CENTER PROBLEM
FOR TRIGONOMETRIC LIÉNARD SYSTEMS

ARMENGOL GASULL, JAUME GINÉ, AND CLAUDIA VALLS

Abstract. We give a complete algebraic characterization of the non-degenerated centers for planar trigonometric Liénard systems. The main tools used in our proof are the classical results of Cherkas on planar analytic Liénard systems and the characterization of some subfields of the quotient field of the ring of trigonometric polynomials. Our results are also applied to some particular subfamilies of planar trigonometric Liénard systems. The results obtained are reminiscent of the ones for planar polynomial Liénard systems but the proofs are different.

1. Introduction and main results

The aim of this paper is to characterize the non-degenerated centers for the planar systems associated to the second order trigonometric Liénard differential equations $\ddot{\theta} = g(\theta) + f(\theta)\dot{\theta}$, where f, g are trigonometric polynomials with real coefficients and the dot denotes the derivative with respect to the time.

The analysis of equations of this form is motivated by a number of problems resulting from pendulum-like equations appearing in the literature. Equations of this form, like $\ddot{\theta} + \sin(\theta) = \varepsilon\dot{\theta}\cos(n\theta)$, or the Josephson equation $\ddot{\theta} + \sin(\theta) = \varepsilon[a - (1 + \gamma\cos(\theta))\,\dot{\theta}]$, are considered in [9, 16, 17] or [2, 14, 18, 19], respectively. Also the perturbed whirling pendulum, $\ddot{\theta} = \sin\theta(\cos\theta - \gamma) + \varepsilon(\cos\theta + \alpha)\,\dot{\theta}$, see [15] falls in this class. Here a, γ, α and ε are real constants and $n \in \mathbb{N}$.

As usual we will write the above second order trigonometric differential equation as the autonomous planar system

$$\begin{cases} \dot{\theta} = y, \\ \dot{y} = g(\theta) + yf(\theta), \end{cases} \tag{1}$$

and we will assume that $f(0) = 0$, $g(0) = 0$, $g'(0) < 0$, where the prime denotes the derivative with respect to θ. These hypotheses on g imply that the origin is either a center or a weak focus. Our main result is:

Theorem 1. *System* (1) *has a center at the origin if and only if*

(i) Either $f = \alpha g$ for some $\alpha \in \mathbb{R}$, or

(ii) There exist a real trigonometric polynomial p and two real polynomials f_1 and g_1 satisfying $p'(0) = 0$, $g_1(p(0))p''(0) < 0$, and such that

$$f(\theta) = f_1(p(\theta))p'(\theta), \quad g(\theta) = g_1(p(\theta))p'(\theta). \tag{2}$$

2010 *Mathematics Subject Classification.* Primary 34C25. Secondary 37C10; 37C27.

Key words and phrases. Center problem, trigonometric Liénard equation, trigonometric polynomial.

The proof of this result is based on the following tools. First, the characterization of the non-degenerated centers for analytic differential equations given by Cherkas in [3], developed in [5] and later improved for polynomial Liénard systems in [4]. Secondly, a version of Luröth Theorem (see [20] or Theorem 5 below) for trigonometric polynomials given in [7, 13] and stated as Theorem 7. We will recall these results in Section 2.

From condition (2), by introducing the functions

$$F(\theta)=\int_0^\theta f(s)\,ds, \quad G(\theta)=\int_0^\theta g(s)\,ds, \tag{3}$$

if follows that, for trigonometric Liénard systems having a center of type (ii),

$$F(\theta)=F_1(p(\theta)), \quad G(\theta)=G_1(p(\theta)), \tag{4}$$

for some polynomial functions F_1 and G_1, with $p'(0)=0$, $G_1'(p(0))p''(0)<0$. This equivalent expression is commonly used to characterize the centers of polynomial Liénard systems, see [4].

We want to remark that the centers of item (i) in the theorem have explicit global first integrals,

$$H(\theta,y)=\begin{cases} y^2-2G(\theta), & \text{when} \quad \alpha=0,\\ (1+\alpha y)\exp\left(\alpha^2 G(\theta)-\alpha y\right), & \text{when} \quad \alpha\neq 0. \end{cases} \tag{5}$$

On the other hand, the centers of item (ii) correspond to orbitally reversible centers with respect to a given curve, see [4, 6, 21], or the proof of the sufficiency part of the theorem.

Nevertheless the case of item (i) can also be written similarly that the one of item (ii) simply taking $f_1=\alpha$, $g_1=1$, and $p=G$, but notice that in this case G does not need to be necessarily a trigonometric polynomial.

There are several differences between polynomial and trigonometrical polynomials that make that our proof of Theorem 1 is not a simple consequence of the parallel result for the polynomial case (see [4] or Theorem 4 below). Two of the more are:

- The primitive of a polynomial is again a polynomial, while the primitive of a trigonometric polynomial is a trigonometrical polynomial plus $k\theta$ for some $k\in\mathbb{R}$.
- The ring of polynomials is a Unique Factorization Domain while the ring of trigonometric polynomials is not. This can be seen for instance looking at the celebrated identity $\sin^2\theta=1-\cos^2\theta=(1+\cos\theta)(1-\cos\theta)$. It holds that $\sin\theta$ divides the right hand expression but it does not divide either $1+\cos\theta$ or $1-\cos\theta$.

The first difference produces the family (i) in Theorem 1. Notice again that for this family of centers, neither F nor G need to be trigonometric polynomials for a system (1) with a center at the origin. The second difference is overcome by using a isomorphism between the field of quotients of trigonometrical polynomials $\mathbb{R}_t(\theta)$ and the field of rational functions $\mathbb{R}(x)$, see Section 2 for details. As we will see, this transformation allows to work with rational functions instead of dealing with trigonometric polynomials and then use the usual divisibility tools. This approach turns out to be very useful for questions dealing with trigonometric polynomials, see for instance [7, 8, 13, 11] for other situations where it is used.

The explicit characterization of the centers provided in Theorem 1 can be applied to list all the centers for concrete subfamilies of system (1). As usual, the degree of a real trigonometric polynomial is given by the highest harmonic in its Fourier expansion. For instance, we prove:

Corollary 2. *Consider the Liénard differential systems* (1) *with either f or g trigonometric polynomials of prime degree. Then the origin is a center if and only one of the following conditions hold:*
(a) $f(\theta) = \alpha g(\theta)$ for some $\alpha \in \mathbb{R}$,
(b) $f(\theta) = f_1(G(\theta))g(\theta)$, with $G \in \mathbb{R}_t[\theta]$, $f_1 \in \mathbb{R}[x]$,
(c) $g(\theta) = g_1(F(\theta))f(\theta)$, with $F \in \mathbb{R}_t[\theta]$, $g_1 \in \mathbb{R}[x]$ and $g_1(0)f'(0) < 0$,
(d) $f(\theta) = f_1(\cos\theta)\sin\theta$ and $g(\theta) = g_1(\cos\theta)\sin\theta$ with $f_1, g_1 \in \mathbb{R}[x]$ and $g_1(1) < 0$.

