Singular Perturbations of z^n with a Pole on the Unit Circle

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Abstract

We consider the family of complex maps given by $f_{\lambda,a}(z) = z^n + \lambda/(z-a)^d$ where $n, d \geq 1$ are integers, and a and λ are complex parameters such that $|a| = 1$ and $|\lambda|$ is sufficiently small. We focus on the topological characteristics of the Julia and Fatou sets of $f_{\lambda,a}$.

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1 Introduction

This paper is dedicated to Professor Robert L. Devaney on his 60th birthday.

In the last few years a number of papers have appeared that deal with the dynamics of functions obtained by a perturbation of the complex function $z \mapsto z^n$ by adding a pole at the origin [5, 6, 3, 7]. These rational functions are of the form $f_\lambda(z) = z^n + \lambda/z^d$. When $|\lambda| << 1$ we consider this function as a *singular perturbation* of z^n. The reason for this terminology is that when $\lambda = 0$ the map is z^n and the dynamical behavior is well understood. When $\lambda \neq 0$, however, the degree jumps to $n+d$ and the dynamical behavior changes significantly. The interest in this type of perturbation arises from the application of Newton's method to find the roots of a family of polynomials that, at one particular parameter value, has a multiple root. At this parameter value, the Newton iteration function undergoes a similar type of singular perturbation.

In [8] we investigated a more general class of functions for which the pole is not located at the origin but rather is located at some other point in the complex plane that does not lie on the unit circle. In particular, we considered the family of functions given by

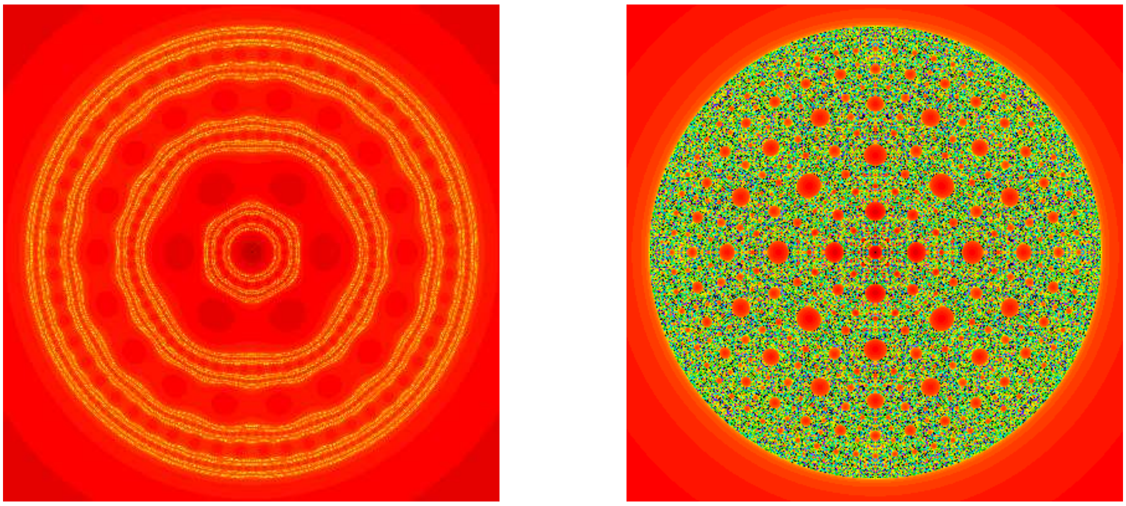
$$f_{\lambda,a}(z) = z^n + \frac{\lambda}{(z-a)^d} \tag{1.1}$$

where $n \geq 2$ and $d \geq 1$ are integers, and a and λ are complex parameters where $|a| \neq 0, 1$ and $|\lambda|$ is sufficiently small.

In this paper we continue the study of the family $f_{\lambda,a}$. In the first part we study the dynamics of Eq. (1.1) when the pole a is on the unit circle and $|\lambda|$ is sufficiently small. In the second part we focus on the dynamics of Eq. (1.1) when $n = 1$, $d \geq 1$ and $a, \lambda \in \mathbb{C}$. Our goal is to describe the topology and dynamics of the *Julia set* of $f_{\lambda,a}$, i.e., the set of points where the family of iterates of $f_{\lambda,a}$ is not a normal family in the sense of Montel. Equivalently, the Julia set is the closure of the set of repelling periodic points of $f_{\lambda,a}$. We denote the Julia set by $J = J(f_{\lambda,a})$. The complement of the Julia set is called the *Fatou set*.

We first consider the case when $n \geq 2$. When $\lambda = 0$, infinity and the origin are superattracting fixed points and the Julia set is the unit circle. When we add the perturbation by setting $\lambda \neq 0$ but very small, several aspects of the dynamics remain the same, but others change dramatically. For example, when $\lambda \neq 0$, the point at ∞ is still a superattracting fixed point and there is an immediate basin of attraction of ∞ that we call $B = B_\lambda$. On the other hand, there is a neighborhood of the pole a that is now mapped d-to-1 onto B. When this neighborhood is disjoint from B we call it the *trap door* and denote it by $T = T_\lambda$. Every point that escapes to infinity and does not lie in B has to do so by passing through T. Since the degree of $f_{\lambda,a}$ changes from n to $n+d$, $2d$ additional critical points are created. The set of critical points includes ∞ and a whose orbits are completely determined, so there are $n+d$ additional "free" critical points. The orbits of these points are of fundamental importance in characterizing the Julia set of $f_{\lambda,a}$.

When λ is sufficiently small and $a \neq 0$ we may find $\delta_1 > 0$ such that, if $|\lambda| < \delta_1$, $f_{\lambda,a}$ still has an attracting fixed point $q = q_\lambda$ near the origin. Throughout this paper, we asume



(a) $n = d = 3$, $a = 0$ and $\lambda = -0.001$. (b) $n = d = 2$, $a = 0$ and $\lambda = -0.001$.

Figure 1: Dynamical plane of $f_{\lambda,a}$ for different values of n, d, a and λ. Points in the basin of attraction of infinity are in different shades of red. Left hand side corresponds to $1/n + 1/d < 1$ and right hand side corresponds to $1/n + 1/d = 1$. In the first case the Julia set is a Cantor set of circles while in the second one the Julia set is a Sierpinski Curve.

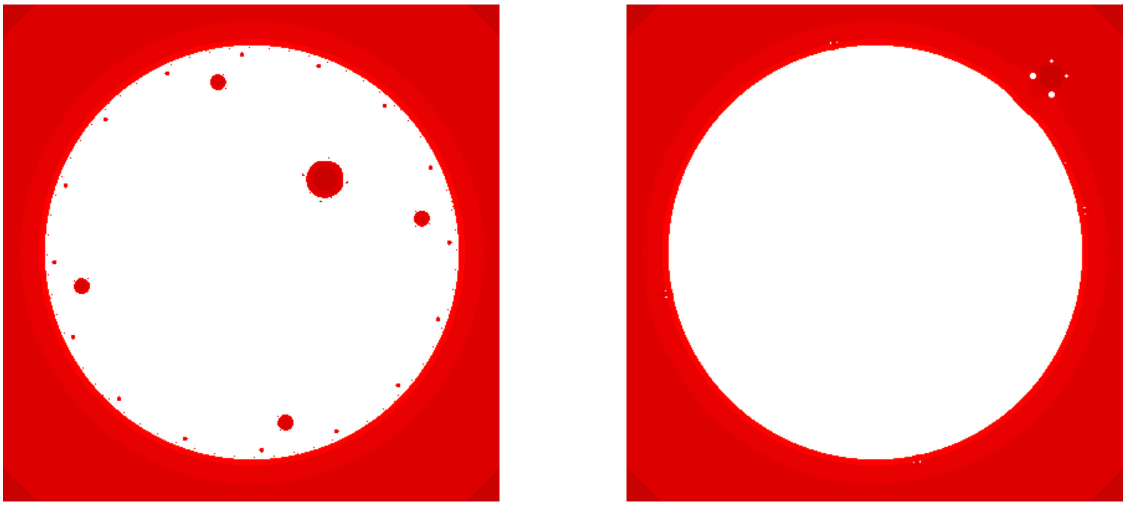
that $|\lambda| < \delta_1$. Let $Q = Q_\lambda$ denote the immediate basin of attraction of q. The set of $n + d$ "free" critical points may be divided into two groups: the first group consists of $n-1$ critical points that are attracted to q. These are the critical points that bifurcate away from the origin when λ becomes nonzero. The remaining $d + 1$ critical points surround the pole a and, for $|\lambda| << 1$, they are mapped close to a^n. It follows that the dynamics of this family of functions is determined by the behavior of this set of $d+1$ critical points and the position of a when $|\lambda|$ is small.

We first review the case when $|a| \neq 1$. When $|\lambda| << 1$ and $0 < |a| < 1$ the orbits of the $d + 1$ critical points that lie around a converge to the fixed point q near the origin, and when $|a| > 1$ they converge to ∞. The following theorem summarizes some of the known results studied in [5, 8, 11].

Theorem 1.1. *(Structure of the Julia and Fatou sets for $|a| \neq 1$) Let $n \geq 2$, $d \geq 1$ and $|a| \neq 1$. Then, for $|\lambda|$ sufficiently small, we have:*

(a) If $a = 0$ and $1/n + 1/d < 1$ the Julia set of $f_{\lambda,a}$ is a Cantor set of simple closed curves that surrounds the origin. The Fatou set consists of two disks (T and B) and infinitely many annuli. In this case Q is empty.

(b) If $|a| \neq 0$ the Julia set of $f_{\lambda,a}$ consists of a countable union of disjoint simple closed curves and an uncountable number of point components that accumulate on those curves. Only one of these curves surrounds the origin. The Fatou set consists of countably many disks and one infinitely connected component (namely, Q if $|a| < 1$ and B if $|a| > 1$).



(a) $n = d = 4$, $a = .5e^{i\pi/4}$ and $\lambda = 0.00007$.

(b) $n = d = 4$, $a = 1.2e^{i\pi/4}$ and $\lambda = 0.00007$.

