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A proof of Perko's conjectures for the Bogdanov–Takens system ☆

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ABSTRACT

The Bogdanov–Takens system has at most one limit cycle and, in the parameter space, it exists between a Hopf and a saddle-loop bifurcation curves. The aim of this paper is to prove the Perko's conjectures about some analytic properties of the saddle-loop bifurcation curve. Moreover, we provide sharp piecewise algebraic upper and lower bounds for this curve.

1. Introduction

The Bogdanov–Takens system

$$\begin{cases} x' = y, \\ y' = -n + by + x^2 + xy, \end{cases}$$

has been introduced in [1,17,18]. It provides a universal unfolding of a cusp point of codimension 2 and it is considered in many basic text books on bifurcation theory; see for instance [3,8,12]. Some

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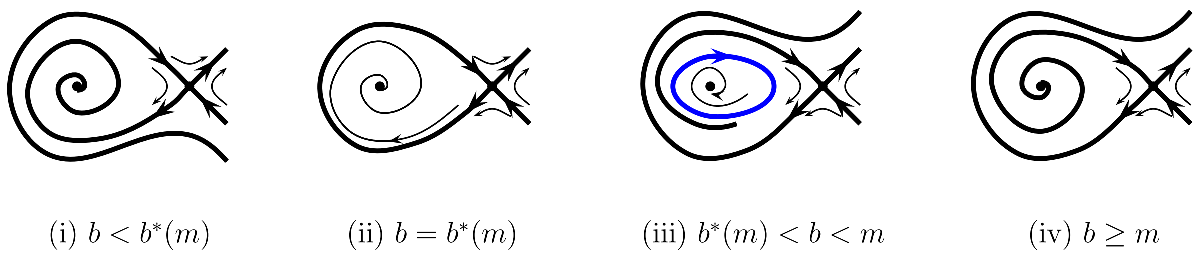

Fig. 1. Phase portraits of system (1).

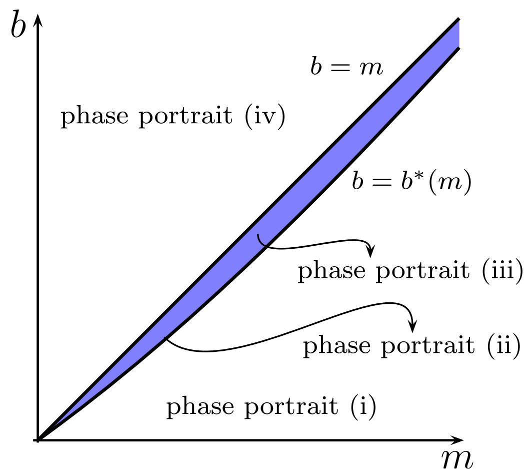

Fig. 2. Sketch of the bifurcation diagram of system (1). The open colored region is the one containing the limit cycle.

global quantitative properties of its bifurcation diagram are not known. In 1992, Perko stated two conjectures about analytic properties of the saddle-loop bifurcation curve in the parameter space; see [13]. The aim of this work is to prove both conjectures.

The interesting bifurcations only appear in the region $n > 0$, because in this case the system has two critical points, $(\pm\sqrt{n}, 0)$, a saddle and a focus. Therefore it is natural to introduce a new positive parameter $m = \sqrt{n}$. So, we will consider the following expression of the above system

$$\begin{cases} x' = y, \\ y' = -m^2 + by + x^2 + xy, \quad \text{with } m > 0. \end{cases} \tag{1}$$

Before presenting our results, we recall the known properties about the bifurcation diagram of system (1). All the qualitative information of this diagram and part of the quantitative one are known; see [10,13,15,16]. In particular, it is proved in [10] that this system has at most one limit cycle and that when it exists it is hyperbolic and unstable. This information, together with the fact that system (1) is a rotated family of vector fields with respect to b, allows to show that the limit cycle exists if and only if $b^*(m) < b < m$, for an unknown function $b^*(m)$. This holds because fixing m and decreasing b, a unique unstable limit cycle is born via a Hopf bifurcation for $b = m$, increases diminishing b, and disappears in a saddle-loop connection for $b = b^*(m)$. The corresponding phase portraits are drawn in Fig. 1 and a sketch of its bifurcation diagram is given in Fig. 2.

Some quantitative information about $b^*(m)$ is given by Perko in [13]:

(i) It is an analytic function.
(ii) It holds that $\max(-m, m-1) < b^*(m) < m$.
(iii) At $m = 0$, $b^*(m) = 5m/7 + O(m^2)$. This term is computed by using the Melnikov method; see also [8].

As usual, we write $f(m) = O(m^p)$ or $g(m) = o(m^p)$ at $m = m_0 \in \mathbb{R} \cup \{\infty\}$ if

$$\lim_{m\to m_0} \frac{f(m)}{m^p} = K \in \mathbb{R}, \quad \text{or} \quad \lim_{m\to m_0} \frac{g(m)}{m^p} = 0.$$

The lower bound given in item (ii) is improved in [9] applying the Bendixson–Dulac Theorem and proving that $\max(m/2, m-1) < b^*(m) < m$.

Item (iii) has been improved recently in [7] using a different approach, based on the construction of algebraic curves with a loop that is without contact for the flow of the system. The authors obtain that, at $m = 0$,

$$b^*(m) = \frac{5}{7}m + \frac{72}{2401}m^2 - \frac{30\,024}{45\,294\,865}m^3 - \frac{2\,352\,961\,656}{11\,108\,339\,166\,925}m^4 + O\big(m^5\big). \tag{2}$$

The Bogdanov–Takens system for parameters in a neighborhood of infinity is also studied in [2]. The aim of that work was to understand the presence of the limit cycle for the system in terms of slow–fast dynamics. No quantitative information about the shape of the curve $b = b^*(m)$ is given there.

In [13] a different, but equivalent, expression of system (1) is considered. Next, Perko's conjectures are translated to the context of system (1).

Perko's Conjectures. *(See [13].) Let $b = b^*(m)$ be the function that corresponds to the saddle-loop bifurcation curve for system (1). Then,*

($\mathcal{I}$) *for m large enough, $b^*(m) = m - 1 + O(\frac{1}{\sqrt{m}})$,*
($\mathcal{II}$) *it holds that* $\max(5m/7, m-1) < b^*(m)$.

To facilitate the reading of this work, original Perko's formulation of above conjectures is recalled in Section 6.

Both conjectures are immediate consequences of Theorems 1 and 2. Moreover, Theorem 2 significatively improves the global lower and upper bounds given above.

Theorem 1. *For m large enough, $b^*(m) = m - 1 + o(\frac{1}{m})$.*

Theorem 2. *It holds that*

$$\max\left(\frac{5m}{7}, m-1\right) < b^*(m) < \min\left(\frac{(5+\frac{37}{12}m)m}{7+\frac{37}{12}m}, m-1+\frac{25}{7m}\right).$$

To prove our results we develop the method introduced in [7], adapting it according to small or large values of m. The basic idea is as follows: for given positive values of b and m such that $b < m$, we want to know if $b > b^*(m)$ or $b < b^*(m)$; or equivalently to prove the existence or non-existence of the limit cycle. Due to the uniqueness and hyperbolicity of the limit cycle these two situations can be distinguished constructing negative or positively invariant regions, as it is shown in Fig. 3, and employing the Poincaré–Bendixson Theorem.

