X + Y), \tfrac{i}{2}(Y - X), \lambda\right) \pm iQ\left(\tfrac{1}{2}(X + Y), \tfrac{i}{2}(Y - X), \lambda\right).$$

In this complex setting we can build a formal series $\tilde{H}(X, Y; \lambda) = XY + \cdots$ such that $\mathcal{X}_\lambda(\tilde{H}) = \sum_{j\geq 1} f_j(\lambda)(XY)^{j+1}$ being $\mathcal{X}_\lambda$ the vector field in $\mathbb{C}^2$ associated to (11) and where $f_j \in \mathbb{R}[\lambda]$ are just the already defined real focus quantities associated to the origin of family (1). Hence (11) with $\lambda = \lambda^* \in \mathbb{C}^{n-1}$ has a formal first integral if and only if $f_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$. Since family (1) with $\lambda_1 = 0$ is linearly conjugate with family (11) we have that the complex family (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$ has a formal first integral $\mathcal{H}(x, y)$ with $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ if and only if $f_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$.

The above arguments lead to conclude that (10) holds in $\mathbb{C}^{n-1}$ whether for any $\lambda^* \in \mathbb{C}^{n-1}$ satisfying $f_{j_1}(\lambda^*) = \cdots = f_{j_s}(\lambda^*) = 0$ one of the following equivalent consequences holds when they are proved using only analytic (not geometric) arguments valid for $(x, y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}^{n-1}$:

(i) there is a formal first integral $H(x, y) = x^2 + y^2 + \cdots$ of (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$.

(ii) there is a formal inverse integrating factor $V(x, y) = 1 + \cdots$ of (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$.

Approach I following the way (i) is used in [8] in the more degenerate context of bounding the cyclicity of some monodromic nilpotent singularities. We observe that if we have the explicit expression of a formal or analytic real first integral of certain subfamily of centers of (1) (this always happens in the Hamiltonian subfamily) we can directly check whether this first integral can be extended to the complex setting concluding that (i) is true. The same remains true changing the above real first integral by a real formal or analytic inverse integrating factor in closed form and non-vanishing at the origin for the second option (ii).

Remark 11. This note concerns on the reversible component of the center variety. Following [14] a complex system $\dot{z} = F(z, w)$, $\dot{w} = G(z, w)$ on $\mathbb{C}^2$ is *time-reversible* if there exists $\gamma \in \mathbb{C}\backslash\{0\}$ such that $F(z, w) = -\gamma G(\gamma w, \gamma^{-1} z)$. In [14] it is showed that every polynomial complex time-reversible of the form $\dot{z} = iz + \cdots$, $\dot{w} = -iw + \cdots$ has a complex center at the origin.

If we complexify a real system as in (6) with $z = x + iy$ by adding to $\dot{z} = F(z, \bar{z})$ the conjugate $\dot{\bar{z}} = G(z, \bar{z}) = \overline{F(z, \bar{z})}$, setting $\gamma = e^{2i\varphi}$ with $\varphi \in \mathbb{R}$ we obtain the time-reversibility condition is $e^{2i\varphi}\overline{F(z, \bar{z})} = -F(e^{2i\varphi}\bar{z}, e^{-2i\varphi}z)$. The geometrical interpretation is that after a rotation $z \mapsto e^{-i\varphi}z$ of angle φ the initial real system is time-reversible with respect to the x-axis, that is, the real system is invariant under the discrete symmetry $(x, y, t) \mapsto (x, -y, -t)$.

Despite the above difficulties one encounters to prove that if (10) holds in $\mathbb{R}^{n-1}$ then it also holds in $\mathbb{C}^{n-1}$, there is a a wide class of systems (1), the time-reversible centers, for which the former is true. We prove this fact in the following proposition.

Proposition 12. *Let system* (1) *with* $(\lambda_1, \lambda) = (0, \lambda^*)$ *and* $\lambda^* \in \mathbb{R}^{n-1}$ *be time-reversible. Then its complex extension to* $(x, y) \in \mathbb{C}^2$ *and* $\lambda^* \in \mathbb{C}^{n-1}$ *possesses a holomorphic first integral near the origin. In particular this* $\lambda^* \in \mathbb{C}^{n-1}$ *vanishes all the real focal values, i.e.,* $f_k(\lambda^*) = 0$ *for all* $k \in \mathbb{N}$.

Proof. Since system (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R}^n$ is time-reversible, after a rotation of angle $\varphi = \varphi(\lambda^*) \in \mathbb{R}$ we can take the x-axis as the symmetry axis, hence the system is invariant under the involution $(x, y, t) \mapsto (x, -y, -t)$. Therefore after doing the linear change of coordinates $(x, y) \mapsto (x\cos\varphi + y\sin\varphi, y\cos\varphi - x\sin\varphi)$ system (1) becomes

$$\dot{x} = -y + yA(x, y^2; \lambda^*), \qquad \dot{y} = x + B(x, y^2; \lambda^*). \tag{12}$$

Clearly this action can be also performed if we make the extension $(x, y) \in \mathbb{C}^2$ and $\lambda^* \in \mathbb{C}^{n-1}$ to the complex setting. The only difference is that now $\varphi(\lambda^*) \in \mathbb{C}$.