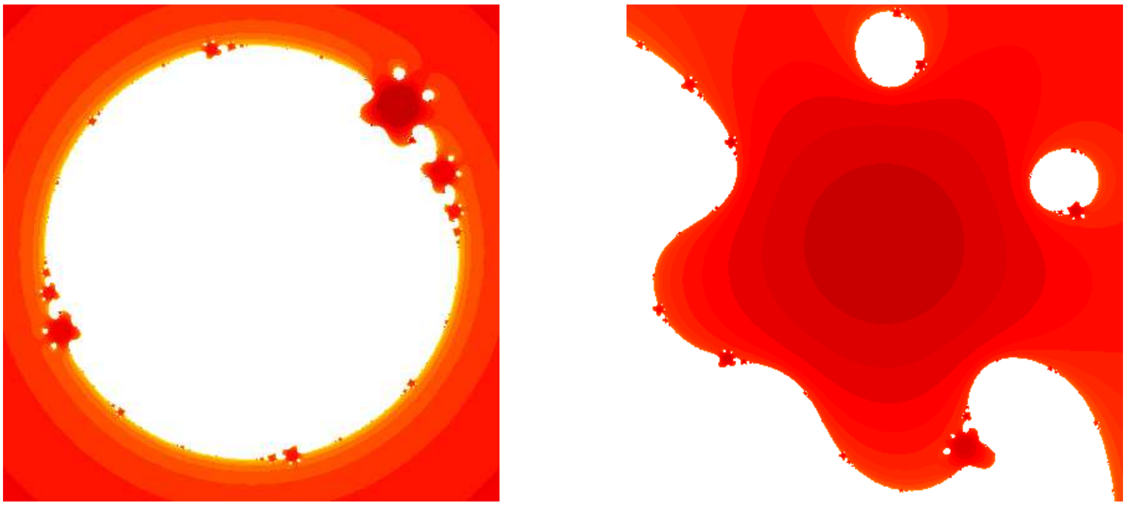
Figure 2: Dynamical plane of $f_{\lambda,a}$ for different values of n, d, a and λ. Points attracted to q are shown in white and points attracted to ∞ are shown in red. Left hand side corresponds to the case where $|a| < 1$ and right hand side corresponds to $|a| > 1$. When $|a| < 1$ (resp. $|a| > 1$) then Q (resp. B) is completely invariant and infinitely connected and B (resp. Q) has infinitely many simply connected preimages.

Notice that in the cases described in the above theorem the Fatou set of $f_{\lambda,a}$ is the union of the basin of attraction of ∞ and the basin of attraction of q. Also, the dynamics of $f_{\lambda,a}$ on J are completely determined by a specific quotient of a subshift of finite type. There is always a neighborhood of the origin in the parameter λ-plane where all these maps have conjugate dynamics on their Julia sets (see [8, 11]).

The case $a = 0$ with $n = d = 2$ is very different. In this case there are infinitely many open sets in any neighborhood of $\lambda = 0$ in which the Julia sets corresponding to these parameters are all Sierpinski curves, but any two such maps whose parameters are drawn from different open sets have non-conjugate dynamics (see [6]). Moreover, in this case when $\lambda \to 0$ the Julia sets of $f_{\lambda,a}$ converge to the unit disk (see [9]). The cases when $a = 0, d = 1$ and $n \geq 2$ are also very different and are still under study.

Figures 1 and 2 show examples of each of the cases discussed above.

The differences between the cases $|a| \neq 1$ and $|a| = 1$ can be explained as follows. For sufficiently small $\lambda \neq 0$ and outside a small neighborhood of the pole a the map $f_{\lambda,a}(z)$ behaves approximately like z^n, since the distance between them is small. Then, the set of $d+1$ critical points that surround a is mapped close to a^n. This implies that when $|a| \neq 1$ the orbits of these critical points behave as 'one' critical orbit. Instead, when $|a| = 1$ the critical points that surround the pole a behave independently. Some of them can converge to q, some of them can converge to ∞ and some of them may be related to a Fatou component different from B and Q, or even belong to the Julia set of $f_{\lambda,a}$. Hence a complete description



(a) $n = 2$, $d = 5$, $a = e^{i\pi/4}$ and $\lambda = 0.00007$.

(b) Same parameters as Figure (a). Magnification around the pole $a = e^{i\pi/4}$.

Figure 3: Dynamical plane of $f_{\lambda,a}$ for different values of n, d, a and λ. Color codes as in Figure 2. These plots represent the typical case when $|a| = 1$ and $d \geq 5$. In this case B is completely invariant and infinitely connected and the basin of attraction of q has infinitely many simply connected components.

of the Julia set can be challenging. However, when these critical points belong to B and Q we can give a detailed description of the Julia and Fatou sets of these maps. Let S_a denote the set of $d + 1$ critical points that surround the pole a when $|\lambda|$ is small.

In Theorem 1.2 we describe some important components of the Fatou set, namely the basins of attraction of q and ∞. These results need no assumptions on the behavior of the critical points in S_a since the order d of the pole a is enough to assure that some of these critical points belong to B.

Theorem 1.2. *(No trap door) Let $|a| = 1$ and $n \geq 2$. Then for $|\lambda|$ sufficiently small we have:*

(a) If $d > 1$ then B is completely invariant and Q is simply connected.

(b) If $d > 4$ (or if $n > 2$ and $d > 3$) then B is infinitely connected and the basin of attraction of q has infinitely many simply connected components.

An important consequence of Theorem 1.2 is the following. When $d > 1$ the pole a lies in B, that is, for $|\lambda|$ sufficiently small these maps have no *trap door* as in the case when $|a| > 1$ (see [8]). Figure 3 displays the dynamical plane of $f_{\lambda,a}$ corresponding to Theorem 1.2.

If the critical points in S_a are distributed between B and Q and, for $|\lambda|$ sufficiently small the number of critical points in B and Q remains constant then we can understand the structure and dynamics on the Julia set of $f_{\lambda,a}$. Let $|S_a \cap B|$ and $|S_a \cap Q|$ denote the number of critical points from S_a that lie in B and Q, respectively. There are two possibilities shown in the next theorem.

Theorem 1.3. *(Structure of the Julia and Fatou sets for $|a| = 1$) Let $n \geq 2, d \geq 1, |a| = 1$ and suppose that for λ sufficiently small $S_a \subset B \cup Q$ and that $|S_a \cap B|$ and $|S_a \cap Q|$ remain constant, then either:*

(a) Exactly one critical point from S_a belongs to B and the Julia set J is a quasi-circle that surrounds the origin where $f_{\lambda,a} : J \mapsto J$ is conjugate to $z \mapsto z^{n+d}$ on the unit circle. The Fatou set consists of two completely invariant disks, namely B and Q; or else,

(b) The Julia set J consists of countably many simple close curves and uncountably many point components that accumulate on each one of these curves. Only one of these curves surrounds the origin. The Fatou set consists of one infinitely connected component and infinitely many disks.

Notice that if $S_a \subset B$ (resp. $S_a \subset Q$) then we are in part (b) of the above theorem. This is exactly what happens in the case when $|a| > 1$ (resp. $|a| < 1$) described in Theorem 1.1. For this reason, in the case $|a| \neq 1$ the situation described in Theorem 1.3 part (a) is not observed. This new possibility when $|a| = 1$ is allowed by the fact the critical points in S_a behave independently and in a very specific manner. We also have:

Theorem 1.4. *(Dynamics on the Julia set) Suppose f_{λ_1,a_1} and f_{λ_2,a_2} are two functions such that they both lie in one of the cases distinguished in Theorem 1.3. In other words, for $|\lambda|$ sufficiently small, the set $S_a \subset B \cup Q$ and the number of critical points in B and Q coincide for both functions but the exact position of the pole a or of these critical points is arbitrary. Then there exists $\epsilon > 0$ such that, for $|\lambda_1|, |\lambda_2| < \epsilon$, these maps are conjugate on their Julia sets. Moreover, the dynamics are determined by a specific quotient of a subshift of finite type.*

We can actually prove the existence of the Julia sets described in Theorem 1.3. Some of the results in the next theorem hold only for *sectors* of values of λ in the parameter $\lambda-$plane.

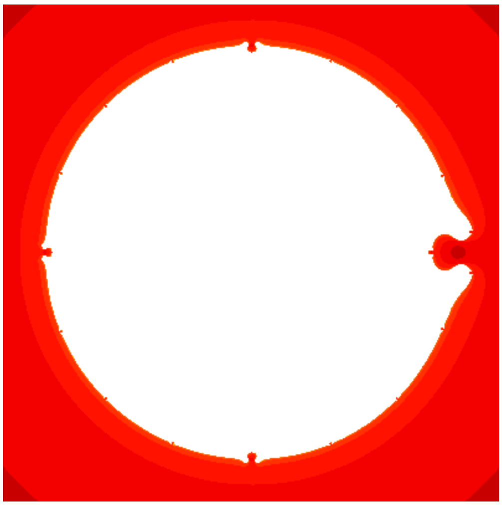
Let $Arg(z)$ denote the argument of the complex number z. Then given two real numbers α and β such that $0 \leq \alpha < \beta \leq 2\pi$, we define a sector $S_{\alpha,\beta}$ of values of the parameter λ in the usual way, that is, $S_{\alpha,\beta} = \{\lambda\,;\, \alpha < Arg(\lambda) < \beta\}$.

Theorem 1.5. *(Existence) Let $n \geq 2$ and $|a| = 1$. For $|\lambda|$ sufficiently small we have:*

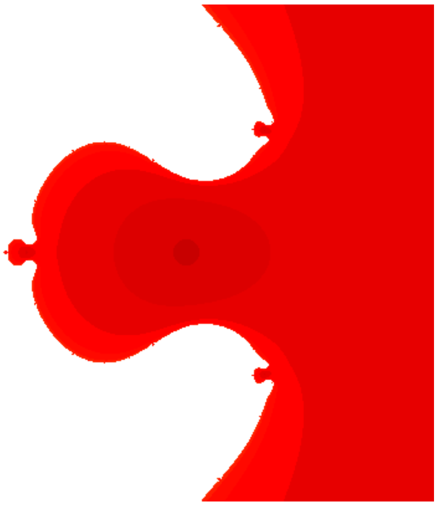
(a) If $d = 1, 2$, then there exists a sector S_{α_d,β_d} in parameter $\lambda-$plane such that, if $\lambda \in S_{\alpha_d,\beta_d}$ then the Julia and Fatou sets of $f_{\lambda,a}$ are as in Theorem 1.3 part (a).

(b) If $d = 2, 3, 4$, then there exists a sector S_{γ_d,δ_d} in parameter $\lambda-$plane such that, if $\lambda \in S_{\gamma_d,\delta_d}$ then the Julia and Fatou sets of $f_{\lambda,a}$ are as in Theorem 1.3 part (b).

Moreover, inside each one of these sectors Theorem 1.4 holds.



(a) $n = 4$, $d = 2$, $a = 1$ and $\lambda = 0.007$.



(b) Same parameters as Figure 4(a). Magnification around the pole $a = 1$.

Figure 4: Dynamical plane of $f_{\lambda,a}$ for different values of n, d, a and λ. Color codes as in Figure 2. These plots represent the case $d = 2$ for some values of λ in S_{α_2,β_2}. In this case Q and B are both completely invariant sets that are therefore, simply connected. Moreover, the Julia set is a quasi-circle which is equal to the common boundary of Q and B.