The most difficult part of this approach consists in constructing these negative or positive invariant regions delimited by loops. In [7], the closed loop around the attracting point to prove (2) is proposed to be the loop of an algebraic self-intersecting curve whose vertex is on the saddle point and whose branches approximate its separatrices. Here we use and develop this idea when m is small. One of the key points for proving that $b = 5m/7$ is a lower bound of the saddle-node bifurcation curve is to consider a special rational parametrization of the straight line $7b - 5m = 0$, $m = m(s)$, $b = b(s)$; see (8). With this parametrization, the coordinates of both critical points on this line and the eigenvalues of

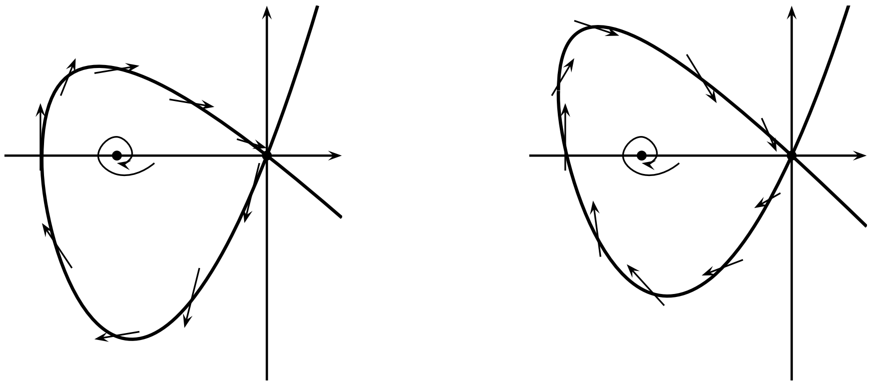

Fig. 3. Negative and positive invariant regions around the focus.

the saddle point are rational functions of s. These facts diminish the computational difficulty helping to prove the result.

When m is large, we adapt the above approach constructing piecewise algebraic closed curves, which also approximate the separatrices of the saddle point and takes into account the region where both separatrices touch, for the first time, the negative x-axis.

The results that involve complicated and long algebraic manipulations are done both with Mathematica and Maple.

The methods introduced in this work can also be useful to quantitatively study the unfolding of other singularities. For instance, the cusp of codimension 3 (our case has codimension 2) considered in [5], the cases considered in [4,6], or the one studied by Takens in [18] and quoted in [15, p. 482] could be approached with our tools.

The paper is organized as follows. In Section 2, changes of variables in phase and parameter spaces are presented to shorten the computations. The new results on the bifurcation curve, close to the origin in the parameter space, are proved in Section 3 and in Section 4 we prove the results for m big enough. Section 5 is devoted to prove Theorems 1 and 2. Finally, in Section 6, Perko's formulation of the conjectures studied in this paper is recalled and it is proved that they are equivalent to the ones stated above.

2. Changes of variables

As usual, to simplify the computations, we introduce changes of variables in both coordinates and parameters. First, we move the saddle point of system (1) to the origin. Second, we change the parameters in such a way that the eigenvalues of the saddle point be rational functions of the new parameters. We obtain in this way simpler expressions for the bifurcation curves.

System (1) is transformed by the change of variables $(x, y) \to (x + m, y)$ into

$$\begin{cases} x' = y, \\ y' = 2mx + (b + m)y + x^2 + xy. \end{cases} \tag{3}$$

In order to obtain rational expressions for the eigenvalues associated to the saddle point we introduce new parameters M and B defined by

$$M^2 = \frac{(b+m)^2 + 8m}{4}, \qquad B = \frac{b+m}{2}. \tag{4}$$

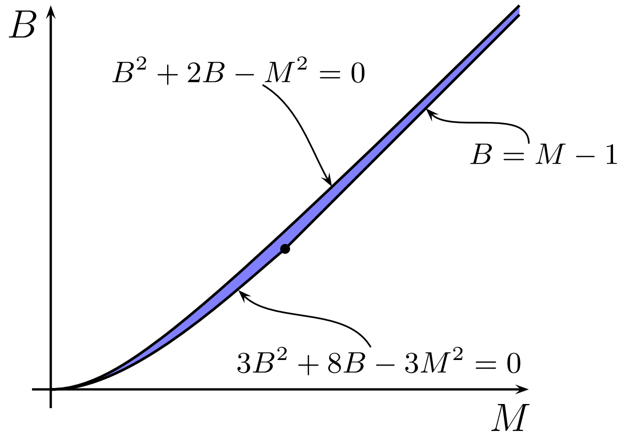

Fig. 4. Curves that define the region $\mathcal{R}$ (colored), that contains $W(M,B)=0$.

Then, system (3) writes as

$$\begin{cases} x' = y, \\ y' = (M^2 - B^2)x + 2By + x^2 + xy. \end{cases} \tag{5}$$

The origin of (5) is a saddle point and the focus of (1), $(-m,0)$, changes to $(B^2-M^2,0)$.

In the parameter space the Hopf bifurcation curve $b=m$ becomes $B^2+2B-M^2=0$. The condition of existence of an invariant straight line $b=m-1$ is moved to $(B+1)^2-M^2=0$ and the lower bound given in [9], $b=m/2$, writes as $3B^2+8B-3M^2=0$. Moreover, the origin goes to the origin and large values of m also correspond with large values of M.

The homoclinic bifurcation curve $b=b^*(m)$ has a new expression for system (5). We denote[1] it by $W(M,B)=0$. As it is located between the curves listed in the previous paragraph, we have that $W(M,B)=0$ is contained in the set

$$\mathcal{R} = \{(M,B)\in\mathbb{R}^2\colon\ B>0,\ M>0,\ 3B^2+8B>3M^2,\ B^2+2B<M^2,\ B>M-1\}, \tag{6}$$

see Fig. 4.

Without loss of generality, we consider system (5) only in the region $\mathcal{R}$. The known results about $b=b^*(m)$ can be translated to analogous results for system (5).

3. Bounds near the origin

This section is devoted to the study of lower and upper bounds for $b=b^*(m)$ for small values of m. The first result proves the lower bound given in Perko's Conjecture $\mathcal{II}$, by using a suitable rational parametrization of the straight line $7b-5m=0$. The second result provides a new algebraic upper bound. In the M,B parameters, this upper bound, $D(M,B)=0$ is given, in an implicit way, by a polynomial D of degree 14. When we transform it into the m,b variables, we get an algebraic curve of degree 25. In Theorem 2 we give a much simpler upper bound, see the details in Section 5.

Notice that the lines $7b-5m=0$ and $b=m-1$ intersect at $(m,b)=(7/2,5/2)$ and for $m>7$ we have $m-1>5m/7$. Recall that it is known that $m-1<b^*(m)$, see [13] or Lemma 6. Hence, in the next result, we only consider $m\leqslant 7/2$.

