Center cyclicity of a family of quintic polynomial vector fields

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Abstract— We present a method for studying the Hopf cyclicity problem for the non-degenerate centers without the necessity of solving previously the Dulac complex center problem associated to the larger complexified family. As application we analyze the Hopf cyclicity of the centers of the quintic polynomial family written in complex notation as $\dot{z} = iz + z\bar{z}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3)$.

Keywords: Cyclicity, limit cycle, center problem.

1 Introduction

We consider a family of planar polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y + P(x, y, \lambda), \\ \dot{y} &= x + \lambda_1 y + Q(x, y, \lambda), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y, \lambda]$ are the polynomial nonlinearities of system (1) and $(\lambda_1, \lambda) = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \Lambda \subset \mathbb{R}^n$ are the parameters of the family. We assume that for some $(\lambda_1, \lambda) = (0, \lambda^*) \in \Lambda$ system (1) has a *center* at the origin. Of course the origin is always a *monodromic* singularity of family (1), i.e., it is a center or a *focus* and clearly when $\lambda_1 \neq 0$ it is a focus.

Using a transversal section $\Sigma = [0, \hat{h})$ with endpoint at the origin of coordinates and parameterized by h where $\hat{h} = \hat{h}(\lambda)$, we have the *displacement map* $d : \Sigma \times \Lambda \to \Sigma \times \Lambda$ defined by $d(h; \lambda) = \Pi(h; \lambda) - h$, where $\Pi : \Sigma \times \Lambda \to \Sigma \times \Lambda$ is the *Poincaré* or *return map*. We note that $\hat{h} > 0$ can be finite or infinite.

Since the differential system (1) is analytic, the displacement map $d(h; \lambda)$ is analytic in the variables $h \in [0, \hat{h})$ and λ. Hence we can expand the displacement function $d(h; \lambda) = \sum_{i \geq 1} v_i(\lambda) h^i$ in Taylor series at $h = 0$ where the coefficients $v_i \in \mathbb{R}[\lambda]$ are called *Poincaré–Liapunov constants*. For $\lambda_1 = 0$ the *Bautin ideal* $\mathcal{B}$ at the origin of system (1) is defined as the ideal generated by all the polynomials $v_i(\lambda)$ with $i \geq 1$ in the polynomials ring $\mathbb{R}[\lambda]$. This ideal $\mathcal{B}$ is Noetherian and then by the Hilbert's basis Theorem it is generated by a finite number of polynomials.

DEFINITION 1 The *minimal basis* of a finitely generated ideal I with respect to an ordered basis $B = \{f_1, f_2, f_3, \dots\}$ is the basis M_I defined by the following procedure:

(a) initially set $M_I = \{f_p\}$, where f_p is the first non-zero element of B;
(b) check successive elements f_j, starting with $j = p + 1$, adjoining f_j to M_I if and only if $f_j \notin \langle M_I \rangle$, the ideal generated by M_I.

We will write

$$\mathcal{B} = \langle v_{i_1}(\lambda), v_{i_2}(\lambda), \dots, v_{i_m}(\lambda) \rangle, \tag{2}$$

where $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \dots, v_{i_m}(\lambda)\} \subset \mathbb{R}[\lambda]$ is a minimal basis of the Bautin ideal $\mathcal{B}$. The cardinality m of this basis is called the *Bautin depth* of $\mathcal{B}$ in [6] and it is associated to the chain of ideals $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}$ where $\mathcal{B}_s = \langle v_1(\lambda), \dots, v_s(\lambda) \rangle$ for certain integer $s \geq 1$.