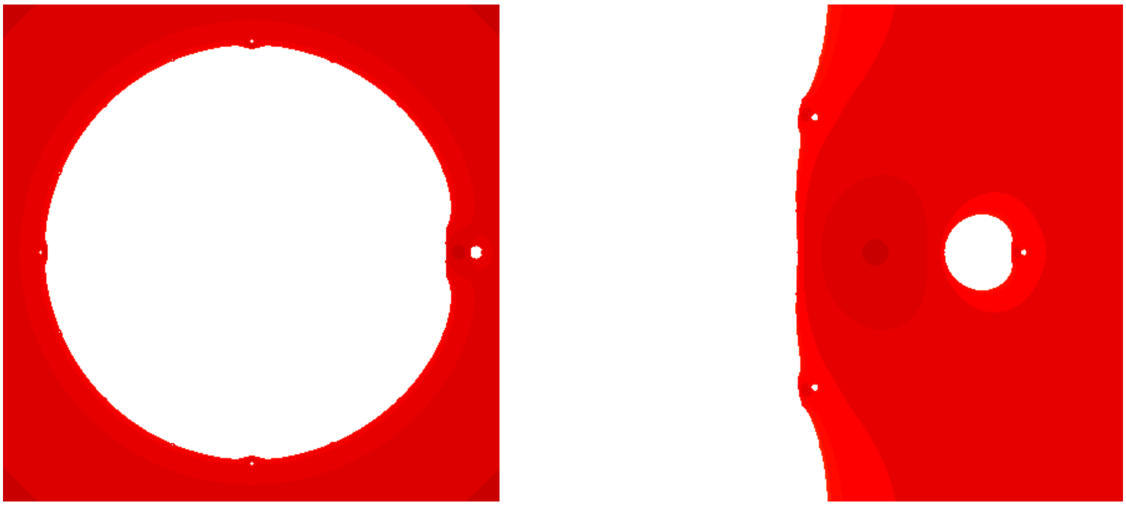
The case when $d = 2$ is very interesting since for some values of λ sufficiently small we can obtain very different topological and dynamical behavior.

The fact that for $|\lambda|$ sufficiently small we have that q and ∞ are attracting fixed points implies that the Julia set cannot be totally disconnected. In other words, the Fatou set consists of at least two disjoint open sets. The minimum of two is attained by part (a) of Theorem 1.5 and, in this case, the Julia set is the common boundary of Q and B.

Remark 1.6. *As we mentioned, when $d > 1$ the basin of attraction of infinity is completely invariant (that is, there is no trap door). By the above theorem we also know that when $d = 1$ there is a sector of parameters in the $\lambda-$plane for which this is also the case. Numerical experiments suggest that when $|a| = 1$ and $|\lambda| << 1$ the basin of attraction of ∞ is always completely invariant.*

Figures 4 and 5 display the dynamical plane of $f_{\lambda,a}$ corresponding to the different cases that appear in Theorem 1.5.

Notice that very interesting bifurcations happen when we fix n and d so that $1/n+1/d < 1$ and we also fix λ sufficiently small and let the parameter a vary. The structure of the Julia set changes dramatically when the pole a moves away from the origin. When $a = 0$ we have that the Julia set of $f_{\lambda,a}$ is a Cantor set of simple closed curves that surrounds the origin (see Figure 1(a)). When $0 < |a| < 1$ there is a neighborhood of the origin in the λ-plane where the Julia set of $f_{\lambda,a}$ consists of countably many simple closed curves only one of which surrounds the origin (namely, ∂B) and uncountably many point components that



(a) $n = 4$, $d = 2$, $a = 1$ and $\lambda = -0.007$.

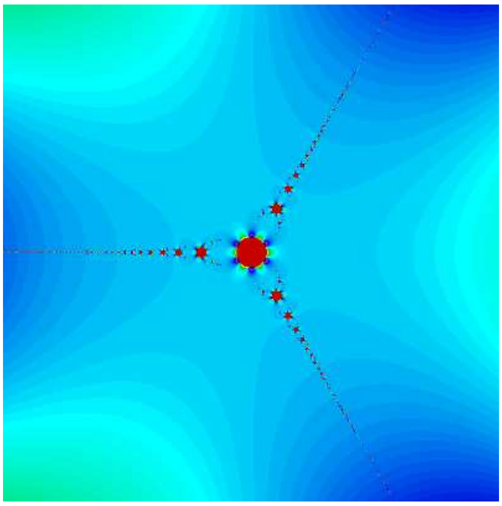
(b) Same parameters as Figure (a). Magnification around the pole $a = 1$.

Figure 5: Dynamical plane of $f_{\lambda,a}$ for different values of n, d, a and λ. Color codes as in Figure 2. These plot represent the case $d = 2$ for some values of λ in S_{γ_2,δ_2}. In this case B is completely invariant and the basin of attraction of q has infinitely many simply connected components.

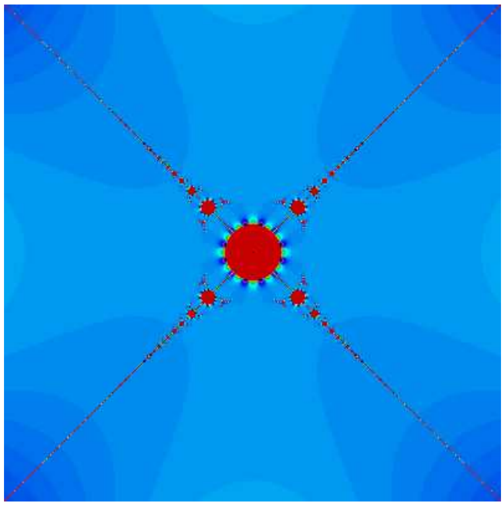
accumulate on these curves. The preimages of ∂B lie inside ∂B (see Figure 2(a)). When $|a| = 1$ we see that there is a sector of parameters in the λ-plane for which the Julia set of $f_{\lambda,a}$ has the same topology as in the previous case but the only curve that surrounds the origin is now ∂Q and the rest of the curves lie outside ∂Q (see Figure 3). When $|a| = 1$ and for some values of n and d (see Theorem 1.5) there is also a sector of parameters in the λ-plane for which the Julia set becomes a simple closed curve that surrounds de origin (see Figure 4). Finally, when $|a| > 1$ there is a neighborhood of the origin in the λ-plane for which the structure of the Julia set again consists of countably many simple closed curves only one of which surrounds the origin (namely, ∂Q) and uncountably many point components that accumulate on these curves (see Figure 2(b)). In between each one of these states the Julia set suffers great transformations due to the fact that the critical points in S_a are now acting independently. A complete description of these transitions between states goes beyond the scope of this paper.

In the second part of the paper we focus on the family given by Eq. (1.1) when $n = 1$. In this case, the point at infinity is always in the Julia set and this causes major changes in the dynamical behavior of $f_{\lambda,a}$. Also, when $n = 1$ we can conjugate $f_{\lambda,a}$ via a Möbius map to make it completely independent of the parameters a and λ. For these reasons the behavior of the map $f_{\lambda,a}$ with $n = 1$ is completely different from the previous cases and the characteristics of the Julia and Fatou sets of $f_{\lambda,a}$ reflect these changes. We have:

Theorem 1.7. *Let $n = 1$ and $d \geq 1$ then, for all parameters $a, \lambda \in \mathbb{C}$ the map $f_{\lambda,a}$ is conformally conjugate to $z + 1/z^d$. In particular, the Julia set of $f_{\lambda,a}$ is connected and the*



(a) $n = 1$ and $d = 2$.



(b) $n = 1$ and $d = 3$.

Figure 6: Dynamical plane of $z \mapsto z + 1/z^d$ for $d = 2, 3$ and $n = 1$. Points in the parabolic basins of infinity are in different shades of blue or green. In each case we can observe $d + 1$ unbounded petals. The complement is the Julia set.

Fatou set contains all the points attracted to the unique parabolic fixed point at infinity. When $d = 1$ *the Fatou set consists of two simply connected regions; otherwise, it consists of infinitely many simply connected components.*

Figure 6 shows examples of the Julia sets of $f_{\lambda,a}$ when $n = 1$ for $d = 2$ and $d = 3$.

The rest of the paper is organized as follows. In Section 2 we obtain some basic results about the function $f_{\lambda,a}$ when $n \geq 2$. In Section 3 we prove Theorems 1.2, 1.3, 1.4, and 1.5. Finally, in Section 4 we study the dynamics of $f_{\lambda,a}$ when $n = 1$ and prove Theorem 1.7.

2 Preliminaries

Let $n \geq 2$, $d \geq 1$ and $|a| = 1$. A straightforward computation shows that, when $\lambda \neq 0$, $f_{\lambda,a}$ has $n + d$ critical points that satisfy the equation

$$z^{n-1}(z - a)^{d+1} = \lambda\, d/n. \tag{2.1}$$

When $\lambda = 0$ this equation has $n + d$ roots, the origin with multiplicity $n - 1$ and a with multiplicity $d + 1$. By continuity, for small enough $|\lambda|$, these roots become simple zeros of $f'_{\lambda,a}$ that are approximately symmetrically distributed around the origin and the pole a. As a consequence, when $|\lambda|$ is small, $n - 1$ of the critical points of $f_{\lambda,a}$ are grouped around 0, near the fixed point q, while $d + 1$ of the critical points are grouped around the pole a.

Let $c = c_\lambda$ be one of the $n + d$ critical points of $f_{\lambda,a}$ given by Eq. (2.1). Replacing z by c in Eq. (2.1) we find $\lambda(c - a)^{-d} = (n/d)c^{n-1}(c - a)$ and so the critical value v corresponding to c is given by

$$v = f_{\lambda,a}(c) = c^n + (n/d)\ c^{n-1}(c - a)$$

or, equivalently,

$$v = c^{n-1}(c\ (1 + n/d) - a\ n/d). \tag{2.2}$$

Note that as $\lambda \to 0$, the fixed point $q_\lambda \to 0$ as well. From Eq. (2.2) it follows that, if $c \to 0$, then $v \to 0$. Similarly, if $c \to a$, then $v \to a^n$. We use S_a to denote the set of $d + 1$ critical points around a and we use S_q to denote the set of $n - 1$ critical points around q. We have:

Lemma 2.1. *When $|\lambda|$ tends to zero the critical values corresponding to the critical points in S_q tend to q and the critical values corresponding to the critical points in S_a tend to a^n.*

To describe the structure of the Julia set, we first need to give an approximate location for this set. Roughly speaking, when $|\lambda|$ is small, the Julia set of $f_{\lambda,a}$ lies in a small annulus around the unit circle.

Proposition 2.2. *Suppose $n \geq 2$, $d \geq 1$ and $|a| = 1$.*

(a) Let $0 < s \leq 1$ and suppose that $|z| = 1 - s$ then, for sufficiently small λ, $z \in Q$.