Proposition 3. *For all $m\leqslant 7/2$ it holds that $b^*(m)>5m/7$.*

[1] Although numerically it seems that we can write it as $B=B^*(M)$ we have no enough information on $b=b^*(m)$ to ensure this fact.

Proof. Recall that system (5) has a saddle point at the origin and the linear approximation of its separatrices is given by the equation

$$C_2(x,y) := \bigl(y-(B+M)x\bigr)\bigl(y-(B-M)x\bigr)=0, \tag{7}$$

which defines two straight lines. The one with slope $B+M$ (resp. $B-M$) is tangent at the origin to the unstable (resp. stable) separatrix.

The curve $7b-5m=0$, for $m,b>0$, can be rationally parametrized as

$$\bigl(m(s),b(s)\bigr)=\left(\frac{7s^2}{2(6s+7)},\frac{5s^2}{2(6s+7)}\right), \tag{8}$$

with $s>0$. From the above expression we have that $M(s)=(7s+3s^2)/(6s+7)$ and $B(s)=3s^2/(6s+7)$. Hence, with this parametrization, the slopes of the separatrices are rational functions of s.

For $N\in\{1,2\}$, let us consider an algebraic curve of degree $2(N+1)$ of the form

$$C(x,y) := C_2(x,y)+\sum_{k=3}^{2(N+1)}\bigl(c_{k,0}x^k+c_{k-1,1}x^{k-1}y+c_{k-2,2}x^{k-2}y^2\bigr)=0 \tag{9}$$

to be determined. Notice that the above curve is quadratic in y. Following the method described in [7], we impose that this expression defines a curve as close as possible to the separatrices. It means that it should coincide at the origin with the separatrices up to the highest possible derivative orders. For this purpose, first we evaluate the Taylor series expansions of both separatrices close to the origin. We express the separatrices as functions of x,

$$y=\Phi^{\pm}(x) := \sum_{k=1}^{\infty}a_k^{\pm}x^k, \tag{10}$$

where the superscript sign determines the separatrix that we approach in each case. Then $a_k^{\pm}$ are real numbers obtained from the identity

$$\frac{\partial(y-\Phi^{\pm}(x))}{\partial x}y+\frac{\partial(y-\Phi^{\pm}(x))}{\partial y}\left(\frac{7s^2}{6s+7}x+\frac{6s^2}{6s+7}y+x^2+xy\right)\bigg|_{y=\Phi^{\pm}(x)}\equiv 0.$$

Straightforward computations show that the first values of $a_k^{\pm}$ are

$$a_1^+=s, \qquad a_1^-=\frac{-7s}{6s+7},$$

$$a_2^+=\frac{(s+1)(6s+7)}{3s(4s+7)}, \qquad a_2^-=\frac{s-7}{3s(2s+7)},$$

$$a_3^+=-\frac{(s+1)(6s+7)^2(5s+14)}{18s^3(4s+7)^2(9s+14)}, \qquad a_3^-=\frac{(7-s)(6s+7)^2(s+2)}{18s^3(2s+7)^2(3s+14)}.$$

In fact all the coefficients of $\Phi^{\pm}$ have rational expressions depending on s with non-vanishing denominators when $s>0$.

Substituting (10) in (9) we get

$$G^{\pm}(x) := C\big(x, \Phi^{\pm}(x)\big) = \sum_{k=3}^{\infty} g_k^{\pm} x^k. \tag{11}$$

From definition (7), $g_k^{\pm}$ vanish for $k = 1, 2$. Imposing that the curve defined by $C(x, y) = 0$ in (9) becomes closer to the separatrices provides extra conditions $g_k^{\pm} = 0$ for higher values of k. We have $6N$ free coefficients $c_{i,j}$. We can fix these coefficients imposing that $g_k^{\pm} = 0$, for $k = 3, 4, \ldots, 3N + 2$ and solving the system. We obtain that all the $c_{i,j}$ are rational functions of s, which are well defined for $s > 0$. We call $U(x, y, s)$ the numerator of $C(x, y)$. We have

$$U(x, y, s) = T_2(x, s)y^2 + T_1(x, s)y + T_0(x, s), \tag{12}$$

where T_0, T_1 and T_2 are polynomials.

The proof continues showing that, for a given set of values of s, the algebraic curve $U(x, y, s) = 0$ defines a positive invariant closed region that contains the focus point; see the right picture in Fig. 3. This assertion follows if we prove:

(I) The curve $U(x, y, s) = 0$ has a loop, as it is shown in Fig. 3, included in the strip $\widetilde{x}(s) < x < 0$ for a given negative value $\widetilde{x}(s)$.
(II) This curve is without contact for the vector field on this strip.
(III) The vector field points in on the loop.

Notice that if we prove these properties for $s \in (0, 7]$ we ensure that the straight line $7b - 5m = 0$ is a lower bound of $b = b^*(m)$ for $m \in (0, 7/2]$.

Taking the curve $U(x, y, s) = 0$ corresponding to $N = 1$, we can only prove the above assertion when $s \in (0, 5]$. When $s \in [5, 7)$ we need to consider $N = 2$. We will detail only the proof for $N = 1$. For the case $N = 2$, we will describe only the differences between the two cases.

So, let us prove (I)–(III) taking $N = 1$ and $s \in (0, 5]$.

Proof of (I). Notice that the curve $U(x, y, s) = 0$ can be written as

$$y = \frac{-T_1(x, s) \pm \sqrt{\Delta(x, s)}}{2T_2(x, s)},$$

where $\Delta := T_1^2 - 4T_2T_0$. Straightforward computations show that

$$\Delta(x, s) = x^2 R_4(x, s), \tag{13}$$

where R_4 is a polynomial of degree 4 in x and of degree 30 in s. Hence, item (I) will follow if we prove that there is a negative value $\widetilde{x}(s)$ such that:

(i) $R_4(\widetilde{x}(s), s) = 0$ and $R_4(x, s) > 0$ in $(\widetilde{x}(s), 0)$.
(ii) $T_2(x, s) \neq 0$ in $[\widetilde{x}(s), 0)$.

We start proving (i). For each s, the coefficients $r_0(s)$ and $r_4(s)$ of minimum and maximum degree of R_4 in x are both positive, it has exactly two simple negative zeros and it has no double zeros. The non-existence of double zeros is due to the fact that

$$\widetilde{R}_4(s) = \operatorname{Res}\big(R_4(x, s), \partial R_4(x, s)/\partial x, x\big) \neq 0, \quad s \in (0, 5],$$

where $\mathrm{Res}(\cdot,\cdot,x)$ denotes the resultant with respect to x; see for instance [11]. These properties follow studying the roots of r_0, r_4 and $\widetilde{R}_4$, that are polynomials with rational coefficients with respective degrees 30, 26 and 190. Their Sturm sequences ensure that they have no positive roots for all $s \in (0,5]$. We take $\widetilde{x}(s)$ to be the maximum of the negative zeros of $R_4(x,s)$.

