ON THE CHEBYSHEV PROPERTY FOR A NEW FAMILY OF FUNCTIONS

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Abstract. We analyze whether a given set of analytic functions is an Extended Chebyshev system. This family of functions appears studying the number of limit cycles bifurcating from some nonlinear vector field in the plane. Our approach is mainly based on the so called Derivation-Division algorithm. We prove that under some natural hypotheses our family is an Extended Chebyshev system and when some of them are not fulfilled then the set of functions is not necessarily an Extended Chebyshev system. One of these examples constitutes an Extended Chebyshev system with high accuracy.

1. Introduction

Given $m+1$ real, analytic and linearly independent functions $\mathcal{F}=\{f_0(x),f_1(x),\ldots,f_m(x)\}$, defined on some open interval I, the problem of estimating the number of real zeroes of any non-zero function $F\in\operatorname{Span}\mathcal{F}$,

$$F(x)=\sum_{j=0}^{m}\lambda_j f_j(x),\qquad \lambda_j\in\mathbb{R},$$

is of wide interest. We will denote by $\mathcal{Z}(F)$ the number of zeroes in I, counted with their multiplicities, of a function F and by

$$\mathcal{Z}(\mathcal{F})=\max_{F\in(\operatorname{Span}\mathcal{F})\setminus\{0\}}\mathcal{Z}(F),$$

whenever they exist. It is easy to see that $\mathcal{Z}(\mathcal{F})\geq m$. A set of functions $\mathcal{F}$ for which $\mathcal{Z}(\mathcal{F})=m$ is usually called an *Extended Chebyshev system* on I and denoted in short as an ET-system. The set of polynomials of degree m, $\{1,x,x^2,\ldots,x^m\}$, on any open interval is a well-known example. Other nice examples are

$$\begin{aligned}
&\big\{1,\log x,x,x\log x,x^2,x^2\log x,\ldots,x^n,x^n\log x\big\}\ \text{ on } (0,\infty);\\
&\{1,\cos x,\cos(2x),\ldots,\cos(mx)\}\ \text{ on } (0,\pi);\\
&\big\{(x+a_0)^{-1},(x+a_1)^{-1},\ldots,(x+a_m)^{-1}\big\}\ \text{ on } (a,\infty),\ \text{ where } a=\max_{j=0,\ldots,m}(-a_j).
\end{aligned}$$

ET-systems are an extension of the so called Chebyshev systems which are sets as above, but of continuous functions and having at most m distinct zeroes in I. See [3, 4, 5, 10,

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14, 16, 17, 18] for general properties of Chebyshev and Extended Chebyshev systems. The "T" stands for Tchebycheff, which is one of the transcriptions of the name of the Russian mathematician.

When $\mathcal{Z}(\mathcal{F}) = m+k$, for some $k>0$, then it is said that $\mathcal{F}$ is an *ET-system with accuracy* k, see [8]. For instance the set $\{1, x, x^2, \ldots, x^{m-1}, x^{m+1}\}$ is an ET-system with accuracy 1 on the whole $\mathbb{R}$.

Chebyshev systems are known to appear in several mathematical problems. The book of Karlin and Studden [10] and the survey of Zalik [16] show many of them. Without aim to be exhaustive, they appear in the theory of approximations, in the study of boundary value problems, in problems involving oscillations properties of zeroes in solutions of n-th order linear differential equations, in the theory of inequalities, ...

In this paper we study the Chebyshev property for the family of functions

$$\mathcal{G} = \{1, x, \ldots, x^{n_0}\} \cup \left(\bigcup_{j=1}^{K} \{(x+a_j)^{\alpha_j},\ x\,(x+a_j)^{\alpha_j}, \ldots, x^{n_j}\,(x+a_j)^{\alpha_j}\} \right), \tag{1}$$

with $n_j \geq 0$ and $\alpha_j \notin \mathbb{Z}$, $j = 1, \ldots, K$, that appears in several problems of nonlinear differential equations providing a new application of the theory of Chebyshev systems. As we will see in Section 5, it controls the bifurcation of isolated periodic orbits (limit cycles) of some planar vector fields.

In a few words, our goal will be to prove that the previous family is only an Extended Chebyshev system when all α_j coincide and all $n_j, j>1$, also coincide. More concretely, we will prove the following result:

Theorem 1.1. *Consider $K \geq 1$, $n \geq 0$, $n_0 \geq -1$ integer numbers, $\alpha \notin \mathbb{Z}$ and $\{a_j\}_{j=1,\ldots,K}$ different real constants and define $a = \max_{j=1,\ldots,K}(-a_j)$. Then the set of $m+1 = n_0+1+K(n+1)$ functions*

$$\mathcal{F} = \{1, x, \ldots, x^{n_0}\} \cup \left(\bigcup_{j=1}^{K} \{(x+a_j)^{\alpha},\ x\,(x+a_j)^{\alpha}, \ldots, x^{n}\,(x+a_j)^{\alpha}\} \right), \tag{2}$$

is an Extended Chebyshev system on $I = (a, \infty)$. By notation, when $n_0 = -1$ the set $\{1, x, \ldots, x^{n_0}\}$ is the empty set.