(b) Let $s > 0$ and suppose that $|z| = 1 + s$ then, for sufficiently small λ, $z \in B$.

Proof. Fix n, d and a. For the first part, we have $|z - a| \geq |a| - |z| = s > 0$ so that

$$|f_{\lambda,a}(z)| \leq |z|^n + \frac{|\lambda|}{|z - a|^d} \leq (1 - s)^n + \frac{|\lambda|}{s^d}.$$

Let $|\lambda| < s^d[(1 - s) - (1 - s)^n]$. Then

$$|f_{\lambda,a}(z)| < (1 - s)^n + \frac{s^d[(1 - s) - (1 - s)^n]}{s^d} = 1 - s.$$

As a consequence, $|f_{\lambda,a}(z)| < |z|$ and so the orbit of z converges to the fixed point q near the origin. Therefore z lies in Q.

For the second part, we have $|z - a| \geq |z| - |a| = s > 0$ so that

$$|f_{\lambda,a}(z)| \geq |z|^n - \frac{|\lambda|}{|z - a|^d} \geq (1 + s)^n - \frac{|\lambda|}{s^d}.$$

Let $|\lambda| < s^d[(1 + s)^n - (1 + s)]$. Then

$$|f_{\lambda,a}(z)| > (1 + s)^n - \frac{s^d[(1 + s)^n - (1 + s)]}{s^d} = 1 + s.$$

Hence $|f_{\lambda,a}(z)| > |z|$ and the orbit of z converges to ∞ so that $z \in B$.

□

The following result gives us a simple procedure to verify when a point belongs to Q or B.

Corollary 2.3. *Suppose $n \geq 2$, $d \geq 1$, $|a| = 1$ and let $s_* = \left(\frac{|\lambda|}{n-1}\right)^{1/(d+1)} > 0$. Then for all $s_* < s < 1$ we have that:*

(a) If $|z| \leq 1 - s$ then, for λ sufficiently small, $z \in Q$; and,

(b) If $|z| \geq 1 + s$ then, for λ sufficiently small, $z \in B$.

Proof. For the first part, notice that for $s > 0$, $(1-s)^n < 1-ns$. Then, simple computations show that

$$(n-1)s^{d+1} < s^d[(1-s) - (1-s)^n].$$

The condition $s > s_*$ is equivalent to $|\lambda| < (n-1)s^{d+1}$ and the result follows from Proposition 2.2. The second part follows in a similar way by noticing that for $s > 0$ we have $1 + ns < (1+s)^n$. $\square$

In order to prove our main theorems we need to obtain more precise results regarding the location of the critical points in S_a and their corresponding critical values. To simplify the notation we introduce two new variables $\delta = \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}$ and $\epsilon = \left(\frac{|\lambda| d}{n}\right)^{\frac{1}{d+n}}$, the first one is a multivalued complex function of λ and a and the second one is a real function of $|\lambda|$. Both parameters play a major role in the rest of this paper.

Let c be a critical point of $f_{\lambda,a}$ in S_a and let $v = f_{\lambda,a}(c)$ be its corresponding critical value. We need four lemmas all of which hold for $|\lambda|$ sufficiently small. In Lemma 2.4 we prove that the distance between the critical point c and the pole a is bounded by ϵ. In Lemma 2.5 we find and approximation for c that we denote by $\tilde{c}$. In Lemma 2.6 we obtain an approximation for v that we denote by $\tilde{v}$; the distance between v and $\tilde{v}$ will be proved to be smaller or equal to $\epsilon|\delta|$. Finally, in Lemma 2.7 we find a criterion to prove when the critical value v belongs to B or Q.

Lemma 2.4. *Let $n \geq 2$, $d \geq 1$, $|a| = 1$ and $\epsilon = \left(\frac{|\lambda| d}{n}\right)^{\frac{1}{d+n}}$. For $|\lambda|$ sufficiently small we have that, if c is a critical point in S_a then $|c - a| \leq \epsilon$.*

Proof. Fix n, d and a. Let $R_0 > 0$ and $R_a > 0$ be two real numbers such that $R_0^{n-1} R_a^{d+1} = |\lambda| d/n$. Consider the closed disk of radius R_0 centered at the origin, that is, $D_0 = \{z : |z| \leq R_0\}$, and the closed disk of radius R_a centered at a, that is, $D_a = \{z : |z - a| \leq R_a\}$. The critical points of $f_{\lambda,a}$ belong to $D_0 \cup D_a$ since all points outside this union verify $|z|^{n-1}|z - a|^{d+1} > |\lambda|\, d/n$ (see Eq. (2.1)). By hypothesis $c \in S_a$ and since $|a| = 1$, for $|\lambda|$ sufficiently small, we have that $|c| > R_0$, obtaining thus that $|c - a| \leq R_a$. Now let $R_0 = (|\lambda|\, d/n)^{1/(d+n)}$ and $R_a = (|\lambda|\, d/n)^{1/(d+n)}$ and the lemma follows. $\square$

Lemma 2.5. *Let $n \geq 2$, $d \geq 1$, $|a| = 1$, $\delta = \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}$ and $\epsilon = \left(\frac{|\lambda| d}{n}\right)^{\frac{1}{d+n}}$. For $|\lambda|$ sufficiently small we have that the $d+1$ critical points in S_a can be approximated by $\tilde{c} = a + \delta$. These points are the vertices of a regular polygon of $d+1$ sides centered at a. Moreover, if c and $\tilde{c}$ are a critical point and its approximation we have*

$$|c - \tilde{c}| \leq \frac{n-1}{d+1} 2^{\frac{n+d}{d+1}} \epsilon\, |\delta|.$$

Proof. Fix n, d and a. To find approximations of the critical points in S_a we use the fact that solving Eq. (2.1) is equivalent to computing fixed points of the multivalued function $T(z)$ defined by

$$T(z) = a + \left(\frac{\lambda d}{n\, z^{n-1}}\right)^{\frac{1}{d+1}}. \tag{2.3}$$

We remark that there are $d+1$ possible different choices for the function T that are the $(d+1)$ branches of the map given by Eq. (2.3). Starting with the initial point a we find an approximate value of $\tilde{c}$ given by

$$\tilde{c} = T(a) = a + \delta = a + \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}.$$

It is clear that the values of $\tilde{c}$ form the vertices of a regular polygon with $d+1$ sides centered at a (see Figure 7). Since c is the fixed point of T we can obtain an upper bound for the distance between the critical point c and the approximate value $\tilde{c}$. We have

$$|c - \tilde{c}| = |T(c) - T(a)| \leq |T'(\xi)||c - a|,$$

where ξ is a point in the segment joining c and a. From Lemma 2.4 we know that the critical points in S_a tend to a when $|\lambda|$ tends to 0. Then, let $|\lambda|$ be small enough so that $|\xi| \geq 1/2$. We get

$$|T'(\xi)| = \frac{n-1}{d+1}\left(\frac{|\lambda| d}{n}\right)^{\frac{1}{d+1}} |\xi|^{-\frac{n+d}{d+1}} \leq \frac{n-1}{d+1}\, 2^{\frac{n+d}{d+1}}\, |\delta|.$$

Then, using Lemma 2.4 and the above inequality we obtain

$$|c - \tilde{c}| \leq \frac{n-1}{d+1} 2^{\frac{n+d}{d+1}} \epsilon\, |\delta|$$

as we wanted to show. □

Lemma 2.6. *Let $n \geq 2$, $d \geq 1$, $|a| = 1$, $\delta = \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}$ and $\epsilon = \left(\frac{|\lambda| d}{n}\right)^{\frac{1}{d+n}}$. Assume that $v = f_{\lambda,a}(c)$ is a critical value of $f_{\lambda,a}$ corresponding to a critical point c in S_a. Let $\tilde{v} = f_{\lambda,a}(\tilde{c})$ where $\tilde{c}$ is as given in Lemma 2.5. Then, for $|\lambda|$ sufficiently small, $\tilde{v}$ is an approximation of v such that $|v - \tilde{v}| \leq \epsilon\, |\delta|$.*

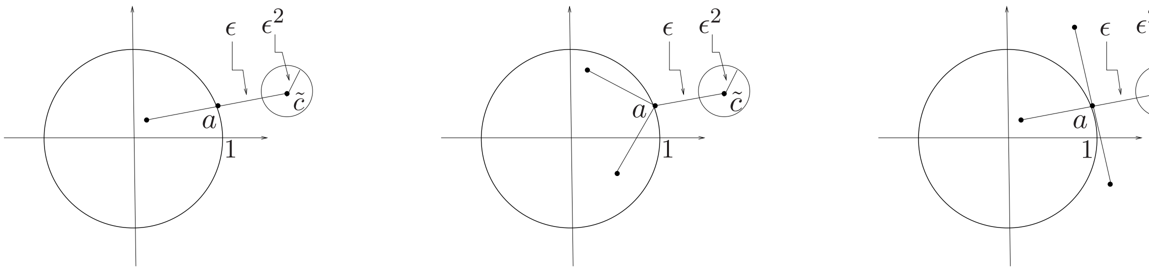


Figure 7: Sketch of the position of $\tilde{c}$.

Proof. Fix n, d and a. In order to control the distance between $v = f_{\lambda,a}(c)$ and $\tilde{v} = f_{\lambda,a}(\tilde{c})$ we use the Taylor expansion for $f(c)$ in terms of $\tilde{c}$. We have

$$f_{\lambda,a}(c) = f_{\lambda,a}(\tilde{c}) + f'_{\lambda,a}(\tilde{c})(c - \tilde{c}) + \mathcal{O}((c - \tilde{c})^2),$$

obtaining thus that

$$|v - \tilde{v}| \leq |f'_{\lambda,a}(\tilde{c})|\,|c - \tilde{c}| + \mathcal{O}(|c - \tilde{c}|^2).$$

Using the expression of $\tilde{c} = a + \delta = a + (\frac{\lambda d}{n a^{n-1}})^{\frac{1}{d+1}}$ and the fact that $f_{\lambda,a}(z) = z^n + \lambda/(z-a)^d$ we have

$$\begin{aligned} |f'_{\lambda,a}(\tilde{c})| &= \left| n\tilde{c}^{n-1} - \tfrac{\lambda d}{(\tilde{c}-a)^{d+1}} \right| = |n(a\ +\ \delta)^{n-1} - n a^{n-1}| \\ &\leq n \textstyle\sum_{k=1}^{n-1} \left(\begin{array}{c} n-1 \\ k \end{array} \right) |\delta|^k \ = n(n-1)|\delta| + \mathcal{O}(|\delta|^2). \end{aligned}$$

Using Lemma 2.5 and the above inequality we obtain

$$|v - \tilde{v}| \leq \frac{n(n-1)^2}{d+1} 2^{\frac{n+d}{d+1}}\, \epsilon\, |\delta|^2 + \mathcal{O}(\epsilon\, |\delta|^3).$$

Then, by taking λ sufficiently small we have

$$|v - \tilde{v}| \leq \ \epsilon\, |\delta|,$$

and the lemma follows. $\square$

Lemma 2.7. *Let $n, d \geq 2$, $|a| = 1$ and $\delta = \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}$. Assume that $v = f_{\lambda,a}(c)$ is a critical value of $f_{\lambda,a}$ corresponding to a critical point c in S_a, and let $\tilde{v} = f_{\lambda,a}(\tilde{c})$ be an approximation of v as in Lemma 2.6. Let $k > 1$ be a constant. We have:*

(a) If $|\tilde{v}| \geq 1 + k|\delta| \pm \mathcal{O}(|\delta|^2)$ then, for $|\lambda|$ sufficiently small, the critical value v belongs to B; and,

(b) If $|\tilde{v}| \leq 1 - k|\delta| \pm \mathcal{O}(|\delta|^2)$ then, for $|\lambda|$ sufficiently small, the critical value v belongs to Q.