The polynomial mapping $(x, y) \mapsto (x, u) = (x, y^2)$ transforms $(x, y) = (0, 0)$ into $(x, u) = (0, 0)$ and (12) into a system that after scaling and removing the common factor y in both components becomes

$$\dot{x} = 1 - A(x, u; \lambda^*), \quad \dot{u} = 2(x + B(x, u; \lambda^*)).$$

Since the origin in no longer a singularity, this system has a holomorphic first integral $\hat{H}(x, u; \lambda^*)$. Finally $\hat{H}$ is pulled back to the holomorphic first integral $H(x, y; \lambda^*) = \hat{H}(x, y^2; \lambda^*)$ of (12) with $(x, y) \in \mathbb{C}^2$ and $\lambda^* \in \mathbb{C}^{n-1}$. This implies (undoing the complex rotation) that (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$ possesses a holomorphic first integral near the origin proving the first part. The second part is a consequence of the argument involved in way (i) of Approach I. □

Approach II. This is the main route for bounding the cyclicity of a center at the origin in [14] and is based on the complexification (7) of family (1) with $\lambda_1 = 0$. A necessary condition to follow this route is to have previously solved the associated complex center problem of the larger family (7).

Theorem 13 (Second Radical Ideal Cyclicity Bound Theorem)**.** *Let $\{g_{j_1}(\mu), \ldots, g_{j_m}(\mu)\}$ be a minimal basis of the ideal $\mathcal{I}_{j_m} \subseteq \mathcal{I}$ where $\mathcal{I}$ is the complex Bautin ideal associated to the complexification* (7) *of family* (1)*. Assume that $\mathcal{I}_{j_m}$ is radical and that the complex center problem is solved in the sense that $\mathbf{V}(\mathcal{I}) = \mathbf{V}(\mathcal{I}_{j_m})$. Then $\mathcal{I} = \mathcal{I}_{j_m}$ and, in particular, the cyclicity of any center at the origin in* (1) *is at most m.*

2.2. **Non-radical Bautin ideal.** Suppose the center problem has been already solved in the sense that we know the center variety $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{j_s})$ but $\mathcal{B}_{j_s}$ is not radical. In this case the methods presented in the above subsection are not longer valid. Anyway we can also obtain an upper bound on the cyclicity of the center at the origin of family (1) in some subset of the center variety as shows Theorem 14. It is based on some ideas from [7] and its proof is analogous (with small technical differences) to that presented in [8] for some class of nilpotent monodromic singularities.

Before stating the next theorem we recall that a polynomial ideal $\mathcal{J} \subset \mathbb{K}[\mathbf{x}]$ is *prime* if whenever $p, q \in \mathbb{K}[\mathbf{x}]$ with $p\,q \in \mathcal{J}$ then either $p \in \mathcal{J}$ or $q \in \mathcal{J}$. The ideal $\mathcal{J}$ is *primary* if $p\,q \in \mathcal{J}$ implies either $p \in \mathcal{J}$ or the power $q^s \in \mathcal{J}$ for some positive $s \in \mathbb{N}$. Every radical ideal can be written as the intersection of prime ideals. Also it is known by the Lasker-Noether Theorem (see [5]) that an arbitrary ideal $\mathcal{J}$ can be decomposed as the intersection of a finite number of primary ideals.

Theorem 14. *Assume that the center problem at the origin of family* (1) *has been solved and its center variety $\mathbf{V}(\mathcal{B})$ satisfies that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{j_s})$ as varieties in $\mathbb{C}^{n-1}$. Let $\{f_{j_1}, \ldots, f_{j_s}\}$ be a minimal basis of $\mathcal{B}_{j_s}$ and suppose a primary decomposition of $\mathcal{B}_{j_s}$ can be written as $\mathcal{B}_{j_s} = R \cap N$ where R is the intersection of the ideals in the decomposition that are prime and N is the intersection of the remaining ideals in the decomposition. Then for any system of family* (1) *corresponding to $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(N)$, the cyclicity of the center at the origin is at most s.*

When the complex Bautin ideal $\mathcal{I}$ is not radical and therefore Theorem 13 does not work there is a method developed in [10] which seeks for transforming the problem to a new ring different of $\mathbb{C}[\mu]$ in which Theorem 13 still can be applied. See also [16] for details.

3. Proof of Theorem 2

Let $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda)\}$ be a minimal basis of the Bautin ideal $\mathcal{B}$ associated to the origin of system (1) with $\lambda_1 = 0$. In Chapter 4 of [15] and in Chapter 6 of [14] it is proved that $d(h;\lambda)$ can be written in the form (3). Then $d(h;\lambda) = \sum_{j=1}^{m} v_{i_j}(\lambda) h^{i_j} q_j(h;\lambda)$ where $q_j(h;\lambda)$ are analytic functions in the variables h and λ. It is known that $v_{i_j}(\lambda^*) = 0$ for all $j = 1, \ldots, m$ when the differential system (1) has a center at the origin for $\lambda = \lambda^*$.

We know that $v_1(\lambda) = \lambda_1$ and if $i_1 = 1$ then $q_1(0;\lambda) = (\exp(2\pi\lambda_1) - 1)/\lambda_1$. Also $v_{i_j}(\lambda) \in \mathbb{R}[\lambda_2]$ and $q_j(0;\lambda^*) = 1$ for $j = 1, \ldots, m$.

We have for system (4) a displacement map whose Taylor expansion at $\varepsilon = 0$ is

$$d(h;\lambda(\varepsilon)) = \Pi(h;\varepsilon) - h = M_k(h)\varepsilon^k + \mathcal{O}(\varepsilon^{k+1}), \tag{13}$$

where $M_k(h)$ is the k–th *Melnikov function* with $k \geq 1$. The function $M_k(h)$ is defined and analytic on the full transversal section Σ. The isolated zeroes of $M_k(h)$ (counted with multiplicity) allow to study the number of limit cycles of system (4).