The proof of this corollary and other applications of Theorem 1 are given in Section 4.

The case of degenerated centers for system (1), that is $g(0) = g'(0) = \cdots = g^{(2k)}(0) = 0$, $g^{(2k+1)}(0) < 0$ for some $k > 0$, together with some monodromy conditions, could be treated with similar tools following the ideas of [10].

2. Preliminary results

Next result of Cherkas ([3]) characterizes theoretically the non-degenerated centers for analytic Liénard systems.

Theorem 3. *The Liénard differential system*

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= g(x) + yf(x), \end{aligned} \tag{6}$$

with f, g real analytic in a neighborhood of zero with $g(0) = 0$, $g'(0) < 0$ has a center at the origin if and only if there exists a real analytic function z defined in a neighborhood of zero with $z(0) = 0$, $z'(0) = -1$ such that

$$F(x) = F(z(x)) \quad G(x) = G(z(x)), \tag{7}$$

where F and G are given in (3).

Moreover, Theorem 3 implies that all centers orbitally reversible, see again [4, 6, 12, 21]. Nevertheless it is difficult to apply it for characterizing the centers for polynomial Liénard systems of a given degree. In this situation, the following result of Christopher ([4]) gives an algebraic and effective solution to the problem.

Theorem 4. *The Liénard differential system* (6) *with f, g real polynomials with $g(0) = 0$, $g'(0) < 0$ has a center at the origin if and only if there exist real polynomials q, f_1 and g_1 satisfying $q'(0) = 0$, $g_1(q(0))q''(0) < 0$, and such that*

$$f(x) = f_1(q(x))q'(x), \quad g(x) = g_1(q(x))q'(x).$$

Theorem 1 can be seen as the trigonometric version of the above result.

To state Lüroth Theorem and its variants, first we introduce some notation. Let $\mathbb{R}(x)$ denote the quotient field of the ring of polynomials $\mathbb{R}[x]$ with coefficients in $\mathbb{R}$ and let $\mathbb{R}_t(\theta)$ denote the quotient field of the ring of trigonometric polynomials $\mathbb{R}_t[\theta]$, also with coefficients in $\mathbb{R}$. It is well-known that $\mathbb{R}_t(\theta)$ is isomorphic to $\mathbb{R}(x)$ by means of the map $\Phi\colon \mathbb{R}_t(\theta) \to \mathbb{R}(x)$ defined by

$$\Phi(\sin\theta) = \frac{2x}{1+x^2} \quad \text{and} \quad \Phi(\cos\theta) = \frac{1-x^2}{1+x^2}. \tag{8}$$

Finally, we will denote by $\mathbb{R}(U_1, U_2, \ldots, U_n)$ the subfield of $\mathbb{R}(x)$ (resp. $\mathbb{R}_t(\theta)$) generated by $U_j \in \mathbb{R}(x)$ (resp. $U_j \in \mathbb{R}_t(\theta)$) for $j = 1, 2, \ldots, n$.

Given a real trigonometric polynomial p we call $\deg(p) = \ell$ the degree of the Fourier series corresponding to p, that is

$$f(\theta) = \sum_{k=-\ell}^{\ell} a_k e^{ki\theta}, \quad a_{-k} = \overline{a}_k, \quad \text{with} \quad a_\ell \neq 0.$$

It holds that

$$\Phi(p(\theta)) = \frac{M(x)}{(1+x^2)^\ell}, \quad \text{with}\, M \in \mathbb{R}[x],\, \deg(M) \leq 2\ell \quad \text{and} \quad \gcd(M(x), 1+x^2) = 1. \tag{9}$$

Moreover, the converse is also true: for each M under the above hypotheses, there exists a trigonometric polynomial, p, of degree ℓ, such that $\Phi(p(\theta)) = \frac{M(x)}{(1+x^2)^\ell}$, see [7, 8].

From (8) it can be seen that

$$\Phi(p'(\theta)) = \frac{\widetilde{M}(x)}{(1+x^2)^\ell} := \frac{\frac{1}{2}\frac{dM(x)}{dx}(1+x^2) - \ell x M(x)}{(1+x^2)^\ell} \tag{10}$$

and the degree of $\widetilde{M}$ is at most 2ℓ, see [11].

To contextualize the result that we need, let us recall first the Lüroth theorem ([20]), then the version used in [4] to prove Theorem 4 and finally the version given in [7, 13], see Theorem 7.

Theorem 5. (Lüroth Theorem) *Let $\mathbb{K}$ be a non-trivial subfield of $\mathbb{R}(x)$. Then $\mathbb{K} = \mathbb{R}(u)$ for some $u \in \mathbb{K}$.*

Theorem 6. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}(x)$ containing a non-constant polynomial. Then $\mathbb{K} = \mathbb{R}(p)$ for some polynomial p.*

Theorem 7. *Let $\mathbb{K}$ be a subfield of $\mathbb{R}_t(\theta)$ containing a non-constant trigonometric polynomial. Then either $\mathbb{K} = \mathbb{R}(\tan(\frac{n\theta}{2}))$ for some $n \in \mathbb{N}$ or $\mathbb{K} = \mathbb{R}(r)$ for some trigonometric polynomial r.*

The possibilities for the generator of $\mathbb{K}$ appearing in Theorem 7 not having an equivalent version in Theorem 6 are essentially due to the well-known identity

$$\frac{1}{1+\tan^2(\theta)} = \cos^2(\theta).$$

Next consequence of Theorem 7 is the one that we will use in this paper.

Proposition 8. *Let $\mathbb{K} = \mathbb{R}(U, V)$ be the subfield of $\mathbb{R}_t(\theta)$ generated by the two elements $U, V \in \mathbb{R}_t(\theta)$. Assume that:*

(a) U is a non-trivial trigonometric polynomial, that is $U \in \mathbb{R}_t[\theta]$ and it is non-constant.