Following Bautin's seminal work [1] in Chapter 4 of [10] and in Chapter 6 of [11] it is proved that when (2) is a minimal basis of the ideal $\mathcal{B}$ then the displacement map $d(h; \lambda)$ can be written in the form

$$d(h; \lambda) = \sum_{j=1}^{m} v_{i_j}(\lambda) h^{i_j} q_j(h; \lambda), \tag{3}$$

where $q_j(h; \lambda)$ are analytic functions in the variables h and λ near $(h, \lambda) = (0, \lambda^*)$ such that $q_j(0; \lambda^*) = 1$. Clearly $v_{i_j}(\lambda^*) = 0$ for all $j = 1, \dots, m$ when the differential system (1) has a center at the origin for $\lambda = \lambda^*$.

The maximum number of small amplitude limit cycles that can bifurcate from a center at the origin of family (1) with $\lambda = \lambda^*$

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under arbitrarily small perturbations inside family (1), that is for $\|\lambda - \lambda^*\| \ll 1$, is called the *Hopf cyclicity* of the center with parameters λ^*.It is well known that the Hopf cyclicity of any center at the origin of (1) is at most the Bautin depth m of $\mathcal{B}$, see Theorem 2.

We will consider the quintic polynomial family written in complex form as

$$(4) \qquad \dot{z} = (i + \lambda_1)z + z\bar{z}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3),$$

with $z = x + iy \in \mathbb{C}$ and parameters $\lambda_1 \in \mathbb{R}$ and $(A, B, C, D) \in \mathbb{C}^4$. The center problem for this family has been solved in [8], but the Hopf cyclicity is only stated for the easier case of having a focus at $z = 0$. In this work we will study the cyclicity problem of the center at $z = 0$ of (4).

The method for bounding the cyclicity does not need to solve previously the Dulac complex center problem associated to the larger complexified family.

2 Background on the cyclicity problem

In this section we summarize several results concerning the cyclicity problem and the approach to that problem using methods from computational commutative algebra. Most of this background can be found in the excellent book [11].

Using the rearrangement (3) of the displacement map $d(h;\lambda)$ and applying Rolle's Theorem several times the following theorem is proved, see for example [1, 6, 11, 10].

THEOREM 2 *Let $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda)\} \subset \mathbb{R}[\lambda]$ be a minimal basis of the Bautin ideal $\mathcal{B}$ associated to the origin of family (1). Then the Hopf cyclicity of any center at the origin in (1) is at most m.*

The Poincaré-Liapunov constants are difficult to work with mainly because to compute them we must perform quadratures. Therefore, instead of working with the Poincaré-Liapunov constants, from the computational point of view it is better to obtain other polynomials $\eta_j(\lambda) \in \mathbb{R}[\lambda]$ that arise as the obstructions in order to get a formal first integral $H(x, y) = x^2 + y^2 + \cdots$ of family (1) with $\lambda_1 = 0$ which is another characterization of centers, see Poincaré [9] and Liapunov [7].

Using the complex coordinate $z = x + iy \in \mathbb{C}$ family (1) with $\lambda_1 = 0$ can be written into the form $\dot{z} = iz + F(z, \bar{z}, \lambda)$ where $\bar{z} = x - iy$ and F is given by the polynomial $F(z, \bar{z}, \lambda) = P\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right) + iQ\left(\frac{1}{2}(z + \bar{z}), \frac{i}{2}(\bar{z} - z), \lambda\right)$. We can adjoin to this complex polynomial differential equation its complex conjugate forming thus the complex system

$$(5) \quad \begin{aligned} \dot{z} &= iz + F(z, \bar{z}, \lambda) = iz + \textstyle\sum_{j+k=2}^N a_{j,k}(\lambda) z^j \bar{z}^k, \\ \dot{\bar{z}} &= -i\bar{z} + \bar{F}(z, \bar{z}, \lambda) = -i\bar{z} + \textstyle\sum_{j+k=2}^N \bar{a}_{j,k}(\lambda) \bar{z}^j z^k. \end{aligned}$$

Replacing the conjugates $\bar{z}$ and $\bar{a}_{j,k}$ by new independent complex state variable and complex parameters, say w and $b_{j,k}$ respectively, yields a larger complex family of systems

$$(6) \qquad \dot{z} = iz + \sum_{j+k=2}^N a_{j,k} z^j w^k, \;\; \dot{w} = -iw + \sum_{j+k=2}^N b_{j,k} w^j z^k,$$

defined in $\mathbb{C}^2$ with complex parameters $\mu = (a_{j,k}, b_{j,k})$. Family (6) is called the *complexification* of family (1) with $\lambda_1 = 0$.