Let us prove (ii). The polynomial $T_2(x,s)$ has degrees 2 and 14 in x and s, respectively. Straightforward computations, using its Sturm sequence, show that the discriminant with respect to x of T_2 is a polynomial with rational coefficients of degree 26 in s without positive zeros. Hence, it does not vanish when $s \in (0,5]$. It is easy to see that $T_2(0,s) \neq 0$ when $s > 0$ because it is a polynomial of degree 26 with positive coefficients. For proving that $T_2(\widetilde{x}(s),s) \neq 0$ for $s \in (0,5]$ first we show that $\mathrm{Res}(R_4(x,s), T_2(x,s), x) \neq 0$ for every s in this interval. In fact, it is a polynomial of degree 106 in s with no real roots in $s \in (0,5]$. Therefore, for these values of s, the number of real roots of $T_2(x,s)$ in $[\widetilde{x}(s),0)$ does not depend on s. Studying for instance the case $s=1$ we can easily verify that $T_2(x,1)$ has no real roots in $[\widetilde{x}(1),0)$ and then we have the desired result.

Proof of (II). We have to show that the vector field (5) is never tangent to $U(x,y,s)=0$. To prove this fact we study the common zeros between U and its derivative with respect to the vector field

$$\dot{U}(x,y,s) := \frac{\partial U(x,y,s)}{\partial x} y + \frac{\partial U(x,y,s)}{\partial y}\left(\frac{7s^2}{6s+7}x + \frac{6s^2}{6s+7}y + x^2 + xy\right).$$

We get

$$\mathrm{Res}\big(U(x,y,s), \dot{U}(x,y,s), y\big) = x^{12} S_4(x,s), \tag{14}$$

where $S_4(x,s)$ is a polynomial of degree 4 in x and of degree 65 in s. We want to prove that S_4 does not change sign for $x \in [\widetilde{x}(s),0)$ and $s \in (0,5]$.

For a given value of s, say $s=1$, the result follows directly from the Sturm method. After that, we proceed like in the proof of (I). We compute

$$\mathrm{Res}\big(R_4(x,s), S_4(x,s), x\big) \quad \text{and} \quad \mathrm{Res}\big(S_4(x,s), \partial S_4(x,s)/\partial x, x\big).$$

We obtain two polynomials in s of degrees 362 and 438, respectively. In the interval $(0,5]$, the former polynomial does not vanish and the latter has six different real roots. Then, as can be seen with the Sturm method, in each of the seven intervals defined by these roots the relative position of the maximum negative zero of $R_4(x,s)$ with respect to the negative zeros of $S_4(x,s)$ does not change. Choosing s in each interval we can check that there are no zeros of S_4 in $[\widetilde{x}(s),0)$ when $s \in (0,5]$, as we wanted to prove. Moreover, $S_4(0,s)$ does not vanish on $s \in (0,5]$. Then the curve $U(x,y,s)=0$ is without contact in the region where the loop is defined.

We remark that this step is the one that does not work in the whole interval $(0,7)$, taking $N=1$, because $S_4(0,s)$ has a zero close to 5.08.

Proof of (III). We finish the case $N=1$ checking that the vector field points in at the intersection point $(x_0(s),0)$, of the loop contained in the curve $U(x,y,s)=0$ with the x-axis. We prove that $\dot{U}(x_0(s),0,s)$, $\frac{\partial U}{\partial x}(x_0(s),0,s)$ and $\frac{\partial U}{\partial y}(x_0(s),0,s)$ are all negative for $s \in (0,5]$, where $x_0(s)$ is the biggest negative zero of $U(x,0,s)$. As in the above items, it is enough to check the inequalities for a concrete value of s. This holds because, straightforward computations show that $\mathrm{Res}(U(x,0,s)/x^2, \dot{U}(x,0,s)/x^2, x)$, which is a polynomial of degree 69 in s, does not vanish in $(0,5]$, as can be verified with the Sturm method.

As we have already said, the proof finishes taking $N=2$ in (9) and following the above procedure for $s \in [5,7)$. All the qualitative properties remain unchanged. In fact, expressions (13) and (14) change to $\Delta(x,s) = x^2 R_8(x,s)$ and $\mathrm{Res}(U(x,y,s), \dot{U}(x,y,s), y) = x^{18} S_8(x,s)$, respectively. Here R_8 and S_8 are now polynomials of degree 8 in x and 98 and 229 in s, respectively. $\square$

Proposition 4. *For all $0 < M \leqslant 30$, the graph of $W(M,B)=0$ is below the graph of the function defined by the branch of the algebraic curve $D(M,B)=0$, that writes as*

$$\begin{aligned}
&-383\,292M^{14} - 1\,910\,439M^{13}B - 3\,223\,665M^{12}B^2 - 314\,748M^{11}B^3 + 5\,603\,940M^{10}B^4\\
&\quad + 5\,541\,141M^9B^5 - 2\,323\,401M^8B^6 - 7\,154\,664M^7B^7 - 3\,397\,092M^6B^8 + 2\,020\,587M^5B^9\\
&\quad + 3\,218\,997M^4B^{10} + 1\,742\,052M^3B^{11} + 499\,644M^2B^{12} + 76\,071MB^{13} + 4869B^{14}\\
&\quad - 500\,742M^{13} - 787\,023M^{12}B + 4\,493\,070M^{11}B^2 + 14\,795\,091M^{10}B^3 + 11\,566\,572M^9B^4\\
&\quad - 11\,585\,754M^8B^5 - 24\,443\,044M^7B^6 - 8\,307\,134M^6B^7 + 12\,772\,706M^5B^8\\
&\quad + 16\,289\,545M^4B^9 + 8\,662\,166M^3B^{10} + 2\,509\,195M^2B^{11} + 388\,536MB^{12} + 25\,344B^{13}\\
&\quad - 174\,798M^{12} + 1\,420\,524M^{11}B + 7\,005\,177M^{10}B^2 + 4\,483\,350M^9B^3 - 16\,943\,919M^8B^4\\
&\quad - 28\,501\,282M^7B^5 - 3\,691\,132M^6B^6 + 27\,741\,570M^5B^7 + 31\,479\,694M^4B^8 + 16\,742\,014M^3B^9\\
&\quad + 4\,938\,651M^2B^{10} + 781\,536MB^{11} + 52\,119B^{12} - 48\,600M^{11} + 855\,846M^{10}B + 404\,136M^9B^2\\
&\quad - 8\,473\,533M^8B^3 - 14\,178\,838M^7B^4 + 2\,903\,273M^6B^5 + 26\,313\,718M^5B^6 + 28\,894\,211M^4B^7\\
&\quad + 15\,714\,446M^3B^8 + 4\,769\,205M^2B^9 + 775\,554MB^{10} + 53\,046B^{11} + 226\,800M^9B\\
&\quad - 1\,138\,914M^8B^2 - 2\,990\,748M^7B^3 + 2\,116\,351M^6B^4 + 10\,975\,549M^5B^5\\
&\quad + 12\,542\,602M^4B^6 + 7\,156\,446M^3B^7 + 2\,261\,723M^2B^8 + 380\,289MB^9 + 26\,766B^{10}\\
&\quad - 264\,600M^7B^2 + 218\,442M^6B^3 + 1\,575\,182M^5B^4 + 2\,042\,992M^4B^5 + 1\,262\,428M^3B^6\\
&\quad + 421\,554M^2B^7 + 73\,806MB^8 + 5364B^9 = 0,
\end{aligned}$$

and whose series expansion at $M=0$ is

$$B = \frac{3}{7}M^2 - \frac{180}{2401}M^4 + \frac{2\,366\,307}{90\,589\,730}M^6 + O\left(M^7\right).$$

Proof. It follows using the same ideas and techniques developed in the proof of Proposition 3 but with bigger computational difficulties. We only comment the main differences avoiding the details.