Proof. Fix n, d and a. For part (a) we first we prove that $\tilde{v}$ belongs to B and then we verify that this implies that v also belongs to B. To prove that $\tilde{v}$ belongs to B we show that for $|\lambda|$ sufficiently small the condition $|\tilde{v}| > 1 + s_*$ with $s_* = \left(\frac{|\lambda|}{n-1}\right)^{\frac{1}{d+1}}$, is satisfied. Then Corollary 2.3 implies the result.

From the definitions of $|\delta|$ and s_* it follows that when $n, d \geq 2$ we have $|\delta| \geq s_*$. Then it is enough to show that $|\tilde{v}| > 1 + |\delta|$; from hypothesis, this reduces to show

$$|\tilde{v}| \geq 1 + k|\delta| \pm \mathcal{O}(|\delta|^2) > 1 + |\delta|.$$

For $|\lambda|$ sufficiently small and $k > 1$ we have that $(k-1)|\delta| \pm \mathcal{O}(|\delta|^2) > 0$. Thus we conclude that $\tilde{v}$ belongs to B.

Finally, we need to show that $v = f_{\lambda,a}(c)$ and $\tilde{v} = f_{\lambda,a}(\tilde{c})$ are close enough to assure that v also belongs to B. It follows from Lemma 2.6 that the distance between v and $\tilde{v}$ is bounded by $\epsilon\,|\delta|$. Hence we have that

$$|v| \geq |\tilde{v}| - |v - \tilde{v}| \geq 1 + k|\delta| \pm \mathcal{O}(|\delta|^2) - \epsilon\,|\delta|.$$

Using the fact that $k > 1$ we conclude that, for $|\lambda|$ sufficiently small, v also belongs to B and the first part follows.

The second part follows in a similar way. The details are left to the reader. $\square$

In the next proposition we prove that controlling the behavior of some critical points we can obtain information about the topology of the basin of attraction of ∞ and q. We have:

Proposition 2.8. *Let $n \geq 2$, $d \geq 1$ and $|a| = 1$. Then, for $|\lambda|$ sufficiently small:*

(a) If one or more critical values corresponding to the critical points in S_a belong to B then B is completely invariant.

(b) Q and B are completely invariant and simply connected if and only if one critical point in S_a belongs to B and the remaining d critical points in S_a belong to Q.

(c) If two or more critical values corresponding to critical points in S_a belong to B then B is infinitely connected and the basin of attraction of q has infinitely many simply connected components.

Proof. (a) We prove the result by contradiction. Let c be a critical point in S_a such that its corresponding critical value $v = f_{\lambda,a}(c)$ belongs to B. Assume that B is not completely invariant. Then there exists a preimage of B disjoint from B; we call this preimage T. Since the only preimage of ∞ different from itself is a we conclude that $a \in T$. Now B is mapped to itself at least n-to-1 because $\infty \in B$, and T is mapped to B at least d-to-1 because $a \in T$. Since the map is of degree $n + d$ we conclude that $c \notin T, B$. This is a contradiction to the fact that $v \in B$. Then B is completely invariant as we wanted to show.

(b) First assume that B and Q are completely invariant and simply connected sets. This implies that the pole a belongs to B. We have that $f_{\lambda,a}$ maps Q to itself, and B to itself both in an $(n+d)$-to-1 fashion. On the one hand, the connectivity of Q is $m=1$, the degree of the map is $\sigma = n+d$ and the number of critical points in Q is $N = n-1+x$; that is, $n-1$ close to 0 and x close to a. By the Riemann-Hurwitz formula, we get

$$m-2 = \sigma(m-2)+N, \text{ so that } x = d.$$

On the other hand, the connectivity of B is $m=1$, the degree of the map is $\sigma = n+d$ and the number of critical points in B is $N = n-1+d-1+y$; that is, $n-1$ at ∞, $d-1$ at the pole a and y close to a. By the Riemann-Hurwitz formula, we get

$$m-2 = \sigma(m-2)+N, \text{ so that } y = 1,$$

and one direction of the implication follows.

For the other direction of the implication assume that one critical point in S_a belongs to B and the other d critical points in S_a belong to Q. It follows from part (a) that B is completely invariant. The fact that B is completely invariant implies that Q is simply connected. To see this notice that when B is completely invariant then the Julia set is equal to the boundary of B. Now consider any Jordan curve γ completely contained in Q. Let U_0 and U_1 be the two components of $\mathbb{C}\setminus\gamma$. Without loss of generality we can assume that B is contained in U_0. By construction, U_1 cannot contain points in the Julia set, hence U_1 is contained in Q proving thus that Q is simply connected.

Now consider $f_{\lambda,a}: Q \to Q$. The connectivity of Q is $m=1$, the number of critical points in the domain is $N = n-1+d$; that is, $n-1$ close by 0 and d close to a, and the degree of the map is σ. By the Riemann-Hurwitz formula, we get

$$m-2 = \sigma(m-2)+N, \text{ so that } \sigma = n+d,$$

and then Q is completely invariant. Finally, since Q is completely invariant we have that B is simply connected and the result follows.

(c) Since B contains at least two critical points from S_a, it follows from part (a) that B is completely invariant. Then Q is simply connected. Since B is the immediate basin of attraction of a superattracting fixed point then B is either simply connected or infinitely connected. This follows from a well known result in complex dynamics (see Theorem 5.2.1 in [1]). It follows from part (b) that B is infinitely connected and Q is not completely invariant. Therefore, the basin of attraction of q has infinitely many simply connected components.

□

3 The case $n \geq 2$.

In this section we prove the results concerning the topological characteristics of the Julia and the Fatou sets as well as the dynamics of $f_{\lambda,a}$ on its Julia set when $n \geq 2$. These results are Theorems 1.2, 1.3, 1.4 and 1.5 stated in Section 1.

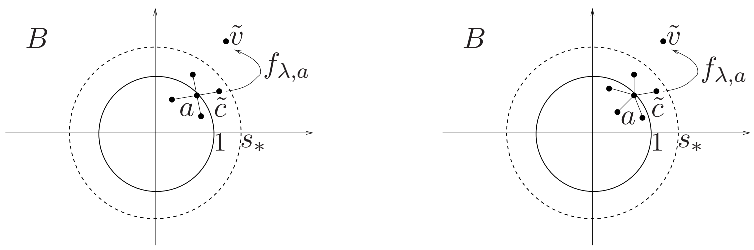
3.1 Proof of Theorem 1.2

Theorem 1.2 shows that in most cases the basin of attraction of infinity is completely invariant, that is, there is no trap door.

Proof. *(a)* The idea of the proof is to find a critical point c in S_a such that its corresponding critical value $v = f_{\lambda,a}(c)$ belongs B. To do this, we prove that for $|\lambda|$ sufficiently small, $|v| > 1 + s_*$ with $s_* = \left(\frac{|\lambda|}{n-1}\right)^{\frac{1}{d+1}}$, and then the result follows from Corollary 2.3. Finally, by Proposition 2.8 part (a) we conclude that B is completely invariant. Since we do not know the values of c and v, we use the approximations $\tilde{c}$ and $\tilde{v} = f_{\lambda,a}(\tilde{c})$ defined in Lemmas 2.5 and 2.6.

Fix $n, d \geq 2$ and a. We denote by c_m the critical point in S_a with the largest magnitude; that is, $|c_m| \geq |c|$ for all $c \in S_a$. Let $v_m = f_{\lambda,a}(c_m)$ be the critical value corresponding to c_m. Also, let $\tilde{c}_m$ and $\tilde{v}_m = f_{\lambda,a}(\tilde{c}_m)$ be approximations of c_m and v_m, respectively. As we have shown in the proof of Lemma 2.5, the $d+1$ values of $\tilde{c}$ are given by

$$\tilde{c} = a + \delta = a + \left(\frac{\lambda d}{n\, a^{n-1}}\right)^{\frac{1}{d+1}}.$$



(a) $d = 3$ (b) $d = 4$

Figure 8: Sketch of the relevant objects in the proof of Theorem 1.2 part (a). In this case it is enough to prove that one critical point belongs to B.

The fact that the values of $\tilde{c}$ are located at the vertices of a regular polygon centered at a and the condition that c_m is the value of c with the largest modulus imply that (see Figure 8)

$$|\tilde{c}_m| \geq 1 + |\delta| \cos\left(\frac{\pi}{d+1}\right). \tag{3.1}$$

To compute $\tilde{v}_m = f_{\lambda,a}(\tilde{c}_m) = \tilde{c}_m^n + \lambda/(\tilde{c}_m - a)^d$ we note that

$$\tilde{c}_m^n = (a+\delta)^n = a^n + na^{n-1}\delta + \mathcal{O}(|\delta|^2)$$

and

$$\frac{\lambda}{(\tilde{c}_m - a)^d} = \frac{\lambda}{\delta^d} = \frac{n}{d} a^{n-1}\delta.$$

Then $\tilde{v}_m$ can be written as

$$\tilde{v}_m = f_{\lambda,a}(\tilde{c}_m) \;=\; a^{n-1}(a + n(1 + \frac{1}{d})\delta) + \mathcal{O}(\delta^2).$$

By definition, $\tilde{c}_m$ is taken so that it has the largest modulus and from the above expression the same happens with the corresponding value of $\tilde{v}_m$. Then we have

$$|\tilde{v}_m| \geq 1 + n\,(1 + \frac{1}{d})\cos\left(\frac{\pi}{d+1}\right)|\delta| \pm \mathcal{O}(|\delta|^2).$$

When $n, d \geq 2$ we have that $n\,(1 + \frac{1}{d})\cos\left(\frac{\pi}{d+1}\right) > 1$. Then, using Lemma 2.7 we conclude that v_m belongs to B as we wanted to show.

