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Center cyclicity of a family of quintic polynomial vector fields

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Abstract— We present a method for studying the Hopf cyclicity problem for the non-degenerate centers without the necessity of solving previously the Dulac complex center problem associated to the larger complexified family. As application we analyze the Hopf cyclicity of the centers of the quintic polynomial family written in complex notation as $\dot{z} = iz + z\bar{z}(Az^3 + Bz^2\bar{z} + Cz\bar{z}^2 + D\bar{z}^3)$.

Keywords: Cyclicity, limit cycle, center problem.

1 Introduction

We consider a family of planar polynomial differential systems of the form

$$\begin{aligned} \dot{x} &= \lambda_1 x - y + P(x, y, \lambda), \\ \dot{y} &= x + \lambda_1 y + Q(x, y, \lambda), \end{aligned} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y, \lambda]$ are the polynomial nonlinearities of system (1) and $(\lambda_1, \lambda) = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda \subset \mathbb{R}^n$ are the parameters of the family. We assume that for some $(\lambda_1, \lambda) = (0, \lambda^*) \in \Lambda$ system (1) has a *center* at the origin. Of course the origin is always a *monodromic* singularity of family (1), i.e., it is a center or a *focus* and clearly when $\lambda_1 \neq 0$ it is a focus.

Using a transversal section $\Sigma = [0, \hat{h})$ with endpoint at the origin of coordinates and parameterized by h where $\hat{h} = \hat{h}(\lambda)$, we have the *displacement map* $d : \Sigma \times \Lambda \to \Sigma \times \Lambda$ defined by $d(h; \lambda) = \Pi(h; \lambda) - h$, where $\Pi : \Sigma \times \Lambda \to \Sigma \times \Lambda$ is the *Poincaré* or *return map*. We note that $\hat{h} > 0$ can be finite or infinite.

Since the differential system (1) is analytic, the displacement map $d(h; \lambda)$ is analytic in the variables $h \in [0, \hat{h})$ and λ. Hence we can expand the displacement function $d(h; \lambda) = \sum_{i \geq 1} v_i(\lambda) h^i$ in Taylor series at $h = 0$ where the coefficients $v_i \in \mathbb{R}[\lambda]$ are called *Poincaré–Liapunov constants*. For $\lambda_1 = 0$ the *Bautin ideal* $\mathcal{B}$ at the origin of system (1) is defined as the ideal generated by all the polynomials $v_i(\lambda)$ with $i \geq 1$ in the polynomials ring $\mathbb{R}[\lambda]$. This ideal $\mathcal{B}$ is Noetherian and then by the Hilbert's basis Theorem it is generated by a finite number of polynomials.

DEFINITION 1 The *minimal basis* of a finitely generated ideal I with respect to an ordered basis $B = \{f_1, f_2, f_3, \ldots\}$ is the basis M_I defined by the following procedure:

(a) initially set $M_I = \{f_p\}$, where f_p is the first non-zero element of B;
(b) check successive elements f_j, starting with $j = p + 1$, adjoining f_j to M_I if and only if $f_j \notin \langle M_I \rangle$, the ideal generated by M_I.

We will write

$$\mathcal{B} = \langle v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda) \rangle, \tag{2}$$

where $\{v_{i_1}(\lambda), v_{i_2}(\lambda), \ldots, v_{i_m}(\lambda)\} \subset \mathbb{R}[\lambda]$ is a minimal basis of the Bautin ideal $\mathcal{B}$. The cardinality m of this basis is called the *Bautin depth* of $\mathcal{B}$ in [6] and it is associated to the chain of ideals $\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}$ where $\mathcal{B}_s = \langle v_1(\lambda), \ldots, v_s(\lambda) \rangle$ for certain integer $s \geq 1$.

Following Bautin's seminal work [1] in Chapter 4 of [10] and in Chapter 6 of [11] it is proved that when (2) is a minimal basis of the ideal $\mathcal{B}$ then the displacement map $d(h; \lambda)$ can be written in the form

$$d(h; \lambda) = \sum_{j=1}^{m} v_{i_j}(\lambda) h^{i_j} q_j(h; \lambda), \tag{3}$$

where $q_j(h; \lambda)$ are analytic functions in the variables h and λ near $(h, \lambda) = (0, \lambda^*)$ such that $q_j(0; \lambda^*) = 1$. Clearly $v_{i_j}(\lambda^*) = 0$ for all $j = 1, \ldots, m$ when the differential system (1) has a center at the origin for $\lambda = \lambda^*$.

The maximum number of small amplitude limit cycles that can bifurcate from a center at the origin of family (1) with $\lambda = \lambda^*$

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