Following Dulac [3] one can generalize the concept of center singularity of systems in $\mathbb{R}^2$ to systems in $\mathbb{C}^2$. To be specific we say that (6) has a (complex) center at the origin $(z, w) = (0, 0)$ when $\mu = \mu^*$ if and only if it admits a formal (complex) first integral $\hat{H}(z, w; \mu^*) = zw + \cdots$. It is easy to check that system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ has a center at the origin if and only if (5) has a center at the origin for $\lambda = \lambda^*$.

We shall define the *focus quantities* $g_j(\mu) \in \mathbb{C}[\mu]$ with $\mu = (a_{j,k}, b_{j,k})$ of the complexification (6). Denote by $\hat{X}_\mu = (iz + \cdots)\partial_z + (-iw + \cdots)\partial_w$ the family of vector fields in $\mathbb{C}^2$ associated with (6). The focus quantities satisfy that when we look for a formal first integral $\hat{H}(z, w; \mu) = zw + \cdots$ of $\hat{X}_\mu$ then $\hat{X}_\mu(\hat{H}) = \sum_{j\geq 1} g_j(\mu)(zw)^{j+1}$.

Let $\mathcal{I}$ the ideal in $\mathbb{C}[\lambda]$ given by $\mathcal{I} = \langle g_j(\mu) : j \in \mathbb{N}\rangle$. It is evident that (6) has a center at the origin when $\mu = \mu^*$ if and only if $\mu^* \in \mathbf{V}(\mathcal{I})$, the complex variety associated to $\mathcal{I}$. We refer to $\mathcal{I}$ and $\mathbf{V}(\mathcal{I})$ as the *complex Bautin ideal* and *complex center variety* respectively.

In order to avoid solve the Dulac center problem, instead of working with complex focus quantities g_j we will work with the *real focus quantities* $f_j(\lambda)$ for family (1) defined as

$$(7) \qquad f_j(\lambda) = g_j(a_{j,k}(\lambda), \bar{a}_{j,k}(\lambda) \in \mathbb{R}[\lambda].$$

Theorem 6.2.3 of [11] describes the relationship between the Poincaré-Liapunov constants $v_j(\lambda)$ and the real focus quantities $f_j(\lambda)$ for family (1). In summary we can finally obtain an upper bound of the Hopf cyclicity only in terms of the real focus quantities instead of Poincaré-Liapunov constants because Theorem 2 can be restated in terms of a minimal Basis of $\mathcal{B}$ formed by real focus quantities. The key point is that expression (3) of the displacement map can be rewritten as

$$(8) \qquad d(h;\lambda) = \sum_{j=1}^m \tilde{f}_{k_j}(\lambda) h^{2k_j+1} \psi_j(h;\lambda),$$

where $\psi_j(h;\lambda)$ are analytic functions in the variables h and λ near $(h, \lambda) = (0, \lambda^*)$ such that $\psi_j(0;\lambda^*) = 1$. So we shall compute real focus quantities instead of the Poincaré-Liapunov constants due to their computational simplicity.

The problem of finding the depth of the Bautin ideal is in general a difficult task, that is the reason for which the cyclicity problem of a center is not easy to solve. However, the problem becomes easier when the Bautin ideal is radical. Recall that the radical of $\mathcal{B}$ is defined as the ideal $\sqrt{\mathcal{B}} = \{p \in \mathbb{R}[\lambda] : p^s \in \mathcal{B}$ for some $s \in \mathbb{N}\}$ and that clearly $\mathcal{B} \subseteq \sqrt{\mathcal{B}}$. If $\mathcal{B} = \sqrt{\mathcal{B}}$ then $\mathcal{B}$ is called a *radical ideal*.