Here it is sufficient to consider a curve $C(x,y)=0$ like in (9) of degree four. Nevertheless, its coefficients are obtained in a different way. We solve the system given by $g_k^+=0$, $k=3,4,5,6$, and $g_3^-=g_4^-=0$. It means that we match the separatrices with different orders. Finally, using the same notation that in the proof of Proposition 3, the expression of $D(M,B)=0$ is obtained imposing the condition $g_7^+(M,B)=0$, following the same strategy as in [7]. □

Remark 5. In the m,b variables the expression of $D(M,B)=0$ provided in the latter result is transformed into a new algebraic curve of degree 25 with 257 monomials.

4. Bounds up to infinity

For the sake of completeness we also include a proof of the inequality $b^*(m) > m-1$, different to the one given in [13]. It is based on the Bendixson–Dulac criterion, and it is quite simple.

Lemma 6. *For all $m>0$, it holds that $b^*(m) > m-1$.*

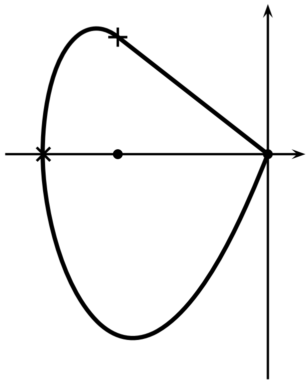

Fig. 5. Piecewise loop for obtaining a negatively invariant region.

Proof. It suffices to prove that, when $b = m - 1$, system (1) has no limit cycles. It writes as

$$\begin{cases} x' = y := P(x, y), \\ y' = -m^2 + (m-1)y + x^2 + xy := Q(x, y), \end{cases}$$

and it has the invariant straight line $\ell := \{x + y - m = 0\}$. Therefore the limit cycles, if they exist, are contained in $\mathcal{U} := \mathbb{R}^2 \setminus \ell$. We have that on $\mathcal{U}$,

$$\operatorname{div}\left(\frac{P(x,y)}{L(x,y)}, \frac{Q(x,y)}{L(x,y)}\right) = \frac{\partial}{\partial x}\left(\frac{P(x,y)}{L(x,y)}\right) + \frac{\partial}{\partial y}\left(\frac{Q(x,y)}{L(x,y)}\right) = \frac{-1}{L(x,y)} \neq 0,$$

where $L(x, y) = x + y - m$. Therefore, we can apply the well-known Bendixson–Dulac criterion [15] to each one of the half planes of $\mathcal{U}$, obtaining that the system has no limit cycles, as we wanted to prove. □

As a first step to obtain the upper bound of $b^*(m)$ for $m \geqslant 7$ given in Theorem 2 we study the curve $W(M, B) = 0$ on regions $\{(M, B)\colon\ M > M_\alpha\}$ that arrive to infinity. The procedure is similar to the one described in Section 3, but in this case only upper bounds will be provided because we only have been able to obtain negatively invariant regions. As we will see, these bounds will be enough to prove Perko's Conjecture $\mathcal{I}$.

Suitable negatively invariant regions are much more difficult to be found than in the previous section. We will construct them using piecewise algebraic curves; see Fig. 5.

Proposition 7. *For every real $\alpha > 0$, there exists $M_\alpha > \sqrt{\alpha} > 0$ such that*

$$\{W(M,B) = 0\} \subset \left\{M - 1 < B < M - 1 + \frac{\alpha}{M^2}\right\} \cap \{M > M_\alpha\}.$$

Proof. We already know that when $B = M - 1$ the system has no limit cycles, so we only need to prove that, for every α, there exist a curve $B = M - 1 + \alpha/M^2$ and a value M_α such that on this curve the system has a limit cycle when $M > M_\alpha$. So we will assume that $B = M - 1 + \alpha/M^2$.

We propose a curve $C(x, y) = 0$ formed by three pieces F_i of different algebraic curves of degree i, for $i = 1, 2, 3$. Fig. 6 shows the shape of the curve $C(x, y) = 0$ and the corresponding pieces. For each i, we use the same F_i to denote the algebraic curve $F_i(x, y) = 0$ that contains the corresponding piece.

The first one is

$$F_1 := \{(x, (B - M)x)\colon\ B^2 - M^2 \leqslant x \leqslant 0\}.$$

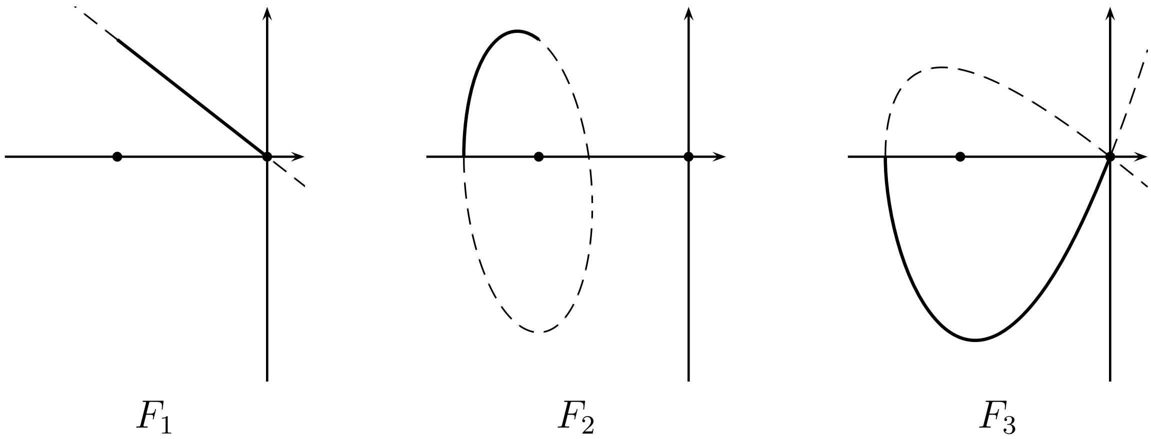

Fig. 6. The different pieces $F_i, i = 1, 2, 3$.

Notice that it is given by the segment of the straight line tangent to the stable separatrix that starts at the origin and ends at the point (x_1, y_1), where $x_1 = B^2 - M^2$ is the x-coordinate of the focus. Hence $F_1(x, y) = y - (B - M)x$.