(b) There exists a real analytic function z, defined in a neighborhood of zero, with $z(0) = 0$, $z'(0) = -1$ and such that

$$U(\theta) = U(z(\theta)), \quad V(\theta) = V(z(\theta)).$$

Then $\mathbb{K} = \mathbb{R}(r)$ for some trigonometric polynomial r satisfying $r'(0) = 0$.

Proof. By Theorem 7, if W is a generator of $\mathbb{K}$ then W is either $\tan(n\theta/2)$, for some $n \in \mathbb{N}$, or a trigonometric polynomial. Notice that the property $U(\theta) = U(z(\theta))$, satisfied also by V, is inherited for all the elements of $\mathbb{K}$. Hence, in particular, it holds that $W(\theta) = W(z(\theta))$.

In any case, taking derivatives at zero,

$$W'(\theta) = W'(z(\theta))z'(\theta) \Rightarrow W'(0) = -W'(0) \Rightarrow W'(0) = 0.$$

Since $\frac{d}{d\theta} \tan(n\theta/2)\big|_{\theta=0} = n/2 \neq 0$, this implies that $W = r$ is a trigonometric polynomial and $r'(0) = 0$, as we wanted to prove. $\square$

3. Proof of Theorem 1

First we prove sufficiency. Under conditions of item (i) the origin is a center because it is a monodromic critical points and the first integrals given in (5) are well defined at this point.

Assume that conditions (2) of item (ii) hold, that is $f(\theta) = f_1(p(\theta))p'(\theta)$ and $g(\theta) = g_1(p(\theta))p'(\theta)$ for some real polynomials f_1 and g_1 and some trigonometric polynomial p, satisfying $p'(0) = 0$, $p''(0)g_1(p(0)) < 0$. Then, by using the non-invertible map

$$(\theta, y) \to \big(\bar{\theta}, \bar{y}\big) = \big(p(\theta) - p(0), y\big) = \Big(\frac{p''(0)}{2}\theta^2 + 0(\theta^3), y\Big),$$

we can rewrite system (1) in the form

$$\dot{\bar{\theta}} = p'(\theta)\bar{y}, \quad \dot{\bar{y}} = p'(\theta)\big(g_1\big(\bar{\theta} + p(0)\big) + \bar{y} f_1\big(\bar{\theta} + p(0)\big)\big) = p'(\theta)\big(\widehat{g}_1(\bar{\theta}) + \bar{y}\widehat{f}_1(\bar{\theta})\big),$$

for some $\widehat{f}_1, \widehat{g}_1 \in \mathbb{R}[x]$. After the new parametrization of the time $\frac{ds}{dt} = p'(\theta)$, the above system is transformed into

$$\frac{d\bar{\theta}}{ds} = \bar{y}, \quad \frac{d\bar{y}}{ds} = \widehat{g}_1(\bar{\theta}) + \bar{y}\widehat{f}_1(\bar{\theta}).$$

This new system is nonsingular at the origin because $\widehat{g}_1(0) = g_1(p(0)) \neq 0$. So, it has a local analytic first integral, and this first integral can be pulled back to a first integral of system (1) around the singularity, producing a center. See [4, Thm. 9b] for more details or [1, Lem. 2] for another application of this idea in a different context for proving the existence of a center.

Now we prove necessity. It follows from Theorem 3 that system (1) has a center at the origin if and only if there is a real analytic function z in a neighborhood of the origin such that $z(0) = 0$ and $z'(0) = -1$ which satisfies (7),

$$F(\theta) = F(z(\theta)), \quad G(\theta) = G(z(\theta)). \tag{11}$$

We consider two different subcases.

Case 1*: Either F or G are trigonometric polynomials.* Assume for instance that G is a trigonometric polynomial and $f \neq 0$. If $f = 0$ the proof is trivial. The case when F is a trigonometric polynomial follows using similar ideas.

We know that $G(\theta) = G(z(\theta))$ with G being a non-trivial trigonometric polynomial because $g'(0) < 0$. Moreover, $f \neq 0$. From (11) we get

$$f(\theta) = f(z(\theta))\, z'(\theta) \quad \text{and} \quad g(\theta) = g(z(\theta))\, z'(\theta), \tag{12}$$

that is, $g(\theta)/f(\theta) = g(z(\theta))/f(z(\theta))$.

Now, consider the subfield of $\mathbb{R}(x)$, $\mathbb{K} = \mathbb{R}\big(G, f/g\big)$. By applying Proposition 8 with $U = G$, $V = f/g$ it holds that $\mathbb{K} = \mathbb{R}(p)$ for some trigonometric polynomial p with $p'(0) = 0$.

As a consequence,

$$G(\theta) = \frac{G_1}{G_2}(p(\theta)) \quad \text{and} \quad \frac{f}{g}(\theta) = \frac{f_3}{f_4}(p(\theta)) \tag{13}$$

with $G_1/G_2 \in \mathbb{R}(x)$, $f_3/f_4 \in \mathbb{R}(x)$ and $\gcd(G_1, G_2) = \gcd(f_3, f_4) = 1$.

Following the ideas in [7] we will prove first that we can choose $G_2 = 1$. From (8) and (9) we have that

$$\frac{G_1}{G_2}\Big(\frac{M}{(1+x^2)^\ell}\Big) = \frac{N}{(1+x^2)^j}, \tag{14}$$

with $M, N \in \mathbb{R}[x]$, $\gcd(M, 1+x^2) = \gcd(N, 1+x^2) = 1$ and $\deg(M) \leq 2\ell$ and $\deg(N) \leq 2j$. Adding, if necessary, a constant to $p(\theta)$ we can assume that $\deg(M) < 2\ell$.

Now suppose, in order to get a contradiction, that $\deg(G_2) \geq 1$. Thus we obtain

$$\frac{(1+x^2)^{d_2\ell}\widehat{G}_1(M, (1+x^2)^\ell)}{(1+x^2)^{d_1\ell}\widehat{G}_2(M, (1+x^2)^\ell)} = \frac{N}{(1+x^2)^j},$$

where $\widehat{G}_1$ and $\widehat{G}_2$ denote the homogenization of G_1 and G_2 and d_1, d_2 are their respective degrees.