But $\mathcal{B}$ is not always radical, of course. Next theorem allows us to obtain an upper bound on the Hopf cyclicity of the center at the origin of family (1) in some subset of the center variety when $\mathcal{B}$ is not radical. It is based on some ideas from [4] and its proof is analogous (with small technical differences) to that presented in [5] for some class of nilpotent monodromic singularities.

Before stating Theorem 3 we recall that, given a ground field $\mathbb{K}$, a polynomial ideal $\mathcal{J} \subset \mathbb{K}[\mathbf{x}]$ is *prime* if whenever $p, q \in \mathbb{K}[\mathbf{x}]$ with $pq \in \mathcal{J}$ then either $p \in \mathcal{J}$ or $q \in \mathcal{J}$. The ideal $\mathcal{J}$ is *primary* if $pq \in \mathcal{J}$ implies either $p \in \mathcal{J}$ or the power $q^s \in \mathcal{J}$ for some positive $s \in \mathbb{N}$. Every radical ideal can be written as the intersection of prime ideals. Also it is known by the Lasker-Noether Theorem (see [2]) that an arbitrary ideal $\mathcal{J}$ can be decomposed as the intersection of a finite number of primary ideals.

THEOREM 3 *Assume that the center problem at the origin of family (1) has been solved and its center variety $\mathbf{V}(\mathcal{B})$ satisfies that $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{j_s})$ as varieties in $\mathbb{C}^{n-1}$. Let $\{f_{j_1}, \ldots, f_{j_s}\}$ be a minimal basis of $\mathcal{B}_{j_s}$ and suppose a primary decomposition of $\mathcal{B}_{j_s}$ can be written as $\mathcal{B}_{j_s} = \mathcal{R} \cap \mathcal{N}$ where $\mathcal{R}$ is the intersection of the ideals in the decomposition that are prime and $\mathcal{N}$ is the intersection of the remaining ideals in the decomposition. Then for any system of family (1) corresponding to $\lambda^* \in \mathbf{V}(\mathcal{B}) \setminus \mathbf{V}(\mathcal{N})$, the Hopf cyclicity of the center at the origin is at most s.*

3 Main Results

It is clear that if you show that $\mathbf{V}(\mathcal{B}_s) = \mathbf{V}(\mathcal{B})$ for some integer $s \geq 1$ then you have solved the center problem of the polynomial family. This is the case of [8] where it is proved that the polynomial differential family

$$(9) \qquad \dot{z} = (i+\lambda_1)z + (z\bar{z})^{\frac{d-3}{2}}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3),$$

with $d \geq 5$ odd and being $A = a_1 + ia_2$, $B = b_1 + ib_2$, $C = c_1 + ic_2$ and $D = d_1 + id_2$ has a center at $z = 0$ if and only if $\lambda_1 = 0$ and one of the following two sets of conditions hold:

(c.1) Integrable case: $b_1 = 3a_1 + c_1 = 3a_2 - c_2 = 0$;

(c.2) Reversible case: $b_1 = a_2c_1 + a_1c_2 = a_1^2d_1 - a_2^2d_1 - 2a_1a_2d_2 = c_1^2d_1 - c_2^2d_1 + 2c_1c_2d_2 = 0$.

We recall that the integrable case (c.1) means that family (9) can be written after rescaling by $|z|^{d-3}$ into the form $\dot{z} = i\partial H/\partial\bar{z}$ where $H(z,\bar{z})$ is a function such that $\exp(H)$ for $d = 5$ and H for $d \geq 7$ odd are both real analytic first integrals in a neighborhood of $(x, y) = (0, 0)$.