Let $\lambda_1^* = 0$ and denote the components of $\lambda^* = (\lambda_2^*, \ldots, \lambda_n^*) \in \mathbb{R}^{n-1}$. Since $\lambda(0) = \lambda^*$ and $v_{i_j}(\lambda^*) = 0$ for all $j = 1, \ldots, m$ we can now do the following expansions

$$\lambda_i(\varepsilon) = \sum_{\ell \geq 0} \lambda_{i,\ell}\, \varepsilon^\ell \ , \ \ v_{i_j}(\lambda(\varepsilon)) = \sum_{r \geq 1} \bar{v}_{i_j,r}\, \varepsilon^r.$$

We do some explicit computations at first order in ε. Since $\lambda(0) = \lambda^*$, $q_j(h;\lambda(\varepsilon)) = q_j(h;\lambda^*) + \mathcal{O}(\varepsilon)$. Additionally with our notation we have $v_{i_j}(\lambda(\varepsilon)) = \bar{v}_{i_j,1}\,\varepsilon + \mathcal{O}(\varepsilon^2)$. Therefore the displacement map of the perturbed system (4) for ε sufficiently small can be written as

$$\begin{aligned} d(h;\lambda(\varepsilon)) = & \sum_{j=1}^{m} v_{i_j}(\lambda(\varepsilon)) h^{i_j} q_j(h;\lambda(\varepsilon)) \\ = & \sum_{j=1}^{m} [\bar{v}_{i_j,1}\,\varepsilon + \mathcal{O}(\varepsilon^2)] h^{i_j} [q_j(h;\lambda^*) + \mathcal{O}(\varepsilon)] \\ = & \left(\sum_{j=1}^{m} \bar{v}_{i_j,1} h^{i_j} q_j(h;\lambda^*) \right) \varepsilon + \mathcal{O}(\varepsilon^2), \end{aligned}$$

with $q_j(0;\lambda^*) = 1$, see [1, 3, 17].

The previous simple computations at first order in ε have been generalized to any order in the recent work [2] where it is proved that there are m linearly independent functions $h^{i_j}Q_j(h)$ which are analytic in the variable h in the whole period annulus and with $Q_j(0) \neq 0$ for $j = 1, \ldots, m$, such that the Melnikov functions satisfy

$$M_k(h) = \sum_{j=1}^{m} \bar{v}_{i_j,k}\, h^{i_j}\, Q_j(h). \tag{14}$$

We will obtain an upper bound for the maximum number of zeroes of $d(h;\lambda(\varepsilon))$ near $(h,\varepsilon) = (0,0)$ where $h > 0$ and $\varepsilon \neq 0$ are sufficiently small, i.e. the maximum number of small amplitude limit cycles of the perturbed system (4) with $|\varepsilon| \neq 0$ small enough that can bifurcate from the center at the origin of system (4) when $\varepsilon = 0$.

First from (3) or (9) we note that if $|h|$ and $|\varepsilon|$ are sufficiently small then the number of local limit cycles of the perturbed system (4) is given by the number of small positive zeroes of

$$\begin{aligned} B(h^2;\lambda(\varepsilon)) &= 2\pi\lambda_1(\varepsilon) + \sum_{j=1}^{m} v_{i_j}(\lambda(\varepsilon))h^{i_j-1} \\ &= 2\pi\lambda_1(\varepsilon) + \sum_{j=1}^{m} g_{k_j}(\lambda(\varepsilon))h^{2k_j}, \end{aligned} \tag{15}$$

that bifurcates from $h = 0$ at $\varepsilon = 0$. We call the polynomial (15) in the variable h the *Bautin polynomial* and we emphasize that the zeroes in h of B come in pairs of opposite sign.

Let k and j^* as in Theorem 2, that is, let first $k \geq 1$ and last j^* with $1 \leq j^* \leq m$ be the smallest integer numbers such that $\bar{v}_{i_{j^*},k} \neq 0$. Perturbing with $\lambda_{1,k} = 0$ the Bautin polynomial (15) is

$$B(h^2;\lambda(\varepsilon)) = B_1(h^2)\varepsilon^k + \varepsilon^{k+1}B_2(h^2;\varepsilon),$$

where

$$B_1(h^2) = \sum_{j=j^*}^{m} \bar{v}_{i_j,k}h^{i_j-1}. \tag{16}$$

We define the *reduced Bautin polynomial* $\hat{B}(h^2;\lambda(\varepsilon)) = B(h^2;\lambda(\varepsilon))/\varepsilon^k$. For $\varepsilon > 0$ sufficiently small $\hat{B}(h;\lambda(\varepsilon))$ has the same roots than the Bautin polynomial $B(h^2;\lambda(\varepsilon))$ and is given by

$$\hat{B}(h^2;\lambda(\varepsilon)) = B_1(h^2) + \varepsilon B_2(h^2;\varepsilon). \tag{17}$$

Recall that by definition of a minimal basis we have the order $i_1 < i_2 < \cdots < i_m$. Hence since i_j are odd, from (17) and (16) and using standard arguments from the bifurcation theory we see that $\hat{B}(h^2;\lambda(\varepsilon))$ can have at most j^* distinct positive real roots $h_1(\varepsilon) > h_2(\varepsilon) > \cdots > h_{j^*}(\varepsilon) > 0$

near $h = 0$ for $|\varepsilon|$ small enough satisfying $\lim_{\varepsilon\to 0} h_s(\varepsilon) = 0$ for all $s = 1, \dots, j^*$. Then $\text{Cycl}(\mathcal{X}_\varepsilon, 0) \leq j^*$. This completes the proof of Theorem 2.