First we show that $\gcd(\widehat{G}_2(M, (1+x^2)^\ell), (1+x^2)^{d_2\ell}\widehat{G}_1(M, (1+x^2)^\ell)) = 1$. To see this we will prove that $\widehat{G}_2(M, (1+x^2)^\ell)$ does not share roots (real or complex) with $(1+x^2)^{d_2\ell}$ or with $\widehat{G}_1(M, (1+x^2)^\ell))$. Indeed, let $z \in \mathbb{C}$ be a root of $\widehat{G}_2(M, (1+x^2)^\ell))$ and suppose first that z is also a root of $1+x^2$. If we write $G_2 = \sum_{j=0}^{d_2} a_j x^j$ with $a_{d_2} \neq 0$ we have

$$\widehat{G}_2(M(x), (1+x^2)^\ell) = \sum_{j=0}^{d_2} a_j M^j(x)(1+x^2)^{(d_2-j)\ell}$$

and, as a consequence, $\widehat{G}_2(M(z), (1+z^2)^\ell) = a_{d_2} M^{d_2}(z) = 0$. Since $a_{d_2} \neq 0$ it holds that $M(z) = 0$ which contradicts to the fact that $\gcd(M, 1+x^2) = 1$. So, $1+z^2 \neq 0$.

Assume now that z is also a root of $\widehat{G}_1(M(x), (1+x^2)^\ell))$. Since $1+z^2 \neq 0$, we obtain that

$$G_1\bigg(\frac{M(z)}{(1+z^2)^\ell}\bigg) = G_2\bigg(\frac{M(z)}{(1+z^2)^\ell}\bigg) = 0$$

which contradicts that $\gcd(G_1, G_2) = 1$.

Therefore, we have that $\widehat{G}_2(M,(1+x^2)^\ell)=(1+x^2)^k$ for some $0\le k\le d_2\ell$. Since $\widehat{G}_2(M,(1+x^2)^\ell)=a_{d_2}M^{d_2}+(1+x^2)^\ell L$ for some $L\in\mathbb{R}[x]$, $a_{d_2}\neq 0$ and $\gcd(M,1+x^2)=1$ we get that $k=0$ and then $\widehat{G}_2(M,(1+x^2)^\ell)$ is a constant polynomial.

If we decompose the homogeneous polynomial $\widehat{G}_2(M,(1+x^2)^\ell)$ in its real irreducible components we will obtain that for each one of them, say T we have

$$T(M,(1+x^2)^\ell)\in\mathbb{R}.$$

If $\deg(T)=2$ this last property is impossible because

$$\big(aM+b(1+x^2)^\ell\big)^2+c^2(1+x^2)^{2\ell}\in\mathbb{R}$$

with $a\neq 0, c\neq 0, b\in\mathbb{R}$, never holds due to $\deg(M)<2\ell$.

If $\deg(T)=1$ then it should happen that $aM+b(1+x^2)^\ell\in\mathbb{R}$ for some $a,b\in\mathbb{R}$. Again using that $\deg(M)<2\ell$ the only possibility is $b=0$ and $M\in\mathbb{R}$.

Then the only irreducible factor of T is x. Hence $G_2(x)=x^{d_2}$ for some $d_2\ge 0$. If $d_2>0$ then, since $\gcd(G_1,G_2)=1$, it holds that $G_1(0)\neq 0$ and $\deg(\widehat{G}_1(M,(1+x^2)^\ell))=2d_1\ell$. Therefore

$$\frac{G_1}{G_2}\Big(\frac{M}{(1+x^2)^\ell}\Big)=\frac{(1+x^2)^{d_2\ell}\widehat{G}_1(M,(1+x^2)^\ell)}{(1+x^2)^{d_1\ell}}=\frac{\widehat{G}_1(M,(1+x^2)^\ell)}{(1+x^2)^{(d_1-d_2)\ell}},$$

with $\deg(\widehat{G}_1(M,(1+x^2)^\ell))=2d_1\ell$, which is in contradiction with (14). Therefore $d_2=0$ and $G_2\in\mathbb{R}$. So we can take $G_2=1$ and this yields that $G(\theta)=G_1(p(\theta))$. Then $g(\theta)=g_1(p(\theta))p'(\theta)$, with $g_1=G_1'\in\mathbb{R}[x]$, as we wanted to show.

Let us prove that f satisfies a similar property. From (13) we get

$$f(\theta)=\frac{f_3(p(\theta))g(\theta)}{f_4(p(\theta))}=\frac{f_3(p(\theta))g_1(p(\theta))}{f_4(p(\theta))}p'(\theta)=\frac{f_1(p(\theta))}{f_2(p(\theta))}p'(\theta) \tag{15}$$

with $f_1,f_2\in\mathbb{R}[x]$ and $\gcd(f_1,f_2)=1$. Now we will show that $f_2=1$.

From (8), (9) and (10) we have that

$$\frac{f_1}{f_2}\Big(\frac{M}{(1+x^2)^\ell}\Big)\frac{\widetilde{M}}{(1+x^2)^\ell}=\frac{N}{(1+x^2)^j},$$

with $M,N\in\mathbb{R}[x]$, $\widetilde{M}(x)=\frac{1}{2}\frac{dM(x)}{dx}(1+x^2)-\ell xM(x)$, $\gcd(M,(1+x^2)^\ell)=\gcd(\widetilde{M},(1+x^2)^\ell)=\gcd(N,(1+x^2)^j)=1$, where recall that $\deg(M)<2\ell$, and $\deg(\widetilde{M})\le 2\ell$ and $\deg(N)\le 2j$. We remark that the polynomial N and the integer j are not necessarily equal to the ones used in the first part of the proof.

Now assume, in order to get a contradiction, that $\deg(f_2)\ge 1$.

Thus we obtain

$$\frac{(1+x^2)^{d_2\ell}\widehat{f}_1(M,(1+x^2)^\ell)\widetilde{M}}{(1+x^2)^{d_1\ell}\widehat{f}_2(M,(1+x^2)^\ell)}=\frac{N}{(1+x^2)^j}, \tag{16}$$

where $\widehat{f}_1$ and $\widehat{f}_2$ denote the homogenization of f_1 and f_2 and d_1, d_2 their respective degrees. Again, these d_j are in general different to the ones used previously in this proof.

