THE ROLE OF THE MULTIPLE ZEROS IN THE AVERAGING THEORY OF DIFFERENTIAL EQUATIONS

ISAAC A. GARCÍA[1], JAUME LLIBRE[2] AND SUSANNA MAZA[1]

Abstract. In this work we improve the classical averaging theory applied to λ-families of analytic T-periodic ordinary differential equations in standard form defined on $\mathbb{R}$. First we characterize the set of points z_0 in the phase space and the parameters λ where T-periodic solutions can be produced when we vary a small parameter ε. Second we expand the displacement map in powers of the parameter ε whose coefficients are the averaged functions. The main contribution consists in analyzing the role that have the multiple zeros $z_0 \in \mathbb{R}$ of the first non-zero averaged function. The outcome is that these multiple zeros can be of two different classes depending on whether the points (z_0, λ) belong or not to the analytic set defined by the real variety associated to the ideal generated by the averaged functions in the Noetheriang ring of all the real analytic functions at (z_0, λ). Next we are able to bound the maximum number of branches of isolated T-periodic solutions that can bifurcate from each multiple zero z_0. Sometimes these bounds depend on the cardinalities of minimal bases of the former ideal. Several examples illustrate our results.

1. Introduction and statement of the results

The method of averaging is a classical tool that allows to study the dynamics of the periodic nonlinear differential systems. It has a long history starting with the intuitive classical works of Lagrange and Laplace. Important advances of the averaging theory were made by Bogoliubov and Krylov, the reader can consult [2] for example. For a more modern exposition of the averaging theory see the book of Sanders, Verhulst and Murdock [11].

In this work we consider a family of T-periodic analytic differential equation in $\Omega \subset \mathbb{R}$ of the form

$$\dot{x} = F(t, x; \lambda, \varepsilon) = \sum_{i \geq 1} F_i(t, x; \lambda)\, \varepsilon^i, \tag{1}$$

where t is the independent variable (here called the *time*), and $x \in \Omega$ is the dependent variable with Ω a bounded open subset, $\lambda \in \mathbb{R}^p$ are the parameters of the family, for all i the function F_i is analytic in its variables and T-periodic in the t variable, and the period T is independent of the small parameter $\varepsilon \in I$ with $I \subset \mathbb{R}$ an interval containing the origin.

For each $z \in \Omega$ we denote $x(t; z, \lambda, \varepsilon)$ the solution of the Cauchy problem formed by the differential equation (1) with the initial condition $x(0; z, \lambda, \varepsilon) = z$. From the

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