4. Proof of Corollary 3

We use the notation of the proof of Theorem 2. Now under the assumptions

$$|v_{i_1}(\lambda(\varepsilon))| \ll |v_{i_2}(\lambda(\varepsilon))| \ll \cdots \ll |v_{i_{j^*}}(\lambda(\varepsilon))| \ll 1,$$

and $v_{i_j}(\lambda(\varepsilon))v_{i_{j+1}}(\lambda(\varepsilon)) < 0$ for $j = 1, \dots, j^* - 1$ it is straightforward to check using again standard arguments in bifurcation theory that the reduced Bautin polynomial $\hat{B}(h^2; \lambda(\varepsilon)) = B_1(h^2) + \varepsilon B_2(h^2; \varepsilon)$ has exactly j^* real positive roots near $h = 0$ for $|\varepsilon|$ small enough.

It is helpful to remember in this argument expression (15) and that v_{i_j} do not depend on $\lambda_1(\varepsilon)$, which is free. Hence in the last perturbation step we take

$$|\lambda_1(\varepsilon)| \ll |v_{i_1}(\lambda(\varepsilon))| \text{ and } \lambda_1(\varepsilon)\, v_{i_1}(\lambda(\varepsilon)) < 0$$

in order to produce the last zero $h_{j^*}(\varepsilon)$.

5. Cubic-like systems

It is clear that if you show that $\mathbf{V}(\mathcal{B}_s) = \mathbf{V}(\mathcal{B})$ for some integer $s \geq 1$ then you have solved the center problem of the polynomial family. This is the case of [12] where it is proved that the polynomial differential family

$$\dot{z} = (i + \lambda_1)z + (z\bar{z})^{\frac{d-3}{2}}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3), \tag{18}$$

with $d \geq 5$ odd has a center at $z = 0$ if and only if $\lambda_1 = 0$ and one of the following two sets of conditions hold:

(c.1) Integrable case: $b_1 = 3A + \bar{C} = 0$;
(c.2) Reversible case: $b_1 = \text{Im}(AC) = \text{Re}(A^2 D) = \text{Re}(\bar{C}^2 D) = 0$.

We recall that the integrable case (c.1) means that family (18) can be written after rescaling by $|z|^{d-3}$ into the form $\dot{z} = i\partial H/\partial\bar{z}$ where $H(z, \bar{z})$ is a function such that $\exp(H)$ for $d = 5$ and H for $d \geq 7$ odd are both real analytic first integrals in a neighborhood of $(x, y) = (0, 0)$.

Writing the center conditions of family (18) in terms of the real parameters $\lambda = (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$ one has

(c.1) Integrable case: $b_1 = 3a_1 + c_1 = 3a_2 - c_2 = 0$;
(c.2) Reversible case: $b_1 = a_2 c_1 + a_1 c_2 = a_1^2 d_1 - a_2^2 d_1 - 2a_1 a_2 d_2 = c_1^2 d_1 - c_2^2 d_1 + 2c_1 c_2 d_2 = 0$.

From now we will focus on (18) with degree $d = 5$, hence we restrict our study to the quintic family (5). First we will see that Approach II does not work in this case. The complexification of family (5) with $\lambda_1 = 0$ is

$$
\begin{array}{rcl}
\dot{z} & = & iz + zw(Az^3 + Bz^2w + Czw^2 + Dw^3), \\
\dot{w} & = & -iw + wz(Ew^3 + Fw^2z + Gwz^2 + Hz^3),
\end{array} \tag{19}
$$

with parameters $\mu = (A, B, C, D, E, F, G, H) \in \mathbb{C}^8$. We have computed the first non vanishing reduced complex focal values obtaining $g_{2j+1}(\mu) \equiv 0$ and up to a multiplicative constant

$$
\begin{array}{rcl}
g_2(\mu) & = & b_1, \\
\tilde{g}_4(\mu) & = & AC - EG, \\
\tilde{g}_6(\mu) & = & 3ADG + DG^2 + C^2H + 3CEH, \\
\tilde{g}_8(\mu) & = & F(9A^2D - DG^2 - C^2H + 9E^2H), \\
\tilde{g}_{10}(\mu) & = & -108A^3DE - 4DEG^3 + 81A^2D^2H - 108AE^3H - \\
 & & 4C^2EGH - 9D^2G^2H - 9C^2DH^2 + 81DE^2H^2, \\
\tilde{g}_{14}(\mu) & = & D^2H^2(9A^2D - DG^2 - C^2H + 9E^2H), \\
\tilde{g}_{16}(\mu) & = & DH(81A^4D^2 - D^2G^4 - C^2DG^2H + 9DE^2G^2H + \\
 & & 9C^2E^2H^2 - 81E^4H^2).
\end{array}
$$

We want to find $k \in \mathbb{N}$ such that $\mathbf{V}(\mathcal{I}) = \mathbf{V}(\mathcal{I}_k)$. Computations show that $\tilde{g}_j(\mu) \in \sqrt{\mathcal{I}_{14}}$ for $j \in \{16, 18, 20, 22, 24\}$ so that we expect that $k = 14$ and $\mathcal{I}_{14} = \langle g_2(\mu), \tilde{g}_4(\mu), \tilde{g}_6(\mu), \tilde{g}_8(\mu), \tilde{g}_{10}(\mu), \tilde{g}_{14}(\mu)\rangle$. But unfortunately $\mathcal{I}_{14}$ is not a radical ideal in the ring $\mathbb{C}[\mu]$ (use for example the `IsRadical` command of Maple which tests whether a given polynomial ideal is radical or not) so that we cannot apply the strategy explained in Approach II and more concretely Theorem 13 in order to get a finite set of generators of the complex Bautin ideal $\mathcal{I}$.