Proceeding as we did for $\widehat{G}_2(M,(1+x^2)^\ell)$ we can show that $\widehat{f}_2(M,(1+x^2)^\ell)$ does not share roots (real or complex) with $(1+x^2)^{d_2\ell}$ or with $\widehat{G}_1(M,(1+x^2)^\ell))$. So, it follows from (16) that

$$Q(x) := \frac{\widetilde{M}(x)}{\widehat{f}_2(M(x),(1+x^2)^\ell)} = \frac{\frac{1}{2}\frac{dM(x)}{dx}(1+x^2) - \ell x M(x)}{\widehat{f}_2(M(x),(1+x^2)^\ell)}$$

has to be a polynomial. If we write $f_2 = \sum_{j=0}^{d_2} b_j x^j$, with $b_{d_2} \neq 0$, we have

$$\widehat{f}_2(M(x),(1+x^2)^\ell) = \sum_{j=0}^{d_2} b_j M^j(x)(1+x^2)^{(d_2-j)\ell}.$$

Assume to arrive to a contradiction that $b_0 = 0$. Then $\widehat{f}_2(M(x),(1+x^2)^\ell) = M(x)L(x)$, for some polynomial L. In particular, the fact that Q is polynomial implies that M divides $\frac{dM(x)}{dx}(1+x^2)$. Since $(M, 1+x^2) = 1$ this is impossible. Hence $b_0 \neq 0$. Therefore, since $\deg(M) < 2\ell$ we have that $\deg(\widehat{f}_2(M,(1+x^2)^\ell)) = 2\ell d_2$.

The degree of $\widetilde{M}$ is at most 2ℓ. Therefore, using again that Q is a polynomial, we conclude that $d_2 \in \{0,1\}$. If $d_2 = 0$ we are done. So we will suppose that $d_2 = 1$, to arrive again to a contradiction. Under this assumption,

$$\begin{aligned} \widehat{f}_2(M(x),(1+x^2)^\ell)) &= b_0(1+x^2)^\ell + b_1 M(x), \\ \widetilde{M}(x) &= \big(b_0(1+x^2)^\ell + b_1 M(x)\big)k, \end{aligned}$$

for some $0 \neq k \in \mathbb{R}$, and $b_1 \neq 0$. By using the expression of $\widetilde{M}$, the second equation above leads to the linear differential equation

$$\frac{dM}{dx} = \frac{a + 2\ell x}{1+x^2} M + b(1+x^2)^{\ell-1},$$

with $a = 2kb_1 \neq 0$ and $b = 2kb_0$. If we write $M(x) = (1+x^2)^\ell P(x)$ then $P(x)$ satisfies

$$\frac{dP}{dx} = \frac{aP + b}{1+x^2}.$$

Solving it we get that $P(x) = -\frac{b}{a} + c\exp\big(a\arctan(x)\big)$, for $c \in \mathbb{R}$. So

$$M(x) = (1+x^2)^\ell\Big(-\frac{b}{a} + c\exp\big(a\arctan(x)\big)\Big).$$

Since M must be a polynomial we get that $c = 0$ but then $M(x) = -b(1+x^2)^\ell/a$ which is not possible because $\gcd(M, 1+x^2) = 1$ and $\ell > 0$.

Hence, $d_2 = 0$ and $\widehat{f}_2(M,(1+x^2)^\ell)$ is a constant. Now proceeding as we did for $\widehat{G}_2(M,(1+x^2)^\ell)$ we get that indeed $f_2 = 1$, as we wanted to prove. Therefore (2) follows with some polynomial p, that satisfies $p'(0) = 0$. In particular $g(\theta) = g_1(p(\theta))p'(\theta)$. Taking derivatives and evaluating on $\theta = 0$ we get $g'(0) = \widetilde{g}_1(p(0))p''(0) < 0$. This completes the proof of the theorem in this case.

Cas 2: Neither G nor F are trigonometric polynomials. We can write F and G as

$$F(\theta) = \alpha\theta + F_1(\theta), \quad G(\theta) = \beta\theta + G_1(\theta)$$

being $\alpha, \beta \in \mathbb{R} \setminus \{0\}$ and $F_1, F_2 \in \mathbb{R}_t[\theta]$. Note that the function

$$H(\theta) = \beta F(\theta) - \alpha G(\theta) = \beta F_1(\theta) - \alpha G_1(\theta)$$

is a trigonometric polynomial. Moreover if $H(\theta) = c \in \mathbb{R}$ we are in case (i) because taking derivatives $0 = \beta f - \alpha g$, that is, $f/g = \alpha/\beta \in \mathbb{R}$ and we are done.

Otherwise H is a non-constant trigonometric polynomial. Moreover, by using (11) and (12) notice that

$$H(\theta) = H(z(\theta)), \quad \frac{f}{g}(\theta) = \frac{f}{g}(z(\theta)).$$

Similarly that in Case 1, we consider the subfield of $\mathbb{R}(x)$, $\mathbb{K} = \mathbb{R}\big(H, f/g\big)$. By applying Proposition 8 with $U = H$, $V = f/g$ it holds that $\mathbb{F} = \mathbb{R}(p)$ for some trigonometric polynomial p, satisfying $p'(0) = 0$.

Now, proceeding as in Case 1 with G replaced by H we conclude that $H(\theta) = H_1(p(\theta))$ with $H_1 \in \mathbb{R}[x]$. Moreover

$$\frac{f}{g}(\theta) = \frac{f_3}{f_4}(p(\theta)) \tag{17}$$

for some $f_3, f_4 \in \mathbb{R}[x]$ and $\gcd(f_3, f_4) = 1$. Since

$$g(\theta)\Big(\beta\frac{f(\theta)}{g(\theta)} - \alpha\Big) = \beta f(\theta) - \alpha g(\theta) = H'(\theta),$$

by using the derivative of $H(\theta) = H_1(p(\theta))$ and (17), it holds that

$$\begin{aligned} g(\theta) &= \frac{H'(\theta)}{\beta\frac{f(\theta)}{g(\theta)} - \alpha} = \frac{H'(\theta)}{\beta\frac{f_3(p(\theta))}{f_4(p(\theta))} - \alpha} = \frac{H'(\theta)f_4(p(\theta))}{\beta f_3(p(\theta)) - \alpha f_4(p(\theta))} \\ &= \frac{H_1'(p(\theta))f_4(p(\theta))}{\beta f_1(p(\theta)) - \alpha f_2(p(\theta))}p'(\theta) = \frac{g_1(p(\theta))}{g_2(p(\theta))}p'(\theta), \end{aligned}$$

with $g_1, g_2 \in \mathbb{R}[x], g_2 \neq 0$ and $\gcd(g_1, g_2) = 1$. Now proceeding as in the study (15) in Case 1, replacing f by g, and $f_j(p(\theta))$ by $g_j(p(\theta))$, for $j = 1, 2$ we can show that $g_2 = 1$ and so $g(\theta) = g_1(p(\theta))p'(\theta)$ with $g_1 \in \mathbb{R}[x]$, as we wanted to see.

