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AN ANALYTIC-NUMERICAL METHOD FOR COMPUTATION OF THE LIAPUNOV AND PERIOD CONSTANTS DERIVED FROM THEIR ALGEBRAIC STRUCTURE.*

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Abstract. We consider the problem of computing the Liapunov and the period constants for a smooth differential equation with a nondegenerate critical point. First, we investigate the structure of both constants when they are regarded as polynomials on the coefficients of the differential equation. Second, we take advantage of this structure to derive a method to obtain the explicit expression of the above-mentioned constants. Although this method is based on the use of the Runge–Kutta–Fehlberg methods of orders 7 and 8 and the use of Richardson's extrapolation, it provides the real expression for these constants.

Key words. center point, Liapunov constants, isochronicity, analytic-numerical method

AMS subject classifications. 65L07, 34D20, 34C25

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1. Introduction and main results. In this paper we deal with the problem of computing the Liapunov and period constants for the differential equations

$$\begin{cases} \dot{x} = -y + P(x,y), \\ \dot{y} = x + Q(x,y), \end{cases}$$

where $P(x,y)$ and $Q(x,y)$ are analytic functions in a neighborhood of the origin and begin, at least, with second order terms. These systems can be expressed in the complex plane using the following notation:

$$\dot{z} = i\,z + F(z,\bar{z}), \tag{1.1}$$

where $F(z,\bar{z}) = \sum_{k\geq 2} F_k(z,\bar{z})$, $F_k(z,\bar{z}) = \sum_{j=0}^{k} f_{k-j,j} z^{k-j}\bar{z}^j$, $f_{k-j,j} \in \mathbf{C}$, and the dot indicates the derivative with respect to t, with $t \in \mathbf{R}$.

The problem of determining whether (1.1) has a center or a focus at the origin can be solved by studying the Poincaré return map. This study can be done (using the power series of the return map) by means of the computation of infinitely many real numbers, $v_{2m+1}, m \geq 1$, called the *Liapunov constants*. In fact, we have that if for some m, $v_3 = v_5 = \cdots = v_{2m-1} = 0$, and $v_{2m+1} \neq 0$, then the origin is a focus of which the stability is determined by the sign of v_{2m+1}, while if all v_{2m+1} are zero, then the origin is a center; see for instance [1].

A closely related problem is the following: Assume that (1.1) has a center at the origin and consider the period of all its periodic orbits. The origin of (1.1) is an *isochronous* center if and only if the period is independent of the orbit. When is the

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