Thus from now on we opted to try Approach I to family (5) with parameters $\lambda = (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$. We have computed the first non vanishing reduced focal values obtaining $f_{2j+1}(\lambda) \equiv 0$ and up to a positive multiplicative constant they are

$$
\begin{aligned}
f_2(\lambda) &= b_1, \\
\tilde{f}_4(\lambda) &= -a_2c_1 - a_1c_2, \\
\tilde{f}_6(\lambda) &= 3a_1c_1d_1 + c_1^2d_1 + 3a_2c_2d_1 - c_2^2d_1 - 6a_2c_1d_2 + 2c_1c_2d_2, \\
\tilde{f}_8(\lambda) &= -b_2(9a_1^2d_1 - 9a_2^2d_1 - c_1^2d_1 + c_2^2d_1 - 18a_1a_2d_2 - 2c_1c_2d_2), \\
\tilde{f}_{10}(\lambda) &= -324a_1^4d_1 + 324a_2^4d_1 + 4c_1^4d_1 - 4c_2^4d_1 + 243a_1^2d_1^3 - 243a_2^2d_1^3 - \\
&\quad 27c_1^2d_1^3 + 27c_2^2d_1^3 + 648a_1^3a_2d_2 + 648a_1a_2^3d_2 + 8c_1^3c_2d_2 + \\
&\quad 8c_1c_2^3d_2 - 486a_1a_2d_1^2d_2 - 54c_1c_2d_1^2d_2 + 243a_1^2d_1d_2^2 - \\
&\quad 243a_2^2d_1d_2^2 - 27c_1^2d_1d_2^2 + 27c_2^2d_1d_2^2 - 486a_1a_2d_2^3 - 54c_1c_2d_2^3, \\
\tilde{f}_{14}(\lambda) &= (d_1^2 + d_2^2)^2(9a_1^2d_1 - 9a_2^2d_1 - c_1^2d_1 + c_2^2d_1 - 18a_1a_2d_2 - 2c_1c_2d_2), \\
\tilde{f}_{16}(\lambda) &= -(d_1^2 + d_2^2)(81a_1^3a_2d_1^2 - 81a_1a_2^3d_1^2 + c_1^3c_2d_1^2 - c_1c_2^3d_1^2 - \\
&\quad 243a_1^2a_2^2d_1d_2 + 81a_2^4d_1d_2 + 3c_1^2c_2^2d_1d_2 - c_2^4d_1d_2 + 162a_1a_2^3d_2^2 + \\
&\quad 2c_1c_2^3d_2^2).
\end{aligned}
$$

We see that $\tilde{f}_{16}(\lambda) \notin \mathcal{B}_{14}$ and also we can check that $\tilde{f}_j(\lambda) \in \mathcal{B}_{16}$ for $j \in \{18, 20, 22, 24, 26\}$ making it probable that the ideal

$$\mathcal{B}_{16} = \langle f_2(\lambda), \tilde{f}_4(\lambda), \tilde{f}_6(\lambda), \tilde{f}_8(\lambda), \tilde{f}_{10}(\lambda), \tilde{f}_{14}(\lambda), \tilde{f}_{16}(\lambda) \rangle$$

is in fact $\mathcal{B}$. Under this hypothesis and since the former generators are a minimal basis of $\mathcal{B}_{16}$ we would obtain that an upper bound on the cyclicity of any center at the origin in family (5) is seven using Theorem 7. Taking into account these facts we have strong evidences to state Conjecture 5.

Unfortunately $\mathcal{B}_{16}$ is not a radical ideal in the ring $\mathbb{R}[\lambda]$ so that we cannot apply Theorem 9 for finding a finite set of generators of the Bautin ideal $\mathcal{B}$.

Now we will prove a proposition that we will need later on for proving Theorem 16.

Proposition 15. *The center variety* $\mathbf{V}(\mathcal{B}) \subset \mathbb{R}^8$ *of family* (5) *is* $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$. *This equality also holds in* $\mathbb{C}^8$.

Proof. Using the routine `minAssChar` in the `primdec.LIB` library of Singular we find that the prime decomposition of $\sqrt{\mathcal{B}_{14}}$ is $\sqrt{\mathcal{B}_{14}} = \cap_{i=1}^3 J_i$ where

$$
\begin{aligned}
J_1 &= \langle b_1, a_2c_1 + a_1c_2, a_1^2d_1 - a_2^2d_1 - 2a_1a_2d_2, c_1^2d_1 - c_2^2d_1 + 2c_1c_2d_2 \rangle, \\
J_2 &= \langle b_1, 3a_1 + c_1, 3a_2 - c_2 \rangle,
\end{aligned}
$$

and we also check that the real variety

$$\mathbf{V}(J_3) = \{\lambda \in \mathbb{R}^8 : A = B = C = D = 0\}$$

corresponds thus to the linear center $\dot{z} = iz$.

Since $\mathbf{V}(J_3) \subset \mathbf{V}(J_k)$ for any $k \in \{1, 2\}$ we have deduced that $\mathbf{V}(\mathcal{B}_{14})$ decomposes as the union of irreducible components as

$$\mathbf{V}(\mathcal{B}_{14}) = \mathbf{V}(\sqrt{\mathcal{B}_{14}}) = \mathbf{V}(J_1) \cup \mathbf{V}(J_2).$$

We also notice that the origin of family (5) is a center if all the generators of either J_1 or J_2 vanish, hence it is established that the center variety is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$ according with the results of [12] (see the former center conditions (c.1) and (c.2)).

In order to prove that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$ also holds in $\mathbb{C}^8$ we will prove that when $\lambda^* \in \mathbf{V}(J_i)$ for $i \in \{1, 2\}$ this forces the existence of a formal first integral $H(x, y)$ of the associated system (1) with $\lambda_1 = 0$ to (5) extended to the complex setting, i.e., with $(x, y) \in \mathbb{C}^2$ and $\lambda = \lambda^* \in \mathbb{C}^8$.