Furthermore, from (17),

$$f(\theta) = \frac{f_3(p(\theta))}{f_4(p(\theta)}g(\theta) = \frac{f_3(p(\theta)g_1(p(\theta))}{f_4(p(\theta))}p'(\theta) = \frac{f_1(p(\theta))}{f_2(p(\theta)}p'(\theta),$$

with $f_1, f_2 \in \mathbb{R}[x]$, $\gcd(f_1, f_2) = 1$, and again we can prove that $f_2 = 1$. Hence the theorem follows as in Case 1.

4. Applications

In the Corollay 2, using Theorem 1, we characterize the Liénard differential systems (1) having a center at the origin when either f or g are trigonometric polynomials of prime degree.

Proof of Corollary 2. By applying Theorem 1 we know that the two families of centers correspond to the ones given in item (i) and (ii) of the theorem. The first case, (a), corresponds to item (i).

Assume now that conditions of item (ii) of the Theorem hold. For convenience we use the following equivalent expressions to conditions (ii) of the theorem, see (4),

$$F(\theta) = F_1(p(\theta)), \quad G(\theta) = G_1(p(\theta)), \tag{18}$$

with $p \in \mathbb{R}_t[\theta]$, $F_1, G_1 \in \mathbb{R}[x]$, $p'(0) = 0$ and $G_1'(p(0))p''(0) < 0$. Notice that $F, G \in \mathbb{R}_t[\theta]$, $\deg(F) = \deg(f)$ and $\deg(G) = \deg(g)$.

From (18) it holds that

$$\deg(f) = \deg(F) = \deg(F_1)\deg(p), \quad \text{and} \quad \deg(g) = \deg(G) = \deg(G_1)\deg(p).$$

Assume first that $\deg(g)$ is prime. Therefore, one of the following situations happens:

(I) $\deg(G_1) = 1$ and $\deg(p) = \deg(g)$, or
(II) $\deg(G_1) = \deg(g)$ and $\deg(p) = 1$.

In case (I), $G_1(x) = ax + b, 0 \neq a \in \mathbb{R}$. Then $G(\theta) = ap(\theta) + b$, and consequently $p(\theta) = (G(\theta) - b)/a$. Hence, by (18),

$$F(\theta) = F_1\Big(\frac{G(\theta) - b}{a}\Big) = \widehat{F}_1(G(\theta)).$$

Taking derivatives in this last expression we get that $f(\theta) = f_1(G(\theta))g(\theta)$, with $f_1 = \widehat{F}_1'$. Therefore we have obtained the centers described in item (b).

In case (II), $p(\theta) = a\cos\theta + b\sin\theta + c$ with $a, b, c \in \mathbb{R}, a^2 + b^2 \neq 0$. Taking into account that $p'(0) = 0$ we get that $b = 0$. Then $a \neq 0$ and

$$F(\theta) = F_1(a\cos\theta + c) = \widehat{F}_1(\cos\theta).$$

Taking once more derivatives, we obtain the centers of case (d).

The case where $\deg(f)$ is prime gives rise to the centers of item (c), or again, to some centers in case (d). This completes the proof of the corollary. $\square$

The following corollaries give the characterization of the centers of system (1) when f and g are trigonometric polynomials of degree at most 3, in terms of the coefficients of the their respective Fourier series and we also study in detail the degree 4 case.

Corollary 9. *System* (1) *with f and g trigonometric polynomials of degree at most* 3,

$$\begin{aligned} f(\theta) &= a_0 + a_1\cos\theta + a_2\sin\theta + a_3\cos(2\theta) + a_4\sin(2\theta) + a_5\cos(3\theta) + a_6\sin(3\theta), \\ g(\theta) &= b_0 + b_1\cos\theta + b_2\sin\theta + b_3\cos(2\theta) + b_4\sin(2\theta) + b_5\cos(3\theta) + b_6\sin(3\theta), \end{aligned}$$

has a center at the origin if and only if $a_0 + a_1 + a_3 + a_5 = 0$, $b_0 + b_1 + b_3 + b_5 = 0$, $b_2 + 2b_4 + 3b_6 < 0$, *and one of the following conditions holds:*

(I) $a_1 = a_3 = a_5 = b_1 = b_3 = b_5 = 0$*;*
(II) $\big(a_0, a_1, a_2, a_3, a_4, a_5, a_6\big) = \alpha\big(b_0, b_1, b_2, b_3, b_4, b_5, b_6\big)$ *for some* $\alpha \in \mathbb{R}$.

Corollary 10. *System* (1) *with f and g trigonometric polynomials of degree* 4,

$$\begin{aligned} f(\theta) =& a_0 + a_1\cos\theta + a_2\sin\theta + a_3\cos(2\theta) + a_4\sin(2\theta) + a_5\cos(3\theta) \\ &+ a_6\sin(3\theta) + a_7\cos(4\theta) + a_8\sin(4\theta), \\ g(\theta) =& b_0 + b_1\cos\theta + b_2\sin\theta + b_3\cos(2\theta) + b_4\sin(2\theta) + b_5\cos(3\theta) \\ &+ b_6\sin(3\theta) + b_7\cos(4\theta) + b_8\sin(4\theta), \end{aligned}$$

with $b_7^2 + b_8^2 \neq 0$, has a non-degenerated center at the origin if and only if $a_0 + a_1 + a_3 + a_5 + a_7 = 0$, $b_0 + b_1 + b_3 + b_5 + b_7 = 0$ and $b_2 + 2b_4 + 3b_6 + 4b_8 < 0$, and one of the following conditions holds.

(I) $a_1 = a_3 = a_5 = a_7 = b_1 = b_3 = b_5 = b_7 = 0$*;*

(II) $\big(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\big) = \alpha\big(b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8\big)$ *for some* $\alpha \in \mathbb{R}$*.*

(III) There exit a real trigonometric polynomial

$$p(\theta) = p_0 + p_1\cos\theta + p_2\sin\theta + p_3\cos(2\theta) + p_4\sin(2\theta),$$

with $p_3^2 + p_4^2 \neq 0$, $p(0) = p_0 + p_1 + p_3 = 0$, $p'(0) = -p_2 - 2p_4 = 0$ and two real polynomials $f_1(x) = 2\alpha c x + b$ and $g_1(x) = 2cx + d$, such that

$$f(\theta) = f_1(p(\theta))p'(\theta), \quad g(\theta) = g_1(p(\theta))p'(\theta), \tag{19}$$

$c \neq 0$ and $g_1(0)p''(0) = -(p_1+4p_3)d < 0$. In particular, it holds that $\big(a_5, a_6, a_7, a_8\big) = \alpha\big(b_5, b_6, b_7, b_8\big)$.