(b) Fix $n \geq 2$, $d \geq 5$ and a with $|a| = 1$. By Proposition 2.8 part (c) we only need to prove that two critical values corresponding to critical points in S_a belong to B. We denote by c_m and c_l the two critical points in S_a with the largest magnitudes; that is, $|c_m| \geq |c_l| \geq |c|$ for all $c \in S_a$. We also denote by $v_m = f_{\lambda,a}(c_m)$ and $v_l = f_{\lambda,a}(c_l)$ their corresponding critical values. From Theorem 1.2 part (a) we conclude that, for $|\lambda|$ sufficiently small, v_m belongs to B. Then, we only have to prove that v_l also belongs to B. Let $\tilde{c}_l$ and $\tilde{v}_l = f_{\lambda,a}(\tilde{c}_l)$ denote approximations of c_l and v_l, respectively. From Lemma 2.5 we know that $\tilde{c} = a + \delta$ are the vertices of a regular polygon of $d+1$ sides centered at a. The definition of c_m and c_l implies that $\tilde{c}_l$ is adjacent to $\tilde{c}_m$ and then, they are separated by an angle of $2\pi/(d+1)$ measured from a. So, if $\tilde{c}_m$ and a have the same argument, then the modulus of $\tilde{c}_m$ is equal to $1 + |\delta|$ and

$$|\tilde{c}_l| = 1 + |\delta|\cos\left(\frac{2\pi}{d+1}\right).$$

Instead, if $\tilde{c}_m$ and a have different arguments, then one of the vertices adjacent to $\tilde{c}_m$ has modulus larger than the above value. Then, in general, the modulus of $\tilde{c}_l$ satisfies

$$|\tilde{c}_l| \geq 1 + |\delta|\cos\left(\frac{2\pi}{d+1}\right). \tag{3.2}$$

Same computations as in the proof of part (a) show that

$$|\tilde{v}_l| \geq 1 + n\,(1 + \frac{1}{d})\cos\left(\frac{2\pi}{d+1}\right)|\delta| \pm \mathcal{O}(|\delta|^2).$$

It is easy to check that when $n > 1$ and $d > 4$ (or if $n > 2$ and $d > 3$) we have that $n\,(1 + \frac{1}{d})\cos\left(\frac{2\pi}{d+1}\right) > 1$. Then, using Lemma 2.7 we conclude that v_l also belongs to B as we wanted to show.

□

3.2 Proofs of Theorems 1.3 and 1.4

Theorems 1.3 and 1.4 deal with the topology of the Julia and Fatou sets when the critical points in S_a are distributed between B and Q. They also describe the dynamics of $f_{\lambda,a}$ on its Julia set.

First notice that part (a) of Theorem 1.3 follows easily from part (b) of Proposition 2.8 then we focus on part (b) of Theorem 1.3. The proof of Theorem 1.4 will become clear from the proof of Theorem 1.3.

The hypothesis of Theorem 1.3 state that $n \geq 2$, $d \geq 1$, $|a| = 1$ and for λ sufficiently small $S_a \subset B \cup Q$. Also, the number of critical points in Q and B is fixed for $|\lambda|$ sufficiently small. Moreover, we know that either $S_a \cap B = \emptyset$ or there are at least two critical points from S_a in B and the rest lie in Q. From Theorem 1.2 we conclude that the first situation can only happen in the very special case when $d = 1$. Numerical experiments suggest that this rarely happens (if it happens at all). See Remark 1.6. In this particular case we would have the existence of a disjoint preimage of B, that is, a trap door T, and all the critical points in S_a would lie in Q. The proof that the structure of the Julia set is as stated in Theorem 1.3 follows the same lines as in [8] in the case when $|a| < 1$. In this section we focus on the second case, that is, when more than one critical point from S_a belongs to B and the rest belong to Q.

We shall prove that for any a with $|a| = 1$, there exists $\epsilon_a > 0$ such that if $|\lambda| < \epsilon_a$, then the Julia set of $f_{\lambda,a}$ consists of a countable collection of simple closed curves together with an uncountable collection of point components that accumulate on these curves. Only one of these curves surrounds the origin while all others bound disjoint disks that are eventually mapped onto Q. Moreover, any two such maps are topologically conjugate on their Julia sets.

By Proposition 2.8 part (c) we know that for sufficiently small λ, a lies in B, B is infinitely connected and Q is simply connected. Let ∂Q denote the boundary of Q. The holes in B are due to the preimages of Q and uncountably many point components that accumulate on the boundaries of these disks.

Let $d_* = |S_a \cap B| \leq d$ and assume that $d_* \geq 2$. Since there are $d + 1 - d_*$ critical points from S_a in Q and $f_{\lambda,a}$ is of degree $n + d$, it follows by the Riemann-Hurwitz formula that there are $d_* - 1$ disjoint disks that are preimages of Q in the complement of $\overline{Q}$.

Let ρ be a simple closed curve that lies in B and such that the Julia set of $f_{\lambda,a}$ lies in the bounded component surrounded by ρ. This component is an open neighborhood of $\overline{Q}$. Consider the $d_* - 1$ preimages of this neighborhood that contain the preimages of Q. We denote these sets by $I_1, I_2, ..., I_{d_*-1}$ and notice that their boundaries are preimages of the curve ρ. For $|\lambda|$ small enough we can choose ρ so that the I_j's are pairwise disjoint. The set of points whose orbits remain for all iterations in the union of the I_j forms a Cantor set on which $f_{\lambda,a}$ is conjugate to the one-sided shift map on $d_* - 1$ symbols. This follows from standard arguments in complex dynamics [12]. This produces an uncountable number of point components in the Julia set. However, there are many other point components in J as we show below.

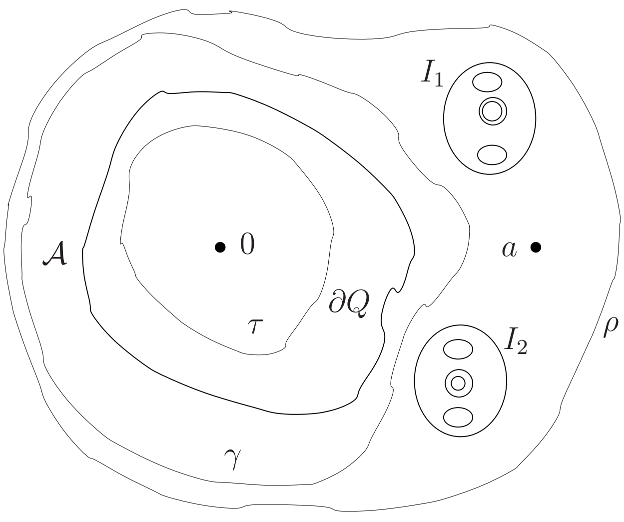


Figure 9: Sketch for the proof of Theorem 1.3. Region $\mathcal{A}$ and the I_j's for $j = 1, ..., d_* - 1$ are depicted. The annulus $\mathcal{A}$ is bounded by the curve τ and the curve γ and contains the boundary of Q.

Let γ be a simple closed curve that lies in B and surrounds Q so that the pole a and the critical points in $S_a \cap B$ lie outside γ. Consider also a simple closed curve τ that lies inside Q and surrounds the origin and such that all the critical points in S_q and the critical points in $S_a \cap Q$ lie in the bounded component surrounded by τ. Notice that $|S_a \cap Q| = d - d_* \geq 0$. Let $\mathcal{A}$ be the annulus bounded by τ and γ. Then, $\partial Q \subset \mathcal{A}$ and notice that each I_j contains a copy of $\overline{Q}$ and a copy of each one of the I_j's. See Figure 9.

To understand the complete structure of the Julia set, we show that J is homeomorphic to a quotient of a subset of a space of one sided sequences of finitely many symbols. Moreover, we show that $f_{\lambda,a}$ on J is conjugate to a certain quotient of a subshift of finite type on this space. Since this is true for λ sufficiently small, this will prove our main results.

To begin the construction of the sequence space, we first partition the annulus $\mathcal{A}$ into n "rectangles" that are mapped over $\mathcal{A}$ by $f_{\lambda,a}$.

Proposition 3.1. *There is an arc ξ lying in $\mathcal{A}$ and having the property that $f_{\lambda,a}$ maps ξ 1-to-1 onto a larger arc that properly contains ξ and such that one of its endpoints lies in τ and the other one in γ. Moreover, ξ meets ∂Q at exactly one point, namely one of the fixed points in ∂Q. With the exception of this point, all other points on ξ lie in the Fatou set.*

Proof. Let $p = p_{\lambda,a}$ be one of the repelling fixed points in ∂Q. Note that p varies analytically with both λ and a. As it is well known, there is an invariant ray in Q extending from p to q (see [12]). Define the portion of ξ in $Q \cap \mathcal{A}$ to be the piece of this ray that lies in $\mathcal{A}$.

To define the piece of ξ lying outside ∂Q, let U be an open set that contains p and meets some portion of γ and also has the property that the branch of the inverse of $f_{\lambda,a}$ that fixes p is well-defined on U. Let $f^{-1}_{\lambda,a}$ denote this branch of the inverse of $f_{\lambda,a}$. Let $w \in \gamma \cap U$ and choose any arc in U that connects w to $f_{\lambda,a}^{-1}(w)$. Then we let the remaining of the curve ξ be the union of the pullbacks of this arc by $f_{\lambda,a}^{-k}$ for all $k \geq 0$. Note that this curve limits on p as $k \to \infty$. $\square$

We now partition $\mathcal{A}$ into n rectangles using the n preimages of $f_{\lambda,a}(\xi)$ that lie in $\mathcal{A}$. Denote these preimages by $\xi_1, ..., \xi_n$ where $\xi_1 = \xi$ and the remaining ξ_j's are arranged counterclockwise around $\mathcal{A}$. Let A_j denote the closed region in $\mathcal{A}$ that is bounded by ξ_j and ξ_{j+1}, so that A_n is bounded by ξ_n and ξ_1. By construction, each A_j is mapped 1-to-1 over $\mathcal{A}$ except on the boundary arcs ξ_j and ξ_{j+1}, which are each mapped 1-to-1 onto $f_{\lambda,a}(\xi_1) \supset \xi_1$.

The only points whose orbits remain for all iterations in $\mathcal{A}$ are those points on the simple closed curve ∂Q. Let $z \in \partial Q$. We may attach a symbol sequence $S(z)$ to z as follows. Consider the n distinct symbols $\alpha_1, \ldots, \alpha_n$ taken from $\mathbb{Z} \setminus \{1, 2, \ldots, d_* - 1\}$. Define $S(z) = (s_0 s_1 s_2 \ldots)$ where each s_j is one of the symbols $\alpha_1, \ldots, \alpha_n$ and $s_j = \alpha_k$ if and only if $f_{\lambda,a}^j(z) \in A_k$. Note that there are two sequences attached to p, the sequences $(\overline{\alpha_1})$ and $(\overline{\alpha_n})$. Similarly, if $z \in \xi_k \cap \partial Q$, then there are also two sequences attached to z, namely $(s_0 s_1 ... s_{j-1} \alpha_{k-1} \overline{\alpha_n})$ and $(s_0 s_1 ... s_{j-1} \alpha_k \overline{\alpha_1})$.