We take F_2 as a portion of the quadratic curve $F_2(x, y) = 1 + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 = 0$ that passes trough (x_1, y_1), is tangent to F_1 at this point, passes also by the point $(x_2, 0)$, with $x_2 = -3M$, and coincides at this point, until second order derivatives, with the solution of the differential equation. More concretely, F_2 is the piece between (x_1, y_1) and $(x_2, 0)$. The choice of this value for x_2 is motivated in Remark 8.

The third piece F_3 is contained in a cubic curve similar to the one defined in (9). We consider

$$F_3(x, y) = C_2(x, y) + c_{3,0}x^3 + c_{2,1}x^2y + c_{1,2}xy^2 + c_{0,3}y^3 = 0.$$

This curve is tangent to both separatrices at the origin. The four coefficients of the homogeneous part of degree three will be fixed to get that F_3 approaches two times more the unstable separatrix at the origin, passes trough $(x_2, 0)$ and is tangent to the solution of the differential equation passing by this point. In fact, F_3 is the piece of the curve between the origin and $(x_2, 0)$ which is contained in the third quadrant.

Let us study now the behavior of the vector field on the three pieces. The gradient of F_1 at (x_1, y_1) is $(M - B, 1)$, the gradients of F_2 and F_3 at $(x_2, 0)$ are $(\frac{2(-B^2-3M+M^2)^3}{3(B+M)^2(M-B)^4M}, 0)$ and $(-3(B + M)(M - B)M, 0)$, respectively. If $M > \sqrt{\alpha}$ we can conclude that the gradient of C points to the exterior of the closed curve $C(x, y) = 0$. Thus, if we prove that the algebraic curves defined by $F_i(x, y) = 0$ and $\dot{F}_i(x, y) = 0$, the derivative of F_i with respect to the vector field, have no common points, we can easily conclude that the vector field points to the exterior of the curve $C(x, y) = 0$ for $M > \sqrt{\alpha}$.

The result for the segment of straight line F_1 is straightforward.

To study the vector field on F_2 we compute $R_2 := \operatorname{Res}(F_2, \dot{F}_2, y)$. We get that $R_2 = (x - x_2)^2 p_4(x, M, \alpha)$, where p_4 is a polynomial of degree four in x. At $M = \infty$, this polynomial has an asymptotic expansion with dominant term $-M^{-6}(2M + x)^3(4M + x)/46\,656$. Hence, the dominant terms of the asymptotic expansions of the corresponding roots are $-4M$ and $-2M$. This latter value corresponds to a triple root. A more detailed computation, considering the next significant term in each coefficient of p_4, shows that the triple root splits into a couple of complex conjugated roots and a real one, $\widehat{x}_1$, with asymptotic expansion

$$\widehat{x}_1(M, \alpha) = -2M + \frac{2\sqrt[3]{\alpha}}{\sqrt[3]{9}} M^{2/3} + o\big(M^{2/3}\big).$$

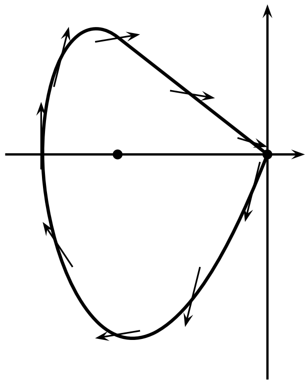

Fig. 7. The negatively invariant region corresponding to the piecewise loop.

Since the other real root of p_4 has asymptotic expansion $\widehat{x}_2(M,\alpha) = -4M + o(M)$,

$$x_1 = B^2 - M^2 = \left(M - 1 + \frac{\alpha}{M^2}\right)^2 - M^2 = -2M + 1 + O\left(\frac{1}{M}\right),$$

and $x_2 = -3M$, it holds that for $M > \widetilde{M}_2(\alpha)$ big enough,

$$\widehat{x}_2(M,\alpha) < x_2 < x_1 < \widehat{x}_1(M,\alpha).$$

Therefore, for $M > \widetilde{M}_2(\alpha)$, R_2 does not change sign on (x_2, x_1), as we wanted to prove. Moreover, it is easy to see that the vector field points to the exterior of the loop on F_2.

The resultant R_3 of F_3 and $\dot{F}_3$ with respect to x is of the form $y^9 q_3(y, M, \alpha)$, where q_3 is a polynomial of degree three in y. The asymptotic expansion of the ordered coefficients of q_3 at $M = \infty$ shows the sign configuration $[+,-,+,-]$ for $M > \widetilde{M}_3(\alpha)$ large enough. Thus, it is clear that q_3 has no negative zeros. Then, for $M > \widetilde{M}_3(\alpha)$, R_3 does not vanish in the half-plane $y < 0$ where F_3 is defined. Similarly to the previous cases, we can check that the vector field points to the exterior of $C(x, y) = 0$ along F_3.

The proof finishes taking M_α as the maximum of $\widetilde{M}_2(\alpha)$, $\widetilde{M}_3(\alpha)$ and $\sqrt{\alpha}$. Then we have the situation given in Fig. 7. □

The next remark clarifies some details about the previous proof. Lemma 10 will follow directly applying the above result for a concrete value of α. The corresponding M_α is also given.

Remark 8. The choice of the point $(x_2, 0) = (-3M, 0)$ in the proof of Proposition 7 is motivated by some numerical computations. We wanted to choose a point on the x-axis that was between the first crossing point of the separatrices of the saddle point with the negative x-axis. Let us call $(P_s, 0)$ the first crossing point for the stable separatrix, and $(P_u, 0)$ for the unstable one. For several values of α, using a Runge–Kutta–Fehlberg 4-5 method, together with degree four interpolation, we obtain numerical approximations of the points P_s and P_u for every M. Fig. 8(b) shows the plots of $-P_s/M$ and $-P_u/M$ together with the corresponding value of the abscissa of the focus point. We remark that for the values of α that we have checked the limit behavior is always the same. Hence, the asymptotic expansions at $M = \infty$ of $(P_s, 0)$ and $(P_u, 0)$ seem to be $(-2M, 0)$ and $(-4M, 0)$, respectively; see Fig. 8. So, our choice of $x_2 = -3M$ is quite natural.

5. Proof of Theorems 1 and 2

To prove Theorem 1 we need to translate the results of Proposition 7 to the m, b parameters.

