CENTER PROBLEM FOR TRIGONOMETRIC LIÉNARD SYSTEMS

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Abstract. We give a complete algebraic characterization of the non-degenerated centers for planar trigonometric Liénard systems. The main tools used in our proof are the classical results of Cherkas on planar analytic Liénard systems and the characterization of some subfields of the quotient field of the ring of trigonometric polynomials. Our results are also applied to some particular subfamilies of planar trigonometric Liénard systems. The results obtained are reminiscent of the ones for planar polynomial Liénard systems but the proofs are different.

1. Introduction and main results

The aim of this paper is to characterize the non-degenerated centers for the planar systems associated to the second order trigonometric Liénard differential equations $\ddot{\theta} = g(\theta) + f(\theta)\dot{\theta}$, where f, g are trigonometric polynomials with real coefficients and the dot denotes the derivative with respect to the time.

The analysis of equations of this form is motivated by a number of problems resulting from pendulum-like equations appearing in the literature. Equations of this form, like $\ddot{\theta} + \sin(\theta) = \varepsilon\dot{\theta}\cos(n\theta)$, or the Josephson equation $\ddot{\theta} + \sin(\theta) = \varepsilon[a - (1 + \gamma\cos(\theta))\,\dot{\theta}]$, are considered in [9, 16, 17] or [2, 14, 18, 19], respectively. Also the perturbed whirling pendulum, $\ddot{\theta} = \sin\theta(\cos\theta - \gamma) + \varepsilon(\cos\theta + \alpha)\,\dot{\theta}$, see [15] falls in this class. Here a, γ, α and ε are real constants and $n \in \mathbb{N}$.

As usual we will write the above second order trigonometric differential equation as the autonomous planar system

$$\begin{cases} \dot{\theta} = y, \\ \dot{y} = g(\theta) + yf(\theta), \end{cases} \tag{1}$$

and we will assume that $f(0) = 0$, $g(0) = 0$, $g'(0) < 0$, where the prime denotes the derivative with respect to θ. These hypotheses on g imply that the origin is either a center or a weak focus. Our main result is:

Theorem 1. *System* (1) *has a center at the origin if and only if*

(i) Either $f = \alpha g$ for some $\alpha \in \mathbb{R}$, or

(ii) There exist a real trigonometric polynomial p and two real polynomials f_1 and g_1 satisfying $p'(0) = 0$, $g_1(p(0))p''(0) < 0$, and such that

$$f(\theta) = f_1(p(\theta))p'(\theta), \quad g(\theta) = g_1(p(\theta))p'(\theta). \tag{2}$$

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