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Integrability of Liénard systems with a weak saddle

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Abstract. We characterize the local analytic integrability of weak saddles for complex Liénard systems, $\dot{x} = y - F(x)$, $\dot{y} = ax$, $0 \neq a \in \mathbb{C}$, with F analytic at 0 and $F(0) = F'(0) = 0$. We prove that they are locally integrable at the origin if and only if $F(x)$ is an even function. This result implies the well-known characterization of the centers for real Liénard systems. Our proof is based on finding the obstructions for the existence of a formal integral at the complex saddle, by computing the so-called resonant saddle quantities.

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1. Introduction and main results

Since the pioneering works of Élie and Henri Cartan [1] and Balthasar van der Pol [25,26] where Liénard-type differential equations [15],

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + f(x)\frac{\mathrm{d}x}{\mathrm{d}t} + g(x) = \ddot{x} + f(x)\dot{x} + g(x) = 0,$$

appear in electrical problems, many other situations have been modeled by differential equations that can be transformed into them. For instance, Liénard equations appear in mechanical problems, in predator–prey systems [9] and chemical or biochemical reactions [17,19]. Recall that Hilbert 16th problem for polynomial differential equations of degree d asks for a bound for their number of limit cycles that depends only on d. It is remarkable that this question restricted to planar polynomial quadratic differential equations ($d = 2$) can be reduced to the study of some Liénard-type differential equations, see [27].

Two of the main questions about them are to know whether they are integrable or not and to give criteria for controlling their number of limit cycles. This paper deals with the first one.

Poincaré proved that if a real planar analytic differential system has a weak focus at a critical point, then this point is a center if and only if the equation has an analytic first integral defined in a neighborhood of this point, see, for instance, [14,18,20,23]. Then, the characterization of the centers for non-degenerate critical points is equivalent to the characterization of the local analytic integrable cases.

Let us recall the simple and well-understood characterization of the centers, and so the local analytic integrable cases, for classical real analytic Liénard systems, $g(x) \equiv x$. If we write these equations in $\mathbb{R}^2$ as

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = y + F(x), \qquad \dot{y} = -x,$$

with $F(x) = -\int_0^x f(s)\,ds$ analytic at zero and $F(0) = F'(0) = 0$, then the origin is a center if and only if F is an even function, see, for instance, [4,7,29]. We include for completeness a very simple proof, which essentially is the one appearing in [4,29]. Write $F = F^o + F^e$, where $F^e(x) = (F(x) + F(-x))/2$ is the even part of F and F^o its odd part. If $F = F^e$, then the origin is a center by the classical criterion of

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