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UPPER BOUNDS FOR THE NUMBER OF LIMIT CYCLES OF SOME PLANAR POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We give an effective method for controlling the maximum number of limit cycles of some planar polynomial systems. It is based on a suitable choice of a Dulac function and the application of the well-known Bendixson-Dulac Criterion for multiple connected regions. The key point is a new approach to control the sign of the functions involved in the criterion. The method is applied to several examples.

1. **Introduction and main results.** One of the few general methods that allows to give upper bounds for the number of limit cycles of planar differential systems

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y)$$

is the use of Dulac functions in multiple connected regions, see [2, 3, 4, 7, 8, 9]. Recall that the primer idea is that when the function $\operatorname{div}(P,Q)$ does not vanish on a simply connected region $\mathcal{U} \subset \mathbb{R}^2$, then the above differential system has no periodic orbit totally contained in $\mathcal{U}$. We state the general Bendixson-Dulac Criterion in next section, see Theorem 2.1. The main difficulty for practical uses of this result is that it is needed to find a (Dulac) function g such that $\operatorname{div}(gP, gQ)$ does not vanish on a suitable set. This paper gives a quite general result for polynomial differential systems, see Theorem A. Its proof is based on a "good" choice of a Dulac function. As we will see in Section 3 this result provides a constructive way for giving upper and lower bounds for the number of limit cycles of a large class of planar polynomial systems.

Given a polynomial $p(s) \in \mathbb{R}[s]$, we will say that the couple $(k, w(r)) \in \mathbb{R}^+ \times \mathbb{R}[r]$ is a *Dulac pair* of $p(s)$ if

$$p_{k,w}(r) := rp(r^2)w'(r) - 2k\left(p(r^2) + r^2p'(r^2)\right)w(r) < 0 \quad \text{for all} \quad r > 0.$$

As we will see in Lemma 2.7 and in the proof of Proposition 2.2, the above inequality implies that the function $|w(r)|^{-1/k}$ is a Dulac function, in any of the connected

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