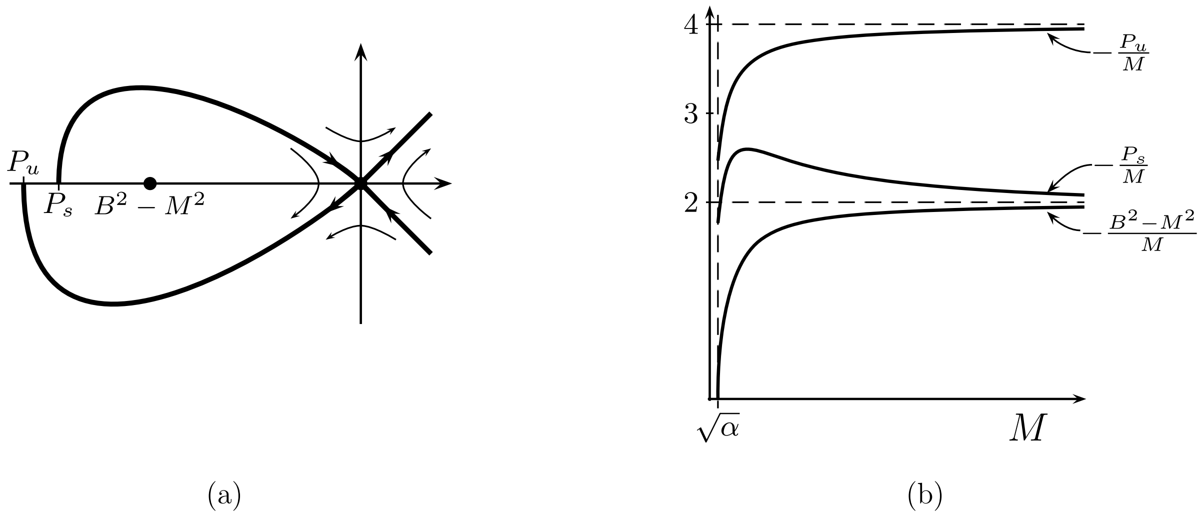

Fig. 8. Separatrices of the saddle point.

Lemma 9. *The change of variables (4) converts the curve $B = M - 1 + \alpha/M^2$ into the branch of the curve*

$$\begin{aligned} &-4m^5 - 12m^4b - 8m^3b^2 + 8m^2b^3 + 12mb^4 + 4b^5 - 60m^4 - 48m^3b + 88m^2b^2 \\ &+ 80mb^3 + 4b^4 + (-192 - 16\alpha)m^3 + (384 - 48\alpha)m^2b + (64 - 48\alpha)mb^2 \\ &- 16\alpha b^3 + (256 - 160\alpha)m^2 - 192\alpha mb - 32\alpha b^2 - 256\alpha m + 64\alpha^2 = 0, \end{aligned} \tag{15}$$

that satisfies

$$b = m - 1 + \frac{2\alpha}{m} - \frac{\alpha}{m^2} + O\left(\frac{1}{m^3}\right), \tag{16}$$

when m goes to ∞. Moreover, it remains below the curve $b = m - 1 + 2\alpha/m$ for every positive m and α.

Proof. Straightforward computations give (15). To prove the last assertion it suffices to see that both curves do not have common points. Clearly, from (16), the result holds for m big enough.

The common points of both curves are characterized by the roots of the polynomial obtained substituting $b = m - 1 + 2\alpha/m$ in (15). Removing the denominator we obtain

$$\begin{aligned} q(m) := {} & 8m^7 + (16\alpha + 12)m^6 + (24\alpha + 6)m^5 + \left(48\alpha^2 - 12\alpha + 1\right)m^4 \\ & - 8\alpha(3\alpha + 1)m^3 + 24\alpha^2(2\alpha + 1)m^2 - 32m\alpha^3 + 16\alpha^4. \end{aligned}$$

The computation of its Sturm sequence evaluated at 0 and at $+\infty$ gives the configurations of signs $[+, -, -, -, +, +, -, -]$ and $[+, +, +, -, -, +, -, -]$, respectively, for each positive α. Hence, $q(m)$ does not have positive roots, and therefore both curves have no common points when $m > 0$ and $\alpha > 0$. $\square$

Proof of Theorem 1. From Proposition 7 and Lemma 9, for each $\alpha > 0$, the curve $b = m - 1 + 2\alpha/m$ is an upper bound of $b = b^*(m)$, for $m > m_\alpha$, where m_α can be obtained applying the transformation (4) to the region $M > M_\alpha$. Therefore, for $m > m_\alpha$,

$$m - 1 < b^*(m) < m - 1 + \frac{2\alpha}{m},$$

which implies

$$\lim_{m\to\infty} \frac{b^*(m) - m + 1}{1/m} = 0,$$

because α is an arbitrary positive number. □

Before proving Theorem 2 we need a preliminary result.

Lemma 10. *It holds that*

$$b^*(m) < m - 1 + \frac{51}{20m},$$

for $m > \widetilde{m}$, *where* $\widetilde{m} \approx 6.93$ *is the unique positive root of the polynomial*

$$\begin{aligned}
&50\,331\,648\,000\,000\,000m^{17} - 243\,269\,632\,000\,000\,000m^{16}\\
&\quad - 2\,129\,238\,425\,600\,000\,000m^{15} - 7\,211\,878\,973\,440\,000\,000m^{14}\\
&\quad + 111\,668\,173\,209\,600\,000\,000m^{13} + 264\,470\,739\,812\,352\,000\,000m^{12}\\
&\quad - 130\,466\,347\,912\,396\,800\,000m^{11} - 9\,197\,101\,546\,824\,499\,200\,000m^{10}\\
&\quad - 6\,302\,900\,112\,535\,388\,160\,000m^{9} + 6\,325\,778\,059\,290\,335\,232\,000m^{8}\\
&\quad + 2\,289\,016\,716\,587\,559\,936\,000m^{7} - 46\,572\,462\,911\,915\,224\,012\,800m^{6}\\
&\quad + 8\,515\,659\,923\,453\,703\,340\,800m^{5} - 4\,901\,243\,812\,728\,523\,876\,800m^{4}\\
&\quad - 45\,716\,337\,137\,659\,722\,706\,080m^{3} + 6\,052\,551\,315\,638\,078\,774\,880m^{2}\\
&\quad - 8\,203\,038\,242\,422\,388\,605\,200m - 79\,536\,086\,493\,533\,822\,540\,007.
\end{aligned}$$

Proof. Proposition 7, with $\alpha = 51/40$, ensures that

$$B = M - 1 + \frac{51}{40M^2}$$

provides an upper bound for $W(M, B) = 0$ when $M > M_{51/40}$. Particularizing the analysis done in the proof of this proposition to this particular α we can obtain an explicit value $M_{51/40}$. It can be taken as the largest positive root of the equation

$$\begin{aligned}
&196\,608\,000\,000M^{17} - 1\,441\,792\,000\,000M^{16} - 535\,756\,800\,000M^{15}\\
&\quad + 1\,480\,294\,400\,000M^{14} - 1\,310\,515\,200\,000M^{13} - 1\,151\,979\,520\,000M^{12}\\
&\quad + 1\,314\,478\,080\,000M^{11} - 727\,741\,440\,000M^{10} - 273\,666\,816\,000M^{9}\\
&\quad + 44\,3460\,096\,000M^{8} - 458\,441\,856\,000M^{7} + 61\,550\,064\,000M^{6} + 227\,310\,753\,600M^{5}\\
&\quad - 162\,364\,824\,000M^{4} - 41\,403\,030\,120M^{3} + 82\,806\,060\,240M^{2} - 17\,596\,287\,801 = 0, \qquad (17)
\end{aligned}$$

which is approximately 7.58.