- When $\lambda^* \in \mathbf{V}(J_2)$ in [12] it is proved that $\exp(H(z, \bar{z}))$ with $H(z, \bar{z}) = \log|z|^2 + i(\bar{A}z\bar{z}^3 - Az^3\bar{z}) + \frac{1}{2}\mathrm{Im}(B)z^2\bar{z}^2 - \frac{i}{4}(D\bar{z}^4 - \bar{D}z^4)$ is a real analytic at $(x, y) = (0, 0)$ (hence formal) first integral of the real system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ associated to (5) which is obviously extended to a formal first integral in the complex setting.
- Let $\lambda^* \in \mathbf{V}(J_1)$. In that case [12] shows that (5) is time-reversible, i.e., after a rotation $z \mapsto \mathrm{e}^{-i\varphi}z$ of some angle $\varphi \in \mathbb{R}$ it is invariant under the symmetry $(z, \bar{z}, t) \mapsto (\bar{z}, z, -t)$. More precisely we have $A = -\bar{A}\exp(-4i\varphi)$, $C = -\bar{C}\exp(4i\varphi)$, $D = -\bar{D}\exp(8i\varphi)$ and $B = -\bar{B}$ for some φ. Therefore from Proposition 12 we deduce the existence of a formal first integral of (5) with $\lambda_1 = 0$ extended to the complex setting.

The proof is done. □

An application of Theorem 14 to family (5) is the following result which proves statement (a) of Theorem 4.

Theorem 16. *For any system in the family* (5) *corresponding to a parameter value* $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \{0\}$ *the cyclicity of the center at the origin is at most* 6.

Proof. One consequence of the work [12] regarding family (5) is that the center variety is given by $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$. Recall that $\tilde{f}_{16}(\lambda) \in \sqrt{\mathcal{B}_{14}}$. A minimal basis of $\mathcal{B}_{14}$ is

$$\{f_2(\lambda), \tilde{f}_4(\lambda), \tilde{f}_6(\lambda), \tilde{f}_8(\lambda), \tilde{f}_{10}(\lambda), \tilde{f}_{14}(\lambda)\},$$

and therefore it contains 6 elements. Now we find the primary decomposition of $\mathcal{B}_{14}$. For this purpose we can use either of the routines `primdecGTZ` or `primdecSY` in the `primdec.LIB` library of SINGULAR. The outcome is that $\mathcal{B}_{14} = \bigcap_{i=1}^{7} I_i$ where I_1, I_2 and I_3 are radical ideals and

$$\begin{aligned}
\sqrt{I_4} &= \langle d_1^2 + d_2^2, c_2 d_1 + c_1 d_2, c_1 d_1 - c_2 d_2, c_1^2 + c_2^2, b_2, -a_2 d_1 + a_1 d_2, \\
&\quad a_1 d_1 + a_2 d_2, a_2 c_1 + a_1 c_2, a_1 c_1 - a_2 c_2, a_1^2 + a_2^2, b_1 \rangle, \\
\sqrt{I_5} &= \langle d_1^2 + d_2^2, -c_2 d_1 + c_1 d_2, c_1 d_1 + c_2 d_2, c_1^2 + c_2^2, b_2, a_2 d_1 + a_1 d_2, \\
&\quad a_1 d_1 - a_2 d_2, a_2 c_1 + a_1 c_2, a_1 c_1 - a_2 c_2, a_1^2 + a_2^2, b_1 \rangle, \\
\sqrt{I_6} &= \langle d_1^2 + d_2^2, b_2, -c_2 d_1 + 3 a_2 d_2, 3 a_2 d_1 + c_2 d_2, 9 a_2^2 + c_2^2, c_1 d_1 + 3 a_1 d_2, \\
&\quad 3 a_1 d_1 - c_1 d_2, a_2 c_1 + a_1 c_2, 9 a_1 a_2 - c_1 c_2, 9 a_1^2 + c_1^2, b_1 \rangle, \\
\sqrt{I_7} &= \langle d_1^2 + d_2^2, b_2, c_2 d_1 + 3 a_2 d_2, 3 a_2 d_1 - c_2 d_2, 9 a_2^2 + c_2^2, -c_1 d_1 + 3 a_1 d_2, \\
&\quad 3 a_1 d_1 + c_1 d_2, a_2 c_1 + a_1 c_2, 9 a_1 a_2 - c_1 c_2, 9 a_1^2 + c_1^2, b_1 \rangle.
\end{aligned}$$

Now we define $N = \bigcap_{i=4}^{7} I_i$ as in Theorem 14. Using the `intersect` command of SINGULAR we get a set of generators of $\sqrt{N}$, namely

$$\sqrt{N} = \bigcap_{i=4}^{7} \sqrt{I_i} = \langle b_1, b_2, d_1^2 + d_2^2, a_2 c_1 + a_1 c_2, c_1^2 + 9 a_1^2 + c_2^2 + 9 a_2^2 \rangle.$$

Finally, taking into account that $\mathbf{V}(N) = \mathbf{V}(\sqrt{N})$ holds in any ground field we obtain

$$\mathbf{V}(N) = \{\lambda \in \mathbb{R}^8 : A = B = C = D = 0\} = \{0\}.$$

This means that $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(N)$ if and only if λ^* corresponds to any nonlinear center of (5) and the result follows as consequence of Proposition 15 and Theorem 14. □

After Theorem 16 it is natural to consider perturbations of the linear center $\dot{z} = iz$ inside family (5). The following result goes in this direction and proves statement (b) of Theorem 4.