Define the reduced focal values as $\widetilde{f}_k \equiv f_k \mod \mathcal{B}_{k-1}$, that is the remainder of f_k upon division by a Gröbner basis of the ideal $\mathcal{B}_{k-1}$. We have computed the first non-vanishing reduced focal values of (4) with parameters $\lambda = (a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2) \in \mathbb{R}^8$ obtaining $f_{2j+1}(\lambda) \equiv 0$ and up to a positive multiplicative constant they are

$$\begin{aligned}
f_2(\lambda) &= b_1,\\
\tilde{f}_4(\lambda) &= -a_2c_1 - a_1c_2,\\
\tilde{f}_6(\lambda) &= 3a_1c_1d_1 + c_1^2d_1 + 3a_2c_2d_1 - c_2^2d_1 -\\
&\quad 6a_2c_1d_2 + 2c_1c_2d_2,\\
\tilde{f}_8(\lambda) &= -b_2(9a_1^2d_1 - 9a_2^2d_1 - c_1^2d_1 + c_2^2d_1 -\\
&\quad 18a_1a_2d_2 - 2c_1c_2d_2),\\
\tilde{f}_{10}(\lambda) &= -324a_1^4d_1 + 324a_2^4d_1 + 4c_1^4d_1 - 4c_2^4d_1 +\\
&\quad 243a_1^2d_1^3 - 243a_2^2d_1^3 - 27c_1^2d_1^3 + 27c_2^2d_1^3 +\\
&\quad 648a_1^3a_2d_2 + 648a_1a_2^3d_2 + 8c_1^3c_2d_2 +\\
&\quad 8c_1c_2^3d_2 - 486a_1a_2d_1^2d_2 - 54c_1c_2d_1^2d_2 +\\
&\quad 243a_1^2d_1d_2^2 - 243a_2^2d_1d_2^2 - 27c_1^2d_1d_2^2 + -\\
&\quad 27c_2^2d_1d_2^2486a_1a_2d_2^3 - 54c_1c_2d_2^3,\\
\tilde{f}_{14}(\lambda) &= (d_1^2 + d_2^2)^2(9a_1^2d_1 - 9a_2^2d_1 - c_1^2d_1 +\\
&\quad c_2^2d_1 - 18a_1a_2d_2 - 2c_1c_2d_2),\\
\tilde{f}_{16}(\lambda) &= -(d_1^2 + d_2^2)(81a_1^3a_2d_1^2 - 81a_1a_2^3d_1^2 +\\
&\quad c_1^3c_2d_1^2 - c_1c_2^3d_1^2 - 243a_1^2a_2^2d_1d_2 +\\
&\quad 81a_2^4d_1d_2 + 3c_1^2c_2^2d_1d_2 - c_2^4d_1d_2 +\\
&\quad 162a_1a_2^3d_2^2 + 2c_1c_2^3d_2^2),
\end{aligned}$$

The following proposition will be needed later on when we use Theorem 3 for proving our main result, Theorem 6.

PROPOSITION 4 *The center variety $\mathbf{V}(\mathcal{B}) \subset \mathbb{R}^8$ of family (4) is $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$. This equality also holds in $\mathbb{C}^8$.*

Notice that $\mathcal{B}$ and $\mathcal{B}_{14}$ are ideals in $\mathbb{R}[\lambda]$ because we are working with real focal values. A key point in Proposition 4 is that the inequality $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_{14})$ must also hold in $\mathbb{C}^8$. Recall that given two ideals I and J in $\mathbb{R}[\lambda]$, it can be that $\mathbf{V}(I) = \mathbf{V}(J)$ as real varieties included in $\mathbb{R}^k$, but $\mathbf{V}(I) \neq \mathbf{V}(J)$ when they are viewed as complex varieties in $\mathbb{C}^k$. At this point we view family (1) as a system on $\mathbb{C}^2$ with complex parameters, i.e., we will study family (1) with $(x, y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}^{n-1}$. Now we do the linear complex change of coordinates $(x, y) \mapsto (X, Y) = (x + iy, x - iy)$. Notice that now $\bar{Y} \neq X$ but anyway (1) is transformed into