Note that if we make the above identifications in the space of all one-sided sequences of the α_j's then this is precisely the same identifications that are made in coding the itineraries of the map $z \mapsto z^n$ on the unit circle. So this sequence space with these identifications and the usual quotient topology is homeomorphic to the unit circle and the shift map on this space is conjugate to $z \mapsto z^n$.

Finally, we extend the definition of $S(z)$ to any point in J that remains in the union of the I_j's by introducing the symbols $1, \ldots, d_* - 1$ and defining $S(z)$ in the usual manner. We identify the sequences of the form $(j\overline{\alpha_1})$ and $(j\overline{\alpha_n})$ as well as $(j\alpha_k\overline{\alpha_n})$ and $(j\alpha_k\overline{\alpha_1})$.

Let Σ' denote the space of one-sided infinite sequences of symbols $\alpha_1, \ldots, \alpha_n, 1, \ldots, d_* - 1$. Let Σ denote the space Σ' with all of the identifications described above and endow Σ with the quotient topology. Then, by construction, the Julia set of $f_{\lambda,a}$ is homeomorphic to Σ and $f_{\lambda,a} \mid J$ is conjugate to the full shift map on Σ.

This finishes the proof of Theorems 1.3 and 1.4.

3.3 Proof of Theorem 1.5

Theorem 1.5 shows the existence of the Julia sets described in Theorem 1.3.

Proof. (a) The idea of the proof is the following. For $d = 1, 2$ and $|\lambda|$ sufficiently small we can always choose the argument of λ such that one critical point in S_a belongs to B and the other d belong to Q. Then from Proposition 2.8 part (b) it follows that B and Q are both completely invariant and simply connected and the Julia sets are as stated in Theorem 1.3 part (a).

Fix n and a. First assume that $d = 1$. In this case there are two values of $\tilde{c}$ (see

Lemma 2.5) which are given by

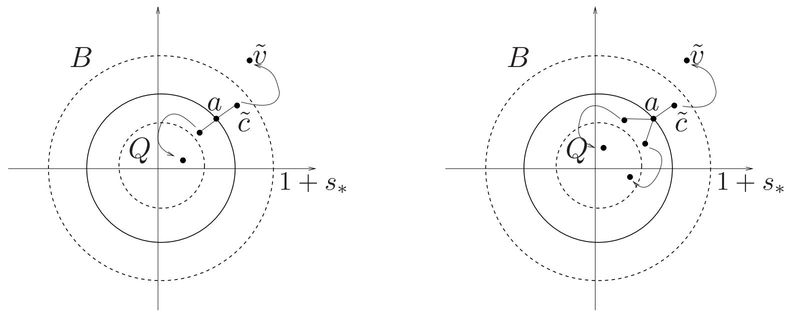
$$\tilde{c}_{\pm} = a \pm \delta = a \pm \left(\frac{\lambda}{n\, a^{n-1}} \right)^{1/2}.$$

We can choose the parameter λ so that a and δ are parallel vectors in the plane (see Figure 10(a)). For this, let $\lambda = \lambda_0$ then, $Arg(\lambda_0)$ has to verify

$$Arg(a) = \frac{Arg(\lambda_0) - (n-1)Arg(a)}{2} \pmod{2\pi}$$

or equivalently,

$$Arg(\lambda_0) = (n+1)Arg(a) \pmod{2\pi}.$$



(a) $d=1$ and $\lambda \in S_{\alpha_1,\beta_1}$

(b) $d=2$ and $\lambda \in S_{\alpha_2,\beta_2}$

Figure 10: Sketch of the relevant objects in the proof of Theorem 1.5 part (a). In this case we must prove that one critical point in S_a belongs to B and the rest d critical points belong to Q.

For $\lambda = \lambda_0$ we have that $|\tilde{c}_{\pm}| = 1 \pm (|\lambda_0|/n)^{1/2}$. Let S_{α_1,β_1} be the sector of parameters λ with $\alpha_1 = Arg(\lambda_0) - \pi/10$ and $\beta_1 = Arg(\lambda_0) + \pi/10$. Then for $\lambda \in S_{\alpha_1,\beta_1}$ we have

$$|\tilde{c}_+| > 1 + \cos(\pi/20)\,|\delta|$$
$$|\tilde{c}_-| < 1 - \cos(\pi/20)\,|\delta|.$$

Simple computations show that $\tilde{v}_{\pm} = f_{\lambda,a}(\tilde{c}_{\pm}) = \tilde{c}_{\pm}^n + \lambda/(\tilde{c}_{\pm} - a)$ can be written as

$$\tilde{v}_{\pm} = a^{n-1}(a + 2n\delta) + \mathcal{O}(\delta^2).$$

For λ in the sector S_{α_1,β_1}, we have that

$$|\tilde{v}_+| > 1 + \cos(\pi/20)\, 2n|\delta| \pm \mathcal{O}(|\delta|^2)$$
$$|\tilde{v}_-| < 1 - \cos(\pi/20)\, 2n|\delta| \pm \mathcal{O}(|\delta|^2).$$

To show that $\tilde{v}_+$ belongs to B and $\tilde{v}_-$ belongs to Q it is enough to prove that $|\tilde{v}_+| > 1 + s_*$ and $|\tilde{v}_-| < 1 - s_*$, where $s_* = \left(\frac{|\lambda|}{n-1}\right)^{1/2}$ (see Corollary 2.3). Then we require

$$\cos(\pi/20)\, 2n \left(\frac{|\lambda|}{n}\right)^{1/2} > \left(\frac{|\lambda|}{n-1}\right)^{1/2}.$$

For $n \geq 2$, we have $\cos^2(\pi/20)\,4n(n-1) > 1$, and then the above inequality is satisfied. Then by Lemma 2.7 it follows that v_+ also belongs to B and v_- belongs to Q.

The case when $d = 2$ is very similar and then we briefly explain the main changes in the above argument. In this case we have to show that one critical point in S_a belongs to B and the other two critical points belong to Q. For this, we pick a value of the parameter λ, that we call λ_0, such that there is only one $\tilde{c}$ with modulus larger than 1 and such that a and δ are parallel vectors in the plane (see Figure 10(b)). The three values of $\tilde{c}$ are given by

$$\tilde{c}_i = a + \delta = a + \left(\frac{2\lambda}{n\,a^{n-1}}\right)^{1/3}, \text{ for } i = 1, 2, 3.$$

The value of $Arg(\lambda_0)$ is the solution of $Arg(a) = Arg(\delta)$, so it has to satisfy

$$Arg(a) = \frac{Arg(\lambda_0) - (n-1)Arg(a)}{3} \pmod{2\pi}.$$

Thus, we have that $Arg(\lambda_0) = (n+2)Arg(a) \pmod{2\pi}$. Now, we define the sector of parameters S_{α_2,β_2} given by $\alpha_2 = Arg(\lambda_0) - \pi/10$ and $\beta_2 = Arg(\lambda_0) + \pi/10$. Then when $\lambda \in S_{\alpha_2,\beta_2}$ we have

$$\begin{aligned} &|\tilde{c}_1| > 1 + \cos(\pi/30)\,|\delta| \\ &|\tilde{c}_i| < 1 - \cos(11\pi/30)\,|\delta| \qquad \text{for } i = 2, 3. \end{aligned}$$

We can rewrite $\tilde{v}_i = f_{\lambda,a}(\tilde{c}_i) = \tilde{c}_i^n + \lambda/(\tilde{c}_i - a)^2$ as

$$\tilde{v}_i = a^{n-1}(a + \frac{3}{2}\,n\delta) + \mathcal{O}(\delta^2) \qquad \text{for } i = 1, 2, 3.$$

Hence for $\lambda \in S_{\alpha_2,\beta_2}$ we obtain

$$\begin{aligned} &|\tilde{v}_1| > 1 + \frac{3}{2}\,n\cos(\pi/30)\,|\delta| \\ &|\tilde{v}_i| < 1 - \frac{3}{2}\,n\cos(11\pi/30)\,|\delta| \qquad \text{for } i = 2, 3. \end{aligned}$$

When $n \geq 2$, we have that $\frac{3}{2}\,n\cos(\pi/30) > 1$ and also $\frac{3}{2}\,n\cos(11\pi/30) > 1$, and Lemma 2.7 implies that v_1 belongs to B and v_2 and v_3 belong to Q as we wanted to show.

(b) The idea to prove this part is the following. When $d = 2, 3, 4$ we can choose $Arg(\lambda)$ such that two critical points in S_a belong to B and the rest belong to Q. Then from Proposition 2.8 part (c) we conclude that B is completely invariant and the basin of attraction of q has infinitely many simply connected components. Then, the structure of the Julia and Fatou sets is as in Theorem 1.3 part (b).

In this case $\tilde{c}$ is given by

$$\tilde{c} = a + \delta = a + \left(\frac{d\lambda}{n\,a^{n-1}}\right)^{\frac{1}{d+1}}.$$

If we impose that $Arg(\delta) = Arg(a) + \frac{\pi}{d+1}$ for some value λ_0, we obtain that $Arg(\lambda_0)$ verifies

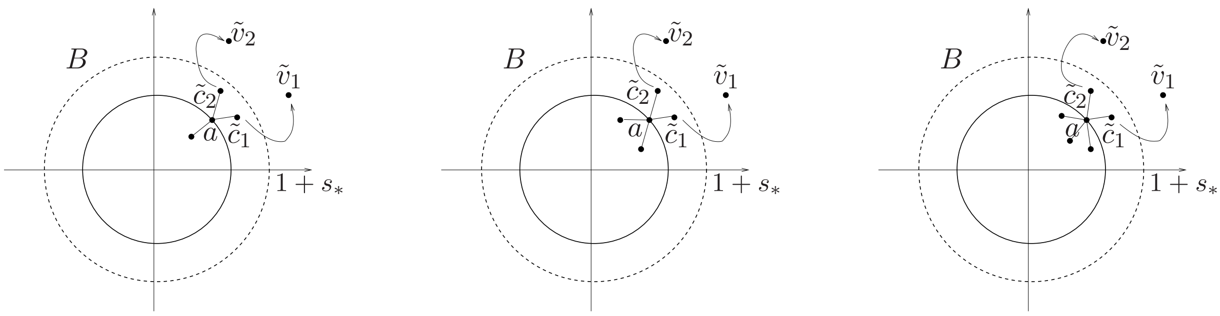
$$Arg(a) + \frac{\pi}{d+1} = \frac{Arg(\lambda_0) - (n-1)Arg(a)}{d+1} \pmod{2\pi}$$

or equivalently,

$$Arg(\lambda_0) = (n+d)Arg(a) + \pi \pmod{2\pi}.$$

For $\lambda = \lambda_0$ there are two values of $\tilde{c}$ that we denote by $\tilde{c}_1$ and $\tilde{c}_2$, such that $|\tilde{c}_1| = |\tilde{c}_2|$ and this is equal to the largest value of the $d+1$ possible values of $\tilde{c}$ (see Figure 11). We have

$$|\tilde{c}_1|, |\tilde{c}_2| = 1 + \cos\left(\frac{\pi}{d+1}\right)|\delta|.$$



(a) $d = 2$ and $\lambda \in S_{\gamma_2,\delta_2}$ (b) $d = 3$ and $\lambda \in S_{\gamma_3,\delta_3}$ (c) $d = 4$ and $\lambda \in S_{\gamma_4,\delta_4}$

Figure 11: Sketch of the relevant objects in the proof of Theorem 1.5 part (b). In this case it is enough to prove that two critical values belong to B.