Then the result follows from Lemma 9. Moreover, the polynomial that defines $\widetilde{m}$ is obtained computing the resultant with respect to b of (15) and the polynomial corresponding to (17) translated to the variables m and b. □

It is useful to introduce the following notation for the lower and upper bounds of $b^*(n)$ given in Theorem 2

$$b_\ell(m) := \max\left(\frac{5m}{7}, m-1\right) \quad \text{and} \quad b_u(m) := \min\left(\frac{(5+\frac{37}{12}m)m}{7+\frac{37}{12}m}, m-1+\frac{25}{7m}\right). \tag{18}$$

Notice that the non-differentiability points of b_ℓ and b_u are at $m = 7/2$ and $m = 7$, respectively.

Proof of Theorem 2. Proposition 3 provides the lower bound given in the statement, when $m \leqslant 7/2$. For $m > 7/2$, Perko gives a proof in [13]. A different one is presented in Lemma 6.

The proof of the second part is done by comparison of the curves in the statement with the ones provided by Proposition 4 and Lemma 10.

For $m \geqslant 7$, $b = b_u(m)$ is an upper bound from Lemma 10 because $\widetilde{m} < 7$ and $25/7 > 51/20$.

When $m \leqslant 7$ the proof starts translating the curve $D(M, B) = 0$ given in Proposition 4 to a new algebraic curve $E(m, b) = 0$, of degree 25 with 257 monomials. Now we compare the curves $(84 + 37m)b - (60 + 37m)m = 0$, corresponding to $b_u(m) = 0$, and $E(m, b) = 0$ when $m, b > 0$. The resultant with respect to b of both polynomials takes the form $m^{15} p_{34}(m)$, where p_{34} is a polynomial of degree 34 with a unique positive zero, m_1, as can be easily seen from the Sturm method. Hence, the curves only intersect at $(0, 0)$ and (m_1, b_1), where $m_1 \approx 7.1$ A local study of the curves close to the origin shows that $b = b_u(m)$ is above $E(m, b) = 0$. Hence, as the relative position of the graphs of both curves does not change when $0 < m \leqslant 7$, $b = b_u(m)$ is also an upper bound of $b = b^*(m)$ in the full interval. □

Corollary 11. *Set $\widetilde{b}(m) = (b_u(m) + b_\ell(m))/2$, where b_ℓ and b_u are given in (18). Then, the absolute and relative errors when we approximate $b^*(m)$ by $\widetilde{b}(m)$, are*

$$\max_{m>0} \left| b^*(m) - \widetilde{b}(m) \right| < \frac{37}{122} < 0.31 \quad \textit{and} \quad \max_{m>0} \left| \frac{b^*(m) - \widetilde{b}(m)}{b^*(m)} \right| < \frac{37}{305} < 0.13.$$

Proof. It is not difficult to see that the maxima of the functions $b_u - b_\ell$ and $(b_u - b_\ell)/b_\ell$ are both at $m = 7/2$. Then

$$\max_{m>0} \left| b^*(m) - \widetilde{b}(m) \right| \leqslant \max_{m>0} \left| \frac{b_u(m) - b_\ell(m)}{2} \right| = \left| \frac{b_u(7/2) - b_\ell(7/2)}{2} \right| = \frac{37}{122}$$

and

$$\max_{m>0} \left| \frac{b^*(m) - \widetilde{b}(m)}{b^*(m)} \right| \leqslant \max_{m>0} \left| \frac{b_u(m) - b_\ell(m)}{2b_\ell(m)} \right| = \left| \frac{b_u(7/2) - b_\ell(7/2)}{2b_\ell(7/2)} \right| = \frac{37}{305}.$$

Hence, the corollary follows. □

Remark 12. From Theorem 2, the family of rational functions

$$b = \frac{5 + \beta m}{7 + \beta m} m,$$

for every $\beta > 0$, approximates $b^*(m)$ in neighborhoods of the origin and the infinity simultaneously. Nevertheless, the upper bound given in Theorem 2, that corresponds to $\beta = 37/12$, is changed in a neighborhood of infinity because the family of functions $b = m - 1 + \gamma/m$ is a much better approximation for m big enough. The concrete values of β and γ are fixed imposing the continuity of b_u and searching nice expressions for the statement of Theorem 2. These values could be changed to obtain slightly better upper bounds.

6. Original formulation of Perko's conjectures

In [13,14], Perko considers the following expression of the Bogdanov–Takens system

$$\begin{cases} u' = v, \\ v' = u(u-1) + \mu_1 v + \mu_2 uv, \end{cases} \tag{19}$$

and proves that it has a homoclinic saddle-loop if and only if $\mu_1 = h(\mu_2)$ for some odd analytic function h. Since $h(-\mu_2) = -h(\mu_2)$ it suffices to study h either for $\mu_2 > 0$ or for $\mu_2 < 0$.

Original formulation of Perko's Conjectures. *(See [13].) Let $\mu_1 = h(\mu_2)$ be the function that gives the saddle-loop bifurcation curve for system (19). Then,*

($\mathcal{I}$) *the curve $\mu_1 = h(\mu_2)$ is asymptotic to the hyperbola $\mu_1 = -1/\mu_2$ for large $|\mu_2|$, i.e., $\mu_2 h(\mu_2) + 1 = O(1/\mu_2)$, as $\mu_2 \to \infty$,*
($\mathcal{II}$) *for each $\mu_2 < 0$, $0 < h(\mu_2) < \min\{-\mu_2/7, -1/\mu_2\}$.*

To see that the above conjectures are equivalent to the ones stated in the introduction we will transform system (1) into (19). If we apply the change of variables $u = (x+m)/(2m)$, $v = y/(2m)^{3/2}$ and consider the new time $s = \sqrt{2m}t$, system (1) writes as

$$\begin{cases} \dot{u} = v, \\ \dot{v} = u(u-1) + \dfrac{b-m}{\sqrt{2m}} v + \sqrt{2m} uv. \end{cases}$$

Hence, we have the following equivalence among the parameters m, b and μ_1, μ_2:

$$\mu_1 = \frac{b-m}{\sqrt{2m}}, \qquad \mu_2 = \sqrt{2m},$$

and every curve of the form $\mu_1 = f(\mu_2)$ is transformed in the variables m and b into $b = m + \sqrt{2m} f(\sqrt{2m})$. In particular, the curves $\mu_1 = 0$, $\mu_1\mu_2 = -1$ and $\mu_1 = -\mu_2/7$ are transformed into the straight lines $b = m, b = m - 1$ and $b = 5m/7$, respectively. This fact shows that both Conjectures $\mathcal{II}$ are equivalent.

To compare both Conjectures $\mathcal{I}$, notice that $\mu_1\mu_2 = b - m$. Therefore, we can write $\mu_2\mu_1 + 1 = O(1/\mu_2)$ as $b - m + 1 = O(1/\sqrt{m})$, as we wanted to prove. In fact, notice that

$$b^*(m) = m + \sqrt{2m} h(\sqrt{2m}).$$

Indeed, since h is analytic and odd this equality proves that $b^*(m)$ is analytic in m.

It is worth to mention that there is a third conjecture in Perko's work: *For $\mu_2 < 0$ the function $h(\mu_2)$ has a unique maximum*. The tools introduced in this paper seem not to be adequate to approach this question.

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