Proof of Corollaries 9 and 10. The equalities $f(0) = a_0 + a_1 + a_3 + a_5 + a_7 = 0$, $g(0) = b_0 + b_1 + b_3 + b_5 + b_7 = 0$ and $g'(0) = b_2 + 2b_4 + 3b_6 + 4b_8 < 0$, are the conditions to have a weak focus at the origin.

Cases (I) correspond to the case (d) of Corollary 2. The second cases (II) correspond to item (a) in Corollary 2. Notice that

$$\begin{aligned} &c_2\sin\theta + c_4\sin(2\theta) + c_6\sin(3\theta) + c_8\sin(4\theta) \\ &\qquad = \big(c_2 - c_6 + 2(c_4 - 2c_8)\cos\theta + 4c_6\cos^2\theta + 8c_8\cos^3\theta\big)\sin\theta. \end{aligned}$$

Finally, the case (III) in Corollary 10 happens when the trigonometric polynomial p of Theorem 1 is of degree 2 and f_1 and g_1 are degree 1 polynomials in p. Notice that this type of cases only appears when the degrees of f and g are not prime.

From Theorem 1 we know that (19) holds for some real trigonometric polynomial

$$\widetilde{p}(\theta) = \widetilde{p}_0 + \widetilde{p}_1\cos\theta + \widetilde{p}_2\sin\theta + \widetilde{p}_3\cos(2\theta) + \widetilde{p}_4\sin(2\theta),$$

of degree at most 2 and for some real polynomials $\widetilde{f}_1(x) = 2\widetilde{a}x + \widetilde{b}$ and $\widetilde{g}_1(x) = 2\widetilde{c}x + \widetilde{d}$. From the condition $b_7^2 + b_8^2 \neq 0$ we have that $\widetilde{p}_3^2 + \widetilde{p}_4^2 \neq 0$ and $\widetilde{c} \neq 0$. Taking $p(\theta) = k(\widetilde{p}(\theta) - \widetilde{p}(0))$ and the corresponding f_1 and g_1 for a suitable $k \in \mathbb{R}$ we get all the conditions (III).

To prove that $\big(a_5, a_6, a_7, a_8\big) = \alpha\big(b_5, b_6, b_7, b_8\big)$ holds notice that

$$f(\theta) = 2\alpha c\, p(\theta)p'(\theta) + b\, p'(\theta) \quad \text{and} \quad g(\theta) = 2c\, p(\theta)p'(\theta) + d\, p'(\theta).$$

Therefore $f(\theta) - \alpha\, g(\theta) = (b - \alpha d)p'(\theta)$. Since p' has degree 2, the above equality implies that the degree 3 and 4 terms of f and g coincide up the multiplicative constant α, as we wanted to prove. □

If one wishes to have the explicit conditions among the coefficients of f and g to characterize the centers of family (III) of Corollary 10, there is a systematic procedure to get them. Define the two trigonometric polynomials

$$s_1(\theta) := f(\theta) - \big(2\alpha c\, p(\theta)p'(\theta) + b\, p'(\theta)\big), \quad s_2(\theta) := g(\theta) - \big(2c\, p(\theta)p'(\theta) + d\, p'(\theta)\big),$$

where α, b, c and d are real constants and in f we take $\big(a_5, a_6, a_7, a_8\big) = \alpha\big(b_5, b_6, b_7, b_8\big)$. Hence these polynomials have 21 free variables: the 9 coefficients of g; the first 5 coefficients of f; three of the coefficients of p (p_1, p_3 and p_4); and α, b, c and d.

By our results the functions s_1 and s_2 must be identically zero. Hence we expand in Fourier series both functions and we take all their coefficients. Finally, with the system generated by equating to zero each of these coefficients, we use the elimination method of parameters to eliminate the 6 parameters of p_1, p_3, p_4, b, c and d. Then we obtain the desired center conditions. We omit them because are really huge and the reader can obtain the conditions following the approach we have described. Instead, we show some subcases of the system considered in Corollary 10 where we apply this approach to characterize all the centers of type (III).

In the first one we assume that $a_3 = a_4 = b_3 = b_4 = 0$. Then, the equations obtained with this elimination method are:

$$\begin{aligned}
&b_0 = 0, \quad a_0 = 0, \quad a_1 + \alpha b_5 + \alpha b_7 = 0, \quad b_1 + b_5 + b_7 = 0,\\
&a_2 b_6 - \alpha b_2 b_6 = 0, \quad a_2 b_8 - \alpha b_2 b_8 = 0, \quad b_5^2 b_7 + b_6^2 b_7 + 6 b_5 b_7^2 + 6 b_6 b_7 b_8 = 0\\
&9 b_2 b_7 - 10 b_5 b_6 - 9 b_6 b_7 - 30 b_5 b_8 - 45 b_7 b_8 = 0,\\
&3 b_2 b_5 + 17 b_5 b_6 + 15 b_6 b_7 + 60 b_5 b_8 + 90 b_7 b_8 = 0,\\
&9 a_2 b_7 - 10 \alpha b_5 b_6 - 9 \alpha b_6 b_7 - 30 \alpha b_5 b_8 - 45 \alpha b_7 b_8 = 0,\\
&3 a_2 b_5 + 17 \alpha b_5 b_6 + 15 \alpha b_6 b_7 + 60 \alpha b_5 b_8 + 90 \alpha b_7 b_8 = 0,\\
&9 b_7^3 - 2 b_6^2 b_7 + 2 b_5 b_6 b_8 - 6 b_6 b_7 b_8 + 6 b_5 b_8^2 + 9 b_7 b_8^2 = 0,\\
&2 b_5^2 b_6 + 12 b_5 b_6 b_7 + 9 b_6 b_7^2 + 6 b_5^2 b_8 + 45 b_5 b_7 b_8 + 54 b_7^2 b_8 = 0,\\
&9 b_6 b_7^3 - 2 b_6^3 b_7 - 18 b_6^2 b_7 b_8 + 9 b_5 b_7^2 b_8 + 54 b_7^3 b_8 - 36 b_6 b_7 b_8^2 = 0,\\
&2 b_6^3 - 9 b_6 b_7^2 + 12 b_5^2 b_8 + 63 b_5 b_7 b_8 + 81 b_7^2 b_8 + 27 b_2 b_8^2 - 9 b_6 b_8^2 = 0,\\
&2 b_5^3 - 18 b_6^2 b_7 - 63 b_5 b_7^2 + 27 b_7^3 - 99 b_6 b_7 b_8 - 18 b_5 b_8^2 - 27 b_7 b_8^2 = 0,\\
&2 b_5 b_6^2 + 18 b_6^2 b_7 - 9 b_5 b_7^2 - 81 b_7^3 + 63 b_6 b_7 b_8 - 18 b_5 b_8^2 - 27 b_7 b_8^2 = 0,\\
&4 b_6^4 b_7 - 36 b_6^2 b_7^3 + 81 b_7^5 + 48 b_6^3 b_7 b_8 - 216 b_6 b_7^3 b_8 + 180 b_6^2 b_7 b_8^2 - 243 b_7^3 b_8^2 + 216 b_6 b_7 b_8^3 = 0,
\end{aligned}$$