Let S_{γ_d,δ_d} be the sector of parameters λ with $\gamma_d = Arg(\lambda_0) - \pi/10$ and $\delta_d = Arg(\lambda_0) + \pi/10$. Then, when $\lambda \in S_{\gamma_d,\delta_d}$ we have

$$|\tilde{c}_1|, |\tilde{c}_2| > 1 + \cos\left(\frac{11\pi}{10(d+1)}\right)|\delta|.$$

Simple computations show that, for $i = 1, 2$, $\tilde{v}_i$ can be written as

$$\tilde{v}_i = a^{n-1}(a + n(1 + \frac{1}{d})\delta) + \mathcal{O}(|\delta|^2) \qquad \text{for } i = 1, 2.$$

Hence for $\lambda \in S_{\gamma_d,\delta_d}$ we have that

$$|\tilde{v}_i| \geq 1 + n(1 + \frac{1}{d})\cos\left(\frac{11\pi}{10(d+1)}\right)|\delta| + \mathcal{O}(|\delta|^2) \qquad \text{for } i = 1, 2.$$

It is easy to check that when $n \geq 2$ and $d = 2, 3, 4$ we have that $n\,(1 + \frac{1}{d})\cos\left(\frac{11\pi}{10(d+1)}\right) > 1$. Then, using Lemma 2.7 we conclude that v_1 and v_2 belong to B.

The fact that the other critical points in S_a belong to Q follows in a similar fashion. For example, when $d = 2$ we have that the third critical point $\tilde{c}_3$ is such that $|\tilde{c}_3| < 1 - |\delta| \cos(\pi/30)$. Then $\tilde{v}_3$ is such that $|\tilde{v}_3| < 1 - 3/2n|\delta| \cos(\pi/30)$. It follows easily that for $n \geq 2$ we have $3/2n \cos(\pi/30) > 1$ and then $v_3 \in Q$. When $d = 3$ we have to check that the other two critical points are in Q and this follows as above.

When $d = 4$ there are actually two different sectors in parameter λ-plane for which part (b) of Theorem 1.5 holds. One case is as shown above when two critical points from S_a belong to B and the other three critical points from S_a belong to Q. The other way occurs when the opposite happens, that is, when there are three critical points from S_a in B and two in Q. This case can be proved in a similar fashion and it is left to the reader.

□

4 The case $n = 1$.

In this part we study the family of complex maps given by

$$f_{\lambda,a}(z) = z + \frac{\lambda}{(z-a)^d}$$

where $d \geq 1$ is an integer and a and λ are complex parameters. As a difference between this case and the case $n > 1$ we observe that the expression of $f_{\lambda,a}(z)$ suggests that the same map could be derived by applying the so called *Newton's Iteration Method*. The Newton iteration function $N(z)$ of a function $h(z)$ is given by

$$N(z) = z - \frac{h(z)}{h'(z)}$$

where $h'(z)$ denotes the derivative of $h(z)$. This method can be used to approximate the roots of polynomials, and has been shown to display very interesting Julia sets when the function $N(z)$ is considered as a map on the Riemann sphere (see, for example, [2, 14]). Indeed, if we let

$$h(z) = ke^{-\frac{(z-a)^{d+1}}{\lambda(d+1)}}$$

where k is an arbitrary constant, then $f_{\lambda,a}(z)$ is the Newton iteration function for $h(z)$[1]. This follows easily since $h(z)$ is the general solution of the differential equation

$$\lambda \frac{dh}{dz} = -(z-a)^d h.$$

The study of the dynamics of $f_{\lambda,a}(z)$ is simplified by the fact that we can conjugate the map to eliminate the parameters λ and a. A simple computation shows that:

[1]As a curiosity, notice that if we let $d = 1$ and $k = 1/\sqrt{\lambda 2\pi}$ then the function $h(z)$ is a Gaussian distribution in the variable z with *expected value* a and *variance* λ.

Lemma 4.1. *The function* $f_{\lambda,a}(z)$ *is conjugate to the function* $g(z) = z + 1/z^d$ *under* $z \mapsto \lambda^{-1/(d+1)}(z-a)$.

Therefore we study the map given by

$$g(z) = z + \frac{1}{z^d}$$

where $d \geq 1$ is an integer. There is no dependence on complex parameters so for each value of d we have just a unique representative of the family $f_{\lambda,a}$.

The following theorem is due to Shishikura [15] (see also [10]) and gives a connection between the number of weakly repelling fixed points of a rational map and the connectivity of the Julia set. Recall that a *weakly repelling fixed point* is a fixed point that is either repelling or parabolic of multiplier 1.

Theorem 4.2. *If the Julia set of a rational map* f *of degree* ≥ 2 *is disconnected, then there exist two weakly repelling fixed points of* f.

It is easy to check that $g(z)$ has only one parabolic fixed point at infinity and then Theorem 1.7 follows as a corollary of Theorem 4.2. In the following paragraphs we describe the symmetries and dynamical behavior of the function $g(z)$.

When $d = 1$ we have that the Julia set of $g(z)$ is the imaginary axis. This follows since the imaginary axis is the smallest closed set with more than two points that is completely invariant under the map. It is easy to check that $|g'(z)| > 1$ for $z \in i\mathbb{R}$ and that every point that is not in the imaginary axis moves away from it under iteration. The Fatou set consists of the two completely invariant half-planes $Re(z) > 0$ and $Re(z) < 0$ for $z \in \mathbb{C}$. Each half of the real axis is forward invariant and every orbit in one of the two half-planes approaches the real axis under iteration and converges to infinity.

When $d > 1$, the Julia set is still connected as we have already shown; however, the Fatou set now consists of infinitely many simply connected components. The degree of $g(z)$ is $d+1$ so the map has $2d$ critical points counted with multiplicity. The pole 0 is a critical point of order $d-1$ and then, there are $d+1$ critical points symmetrically distributed around the origin. The critical points c of g are given by

$$c = d^{\frac{1}{d+1}}. \tag{4.1}$$

Infinity and its preimages lie in the Julia set of $g(z)$. This set includes the *prepoles*, that is, the preimages of the pole at the origin. The prepoles p of $g(z)$ are also symmetrically distributed around the origin and are given by $p = (-1)^{\frac{1}{d+1}}$. The critical points c of $g(z)$ are mapped to the critical values v. We have

$$v = g(c) = d^{\frac{1}{d+1}}(1 + 1/d). \tag{4.2}$$

A straightforward computation shows that each line of the form ωt with $\omega^{d+1} = 1$ and $t > 0$ is forward invariant under $g(z)$. Moreover, every point in one of these lines converges monotonically to infinity under iteration. From Equations (4.1) and (4.2), it follows that the critical points lie in these lines where the orbit of every point converges to infinity. In other words, each one of the critical points lies in a different petal of the flower around infinity.

Figure 6 displays some examples of the Julia sets studied in this section.

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References

[1] Beardon, A. F. *Iteration of rational functions*, Springer, 1991.

[2] Blanchard, P. The dynamics of Newton's method. *Complex dynamical systems*, Cincinnati, OH, 1994, *Proc. Symp. Appl. Math.*, Vol. **49** (AMS, Providence, RI), pp. 139-154.

[3] Blanchard, P.,Devaney, R. L., Look, D. M., Seal, P. and Shapiro, Y. Sierpinski curve Julia sets and singular perturbations of complex polynomials. *Ergodic Theory Dynam. Systems*, Vol. **25** (2005), pp. 1047-1055.

[4] Carleson, L. and Gamelin, Th. *Complex dynamics*, Springer, 1993.

[5] Devaney, R. L., Look, D. M. and Uminsky, D. The escape trichotomy for singularly perturbed rational maps. *Indiana Univ. Math. J.*, Vol. **54** (2005) pp. 1621-1634.

[6] Devaney, R. L., Holzer, M., Look, D. M., Moreno Rocha, M. and Uminsky, D. Singular perturbations of z^n. *Transcendental dynamics and complex analysis*, London Math. Society Lecture Notes Series, Vol. **348**, Cambridge University Press, eds. P. Rippon and G. Stallard, (2008) pp. 111-137.

[7] Devaney, R. L., Moreno Rocha M. and Siegmund S. Rational maps with generalized Sierpinski gasket Julia sets. *Topology Appl.*, Vol. **154** (2007), pp. 11-27.

[8] Devaney, R. L. and Marotta, S. M. Evolution of the McMullen domain for singularly perturbed rational maps. *Topology Proceedings*, Vol. **32** (2008), pp. 301-320.

[9] Devaney, R. L. and Garijo, A. Julia sets converging to the unit disk. *Proc. Amer. Math. Soc.*, Vol. **136** (2008), pp. 981-988.

[10] Fagella, N., Jarque, X and Taixes, J. Connectivity of Julia sets of transcendental meromorphic maps. 2006. Preprint.

[11] McMullen, C. Automorphisms of rational maps. Holomorphic functions and moduli. Vol. 1. Math. Sci. Res. Inst. Publ. 10. Springer, New York, 1988.

[12] Milnor, J. *Dynamics in one complex variable*. Third edition. Princeton University Press, 2006.

[13] Morosawa, S., Nishimura, Y., Taniguchi, M. and Ueda, T. *Holomorphic dynamics*. Cambridge University Press, 2000.

[14] Roberts, G. E. and Horgan-Kobelski, J. Newton's versus Halley's method: a dynamical systems approach. *International Journal of Bifurcation and Chaos*, Vol. **14**, No. 10 (2004) pp. 3459-3475.

[15] Shishikura, M. The connectivity of the Julia set and fixed points. Preprint IHES, M/90/37 (1990).