Journal of Difference Equations and Applications
Vol. 19, No. 1, January 2013, 124–145

Singular perturbations in the quadratic family with multiple poles

Antonio Garijo[a1], Sebastian M. Marotta[b2] and Elizabeth D. Russell[c]*

[a]*Dep. d'Eng. Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda dels Països Catalans, 26, 43007 Tarragona, Spain;* [b]*Department of Mathematics, University of the Pacific, 3601 Pacific Avenue, Stockon, CA 95211, USA;* [c]*Department of Mathematical Sciences, United States Military Academy, West Point, NY 10996, USA*

(*Received 24 May 2011; final version received 4 October 2011*)

We consider the quadratic family of complex maps given by $q_c(z) = z^2 + c$, where c is the centre of a hyperbolic component in the Mandelbrot set. Then, we introduce a singular perturbation on the corresponding bounded super-attracting cycle by adding one pole to each point in the cycle. When $c = -1$, the Julia set of q_{-1} is the well-known basilica and the perturbed map is given by $f_\lambda(z) = z^2 - 1 + \lambda/(z^{d_0}(z+1)^{d_1})$, where $d_0, d_1 \geq 1$ are integers, and λ is a complex parameter such that $|\lambda|$ is very small. We focus on the topological characteristics of the Julia and Fatou sets of f_λ that arise when the parameter λ becomes non-zero. We give sufficient conditions on the order of the poles so that for small λ, the Julia sets consist of the union of homeomorphic copies of the unperturbed Julia set, countably many Cantor sets of concentric closed curves, and Cantor sets of point components that accumulate on them.

Keywords: Complex dynamical systems; dynamics of rational maps; topology; combinatorics

1. Introduction

In the last decade, a number of papers have appeared that deal with rational maps obtained by perturbing a complex polynomial by adding a pole at some point in the Fatou set of the polynomial. This kind of perturbation of a polynomial is usually called a *singular perturbation* since the degree of the resulting rational map increases when the pole is added. As a consequence of this perturbation, the structure of the Julia sets often changes dramatically after the addition of the pole since new critical points appear close to the pole.

The most studied case is the singular perturbation of the polynomial z^n with $n \geq 2$ obtained by adding a pole at the origin. In this case, the corresponding rational map is given by $z^n + \lambda/z^d$, where $d \geq 1$. See, for example, [4–7]. For the unperturbed map (i.e. when $\lambda = 0$), the Julia set is the unit circle. Points with modulus larger than one are attracted to the super-attracting fixed point at infinity, while points with modulus smaller than one are attracted to the super-attracting fixed point at the origin. When $\lambda \neq 0$, the super-attracting fixed point at the origin is replaced by a pole. However, the rational map $z^n + \lambda/z^d$ inherits some properties of the polynomial. For example, infinity is still a super-attracting fixed point. Since the origin is a pole of order d, there is an open neighbourhood of 0 that is mapped onto a neighbourhood of ∞ in a d-to-1 fashion. If the component of the basin of ∞ which contains ∞ is disjoint from this neighbourhood around the origin we call

*Corresponding author. Email: elizabeth.russell@usma.edu

ISSN 1023-6198 print/ISSN 1563-5120 online

http://dx.doi.org/10.1080/10236198.2011.630668
http://www.tandfonline.com