Theorem 17. *There are perturbations of the linear center $\dot{z} = iz$ inside family* (5) *in such a way that six small amplitude limit cycles bifurcate from the origin.*

Proof. First we will see that the point $\lambda^* = 0$ corresponding to the linear center $\dot{z} = iz$ is not isolated from the set of points in the parameter space Λ corresponding to a system in family (5) possessing a sixth order weak focus at the origin. More precisely if we perturb from $\lambda^* = 0$ to $\lambda(\varepsilon) = (\varepsilon/\sqrt{2}, \varepsilon, 0, 0, 0, 0, \varepsilon, \varepsilon)$ with the small real perturbation parameter ε then $f_2 = \tilde{f}_4 = \tilde{f}_6 = \tilde{f}_8 = \tilde{f}_{10} = 0$

and $\tilde{f}_{14} = -18(1 + 2\sqrt{2})\varepsilon^7 \neq 0$, and therefore the perturbed system $\dot{z} = iz + z\bar{z}(A(\varepsilon)z^3 + B(\varepsilon)z^2\bar{z} + C(\varepsilon)z\bar{z}^2 + D(\varepsilon)\bar{z}^3)$ has a sixth order weak focus at the origin. Since the conditions $B(\varepsilon) = C(\varepsilon) = 4|A(\varepsilon)|^2 - 3|D(\varepsilon)|^2 = 0$ and $D(\varepsilon) = \varepsilon(1 + i) \neq 0$ hold it follows from statement (i) in Theorem 5 of [12] that a further arbitrarily small perturbation can produce six limit cycles bifurcating from the focus at the origin. □

5.1. **The cyclicity of some subfamilies.** Although $\mathcal{B}_{16}$ is not radical in the ring $\mathbb{R}[\lambda]$ we notice that when we fix b_1 as a constant (not a parameter) in family (5) then the resulting ideal

$$\mathcal{B}_{16}^{[b_1]} = \langle f_2(\lambda), \tilde{f}_4(\lambda), \tilde{f}_8(\lambda), \tilde{f}_{10}(\lambda), \tilde{f}_{14}(\lambda), \tilde{f}_{16}(\lambda)\rangle$$

in the ring $\mathbb{R}[\lambda\backslash b_1]$ is radical. The same phenomena occurs for the analogous ideals $\mathcal{B}_{16}^{[c_1]}$, $\mathcal{B}_{16}^{[c_2]}$, $\mathcal{B}_{16}^{[A]}$ and $\mathcal{B}_{16}^{[D]}$ in the rings $\mathbb{R}[\lambda\backslash c_1]$, $\mathbb{R}[\lambda\backslash c_2]$, $\mathbb{R}[\lambda\backslash\{a_1, a_2\}]$ and $\mathbb{R}[\lambda\backslash\{d_1, d_2\}]$, respectively. Then we will prove the following results. In particular Proposition 6 is proved.

Proposition 18. *The cyclicity of the center at the origin in the subfamilies of* (5) *obtained by fixing either the parameter* c_1 *or* c_2 *is* 3 *and is* 2 *when we fix* $D = 0$.

Proof. We only prove the first part of the theorem for the subfamily having fixed c_1 because the other case (to fix c_2) is almost identical.

Fixing c_1, hence working in the ring $\mathbb{R}[\lambda\backslash\{c_1\}]$, we get that $\tilde{f}_i \equiv 0$ for $i \in \{8, 10, 14, 16\}$. Moreover $\sqrt{\mathcal{B}_6^{[c_1]}} = \mathcal{B}_6^{[c_1]}$ in that ring. Then Proposition 15 and Theorem 9 gives $\mathcal{B}^{[c_1]} = \mathcal{B}_6^{[c_1]}$. Since $\{\tilde{f}_2(\lambda), \tilde{f}_4(\lambda), \tilde{f}_6(\lambda)\}$ is a minimal basis of the Bautin ideal in this case we get the Bautin depth 3.

For the second subfamily with $D = 0$ it follows that $\tilde{f}_i \equiv 0$ for $i \in \{6, 8, 10, 14, 16\}$ and $\sqrt{\mathcal{B}_4^{[D]}} = \mathcal{B}_4^{[D]}$ in the ring $\mathbb{R}[\lambda\backslash\{d_1, d_2\}]$. Again by Proposition 15 and Theorem 9, the Bautin ideal is given by $\mathcal{B}^{[D]} = \mathcal{B}_4^{[D]} = \langle \tilde{f}_2(\lambda), \tilde{f}_4(\lambda)\rangle$ and has Bautin depth 2.

Now we will see by using Corollary 3 that the former upper bounds 3 and 2 of the cyclicity are sharp. More precisely we will see that there are perturbations $\lambda(\varepsilon)$ of points λ^* on the component $\mathbf{V}(J_2)$ of the center variety of each of the above subfamilies of (5) producing 3 and 2 limit cycles, respectively.

In the subfamily with constant c_1 we take the parameters $\lambda^* = (a_1, a_2, b_1, b_2, c_2, d_1, d_2) = (-c_1/3, a_2, 0, b_2, 3a_2, d_1, d_2) \in \mathbf{V}(J_2)$ and perturb to

$$\lambda(\varepsilon) = \lambda^* + (\lambda_2(\varepsilon), \lambda_3(\varepsilon), \lambda_4(\varepsilon), \lambda_5(\varepsilon), \lambda_7(\varepsilon), \lambda_8(\varepsilon), \lambda_9(\varepsilon)) \in \mathbb{R}^7,$$