Solving them and doing some tedious but straightforward computations we get that when $a_3 = a_4 = b_3 = b_4 = 0$ all the centers are inside classes (I) and (II) of Corollary 10.

Similarly, when we consider the subfamily of the system studied in Corollary 10, with $a_1 = a_2 = a_5 = a_6 = 0$ and $b_1 = b_2 = b_5 = b_6 = 0$, we can prove that all non-degenerated centers are again the ones given in classes (I) and (II).

An example of case (III) is given by the following conditions

$$a_0 = b_0 = 0,\ a_2 = \alpha/2,\ a_3 = \alpha - a_1,\ a_4 = 2a_1 - \alpha,$$
$$\big(a_5, a_6, a_7, a_8\big) = \alpha\big(-3, -3/2, 2, -3/2\big),\ b_1 = (1 + b_4)/2,$$
$$b_2 = 1/2,\ b_3 = (1 - b_4)/2,\ b_5 = -3,\ b_6 = -3/2,\ b_7 = 2,\ b_8 = -3/2,$$

where α and a_1 are arbitrary real constants and $b_4 < 5$ (this condition comes from $g'(0) < 0$). In this exemple $f_1(x) = 2\alpha x + 3\alpha - a_1$, $g_1(x) = 2x + (5 - b_4)/2$ and $p(\theta) = -1 - \sin\theta + \cos(2\theta) + \frac{1}{2}\sin(2\theta)$.

Acknowledgments

The first author is partially supported by the MINECO MTM2013-40998-P and the AGAUR (Generalitat de Catalunya) 2014SGR568 grants. The second author is partially supported by the MINECO/FEDER grant number MTM2014-53703-P and the AGAUR grant number 2014SGR 1204. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2013.

References

[1] A. F. Andreev, A. P. Sadovskii and V. A. Tsikalyuk, *The center-focus problem for a system with homogeneous nonlinearities in the case of zero eigenvalues of the linear pact,* (Russian) Differ. Uravn. **39** (2003), 147–153; translation in Differ. Equ. **39** (2003), 155–164.

[2] V. N. Belykh, N. F. Pedersen and O. H. Soerensen, *Shunted-Josephson-junction model. I. The autonomous case*, Phys. Rev. B **16** (1977), 4853–4859.

[3] L. A. Cherkas, *Conditions for a Lienard equation to have a centre*, Differential Equations **12** (1976) 201–206.

[4] C. Christopher, *An algebraic approach to the classification of centers in polynomial Liénard systems*, J. Math. Anal. Appl. **229** (1999), 319–329.

[5] C. J. Christopher, N. G. Lloyd and J. M. Pearson, *On a Cherkass method for centre conditions*, Nonlinear World **2** (1995) 459–469.

[6] C. J. Christopher and D. Schlomiuk, *On general algebraic mechanisms for producing centers in polynomial differential systems*, J. Fixed Point Theory Appl. **3** (2008) 331–351.

[7] A. Cima, A. Gasull and F. Mañosas, *A simple solution of some composition conjectures for abel equations*, J. Math. Anal. Appl. **398** (2013) 477–486.

[8] A. Cima, A. Gasull and F. Mañosas, *An explicit bound of the number of vanishing double moments forcing composition* J. Differential Equations **255** (2013) 339–350.

[9] A. Gasull, A. Geyer and F. Mañosas, *On the number of limit cycles for perturbed pendulum equations*, J. Differential Equations, **261** (2016) 2141–2167.

[10] A. Gasull and J. Torregrosa, *Center problem for several differential equations via Cherkas' method*, J. Math. Anal. Appl., **228** (1998) 322–343.

[11] A. Gasull, J. Torregrosa and X. Zhang, *The number of polynomial solutions of polynomial Riccati equations*, J. Differential Equations **261** (2016) 5071–5093.

[12] J. Giné, *Center conditions for polynomial Liénard systems*, Qual. Theory Dyn. Syst., to appear (DOI 10.1007/s12346-016-0202-3).

[13] J. Giné, M. Grau and J. Llibre, *Universal centres and composition conditions*, Proc. London Math. Soc. **106** (2013), 481–507.

[14] K. Inoue, *Perturbed motion of a simple pendulum* J. Phys. Soc. Japan **57** (1988), 1226–1237.

[15] H. Lichardová, *Limit cycles in the equation of whirling pendulum with autonomous perturbation*, Appl. Math. **44** (1999), 271–288.

[16] A. D. Morozov, *Limit cycles and chaos in equations of pendulum type* (Russian), Prikl. Mat. Mekh. **53** (1989) 721–730; translation in J. Appl. Math. Mech. **53** (1989) 565–572 (1990).

[17] A. D. Morozov, Quasi-conservative systems. Cycles, resonances and chaos, World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises, 30. World Scientific Publishing Co., Inc., River Edge, NJ, 1998.

[18] J. A. Sanders, *Report on the driven Josephson equation*, Dyn. Syst. Chaos (Luis Garrido, ed.), Lecture Notes in Physics, vol. **179**, Springer Berlin Heidelberg, 1983, pp. 297–298.

[19] J. A. Sanders and R. Cushman, *Limit cycles in the Josephson equation*, SIAM J. Math. Anal. **17** (1986), 495–511.

[20] B. L. van der Waerden, Modern Algebra, Vol. **1**, second ed., Frederick Ungar, New York, 1966.

[21] H. Zoladek, *The classification of reversible cubic systems with center*, Topol. Methods Nonlinear Anal. **4** (1994) 79–136.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: gasull@mat.uab.cat

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69; 25001 Lleida, Catalonia, Spain

E-mail address: gine@matematica.udl.cat

Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

E-mail address: cvalls@math.ist.utl.pt