$$(10) \qquad \dot{X} = iX + \mathcal{F}^+(X, Y; \lambda), \;\; \dot{Y} = -iY + \mathcal{F}^-(X, Y; \lambda),$$

where $\mathcal{F}^{\pm}$ only contains nonlinear terms in X and Y because

$$\mathcal{F}^{\pm}(X,Y;\lambda) = P\left(\frac{1}{2}(X+Y), \frac{i}{2}(Y-X), \lambda\right) \pm iQ\left(\frac{1}{2}(X+Y), \frac{i}{2}(Y-X), \lambda\right).$$

In this complex setting we can build a formal series $\tilde{H}(X,Y;\lambda) = XY + \cdots$ such that $\mathcal{X}_\lambda(\tilde{H}) = \sum_{j\geq 1} f_j(\lambda)(XY)^{j+1}$ being $\mathcal{X}_\lambda$ the vector field in $\mathbb{C}^2$ associated to (10) and where $f_j \in \mathbb{R}[\lambda]$ are just the already defined real focus quantities associated to the origin of family (1). Hence (10) with $\lambda = \lambda^* \in \mathbb{C}^{n-1}$ has a formal first integral if and only if $f_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$. Since family (1) with $\lambda_1 = 0$ is linearly conjugate with family (10) we have that the complex family (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$ has a formal first integral $\mathcal{H}(x,y)$ with $\mathcal{H} : \mathbb{C}^2 \to \mathbb{C}$ if and only if $f_j(\lambda^*) = 0$ for all $j \in \mathbb{N}$.

The above arguments lead to conclude that the equality of the varieties $\mathbf{V}(\mathcal{B}) = \mathbf{V}(\mathcal{B}_k)$ holds in $\mathbb{C}^{n-1}$ whether for any $\lambda^* \in \mathbb{C}^{n-1}$ satisfying $f_1(\lambda^*) = \cdots = f_k(\lambda^*) = 0$ there is a formal first integral $H(x,y) = x^2 + y^2 + \cdots$ of (1) with $(\lambda_1, \lambda) = (0, \lambda^*) \in \mathbb{R} \times \mathbb{C}^{n-1}$ and it is proved using only analytic (not geometric) arguments valid for $(x,y) \in \mathbb{C}^2$ and $\lambda \in \mathbb{C}^{n-1}$:

We observe that if we have the explicit expression of a formal or analytic real first integral of certain subfamily of centers of (1) (this always happens in the Hamiltonian subfamily) we can directly check whether this first integral can be extended to the complex setting concluding that equality of the varieties also holds in the complex setting.

There is a wide class of systems (1), the time-reversible centers, for which the former is true. We prove this fact in the following proposition.

PROPOSITION 5 *Let system (1) with $(\lambda_1, \lambda) = (0, \lambda^*)$ and $\lambda^* \in \mathbb{R}^{n-1}$ be time-reversible. Then its complex extension to $(x,y) \in \mathbb{C}^2$ and $\lambda^* \in \mathbb{C}^{n-1}$ possesses a holomorphic first integral near the origin. In particular, this $\lambda^* \in \mathbb{C}^{n-1}$ vanishes all the real focal values, i.e., $f_k(\lambda^*) = 0$ for all $k \in \mathbb{N}$.*

We end with an application of Theorem 3 to family (4) which is one of our main results. Notice that, when reading the statement of the forthcoming Theorem 6, only remains to obtain a cyclicity upper bound of the linear center $\dot{z} = iz$, that is system (4) with $\lambda = 0$. Anyway our calculations show strong evidences for stating the conjecture that such a bound is seven.

THEOREM 6 *The following statements hold.*

(a) Any nonlinear center at the origin of family (4) has Hopf cyclicity at most 6 when we perturb it inside this family.

(b) There are perturbations of the linear center $\dot{z} = iz$ inside family (4) producing 6 limit cycles bifurcating from the origin.

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