with $\lambda_i(\varepsilon) = \sum_{j\geq 1} \lambda_{i,j}\, \varepsilon^j$ for $i \in \{2, 3, 4, 5, 7, 8, 9\}$. We assume from now that $a_2 \neq 0$ and we choose the perturbation with $\lambda_{4,1} = \lambda_{4,2} = 0$ and $\lambda_{2,1} = -\frac{c_1}{9a_2}(3\lambda_{3,1} - \lambda_{7,1})$, then we obtain $f_2(\lambda(\varepsilon) = \mathcal{O}(\varepsilon^3)$, $f_4(\lambda(\varepsilon) = \mathcal{O}(\varepsilon^2)$ and $f_8(\lambda(\varepsilon) = \mathcal{O}(\varepsilon)$ so that $|f_2(\lambda(\varepsilon)| \ll |f_4(\lambda(\varepsilon)| \ll |f_8(\lambda(\varepsilon)| \ll 1$ for $|\varepsilon|$ sufficiently small. In addition we can take, as a simple example, $a_2 = d_1 = c_1 = d_2 = \lambda_{3,1} = \lambda_{7,1} = \lambda_{2,2} = \lambda_{4,3} = 1$ and $\lambda_{3,2} = \lambda_{7,2} = 0$ yielding $f_2(\lambda(\varepsilon) = \varepsilon^3 + \mathcal{O}(\varepsilon^4)$, $f_4(\lambda(\varepsilon) = -\frac{25}{9}\varepsilon^2 + \mathcal{O}(\varepsilon^3)$ and $f_8(\lambda(\varepsilon) = \frac{4}{3}\varepsilon + \mathcal{O}(\varepsilon^2)$. Hence for $|\varepsilon| \ll 1$ the following alternate signs hold: $f_2(\lambda(\varepsilon) > 0$, $f_4(\lambda(\varepsilon) < 0$ and $f_8(\lambda(\varepsilon) > 0$. From Corollary 3 we get that 3 is the cyclicity in this subfamily.

Analogously for the subfamily with $D = 0$, we choose the initial point $\lambda^* = (a_1, a_2, b_1, b_2, c_1, c_2) = (a_1, a_2, 0, b_2, -3a_1, 3a_2) \in \mathbf{V}(J_2)$ and perturb to $\lambda(\varepsilon) = \lambda^* + (\mathcal{O}(\varepsilon^2), \mathcal{O}(\varepsilon^2), \varepsilon^2, 0, \mathcal{O}(\varepsilon^2), \varepsilon) \in \mathbb{R}^6$. If we take now, for example, $a_1 = 1$ then we obtain $f_2(\lambda(\varepsilon) = \varepsilon^2 + \mathcal{O}(\varepsilon^3)$ and $f_4(\lambda(\varepsilon) = -\varepsilon + \mathcal{O}(\varepsilon^2)$. Thus using Corollary 3 two limit cycles are created finishing the proof. □

Proposition 19. *The cyclicity of the center at the origin in the following three subfamilies of* (5) *obtained by fixing the parameters either (i)* $b_1 = 0$ *or (ii)* $D \neq 0$ *is constant or (iii)* A *is fixed; is bounded by* 6.

Proof. We start the proof of statement (i). Fixing b_1 we get that $\sqrt{\mathcal{B}_{16}^{[b_1]}} = \mathcal{B}_{16}^{[b_1]}$ in the ground ring $\mathbb{R}[\lambda \backslash b_1]$. Now since the unperturbed family has a center, it is clear that initially $b_1 = 0$ because $g_2(\lambda) = b_1$. Since we perturb inside the subfamily with fixed b_1 then we always have $b_1 = 0$. Taking into account the center problem of (5) is already solved we know that $\mathbf{V}(\mathcal{B}^{[b_1]}) = \mathbf{V}(\sqrt{\mathcal{B}_{16}^{[b_1]}})$. Finally taking into account Proposition 15 and Theorem 9 we get $\mathcal{B}^{[b_1]} = \mathcal{B}_{16}^{[b_1]}$. Additionally when $b_1 = 0$ then $g_2(\lambda) \equiv 0$ and a minimal basis of de Bautin ideal $\mathcal{B}^{[b_1]}$ is $\{\tilde{f}_4(\lambda), \tilde{f}_6(\lambda), \tilde{f}_8(\lambda), \tilde{f}_{10}(\lambda), \tilde{f}_{14}(\lambda), \tilde{f}_{16}(\lambda)\}$, thus the Bautin depth is 6.

Let us prove now in a similar way the rest of the parts of the proposition.

(ii) We fix D and therefore we check that $\tilde{f}_{16} \equiv 0$ and that $\sqrt{\mathcal{B}_{14}^{[D]}} = \mathcal{B}_{14}^{[D]}$ in the ring $\mathbb{R}[\lambda \backslash \{d_1, d_2\}]$. Hence $\mathbf{V}(\mathcal{B}^{[D]}) = \mathbf{V}(\sqrt{\mathcal{B}_{14}^{[D]}})$ and from Proposition 15 and Theorem 9 one has that the Bautin ideal is $\mathcal{B}^{[D]} = \mathcal{B}_{14}^{[D]}$ with

$$\{\tilde{f}_2(\lambda), \tilde{f}_4(\lambda), \tilde{f}_6(\lambda), \tilde{f}_8(\lambda), \tilde{f}_{10}(\lambda), \tilde{f}_{14}(\lambda)\} \tag{20}$$

as minimal basis when $D \neq 0$. Then the Bautin depth is 6.

(iii) Fixing A yields $\tilde{f}_{16} \equiv 0$ and $\sqrt{\mathcal{B}_{14}^{[A]}} = \mathcal{B}_{14}^{[A]}$ in the reduced ring $\mathbb{R}[\lambda \backslash \{a_1, a_2\}]$. We therefore have from Proposition 15 and Theorem 9 that $\mathcal{B}^{[A]} = \mathcal{B}_{14}^{[A]}$ being (20) a minimal basis of the Bautin ideal. Then the Bautin depth is 6.

The proof is finished. □

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