In particular this result implies that for a non-zero function of the form

$$F(x) = P^0(x) + \sum_{j=1}^{K} P^j(x)\,(x+a_j)^{\alpha}, \tag{3}$$

where P^j are polynomials of degree n for $j = 1, \ldots, K$ and P^0 of degree n_0,

$$\mathcal{Z}(F) \leq m = n_0 + K(n+1), \tag{4}$$

where $\deg(0) = -1$ and, moreover, that this bound is reached.

On the other hand, we will see that when not all α_j coincide or not all $n_j, j>1$, coincide this family is not usually an ET-system. In Section 3 several examples are presented which are ET-systems with accuracy 1. Theorem 4.1 in Section 4 provides, as well, families of the form (1), with all $\alpha_j = 1/2$, and accuracy k for any $k \leq 16$. All these examples show that the hypotheses in Theorem 1.1 saying that there is a unique exponent α and that all the values n_j, $j \geq 1$, coincide can not be, in general, relaxed.

Observe also that when all $\alpha_j, j = 1, \ldots, K$ coincide it is possible to apply the bound (4) to the family (1) with $n = \max_{j=1,\ldots,K} n_j$. This bound is also given in [7]. Sometimes the upper bound for $\mathcal{Z}(\mathcal{G})$ obtained with this method is optimal and sometimes it is not. Examples of both situations are presented in Section 4.

When, for instance, $\alpha = 1/2$ two other natural approaches can be used to study the real zeroes of

$$F(x) = P^0(x) + P^1(x)\sqrt{x + a_1} + P^2(x)\sqrt{x + a_2} + \cdots + P^K(x)\sqrt{x + a_K}.$$

One of these techniques consists simply on squaring recurrently the equation $F(x) = 0$ and applying, afterwards, the Algebra Fundamental Theorem to the final attained polynomial. Another standard technique, coming from Complex Variable Theory, consists on applying the Principle of the Argument. Table 1 shows, for $n_0 = n$ and $K \leq 5$, the advantage of our theorem (which provides the exact bound for $\mathcal{Z}(\mathcal{F})$) compared with the precedent methods.

K	Squaring	Principle of the Argument	ET-system
1	$2n + 1$	$2n + 1$	$2n + 1$
2	$4n + 2$	$3n + 2$	$3n + 2$
3	$8n + 4$	$5n + 4$	$4n + 3$
4	$16n + 8$	$7n + 6$	$5n + 4$
5	$32n + 16$	$10n + 9$	$6n + 5$

Table 1. Comparative of the bounds provided by the three methods.

The proof of Theorem 1.1, given in Section 2, and the proof that some families $\mathcal{G}$ are ET-systems with positive accuracy, are based on the so called Derivation-Division algorithm, see [13, p. 119]. This algorithm is one of the most common tools used in the study of the cyclicity of the limit periodic sets of planar differential equations. In Subsection 2.1 we discuss the relation between it and other methods frequently used in the study of Chebyshev systems: the integral representation of the set of functions and the relative differentiation, see [10, Chap. XI] and [12, 17].

The rest of the paper is structured as follows: Section 3 provides examples showing the necessity of the hypotheses of Theorem 1.1. Two more related families of examples are studied in Section 4 providing systems $\mathcal{G}$ which are ET-systems with a high accuracy. Finally, in Section 5 we present some applications of our results to the study of the number of periodic orbits of some nonlinear vector fields of the plane.

2. Proof of Theorem 1.1

First we will show that all the functions in $\mathcal{G}$ are linearly independent.

Lemma 2.1. *Consider $n_0, n_j \geq 0$ and $\alpha_j \notin \mathbb{Z}$, $j = 1, \ldots, K$ and $\{a_j\}_{j=1,\ldots,K}$ different real constants and define $a = \max\limits_{j=1,\ldots,K}(-a_j)$. Then the $n_0 + n_1 + \cdots + n_K + K + 1$ functions*

$$\mathcal{G} = \{1, x, \ldots, x^{n_0}\} \cup \left(\bigcup_{j=1}^{K} \{(x + a_j)^{\alpha_j},\ x\,(x + a_j)^{\alpha_j}, \ldots, x^{n_j}\,(x + a_j)^{\alpha_j}\} \right),$$

defined on $I=(a,\infty)$, are linearly independent.

Proof. Define $F_0(x)=P^0_{n_0}(x)$ and $F_j(x)=P^j_{n_j}(x)\,(x+a_j)^{\alpha_j}$, where $P^j_{n_j}$ is a polynomial of degree n_j for $j=0,\dots,K$. Then any $F\in\mathcal{G}$ can be written as

$$F(x)=\sum_{j=0}^{K}F_j(x),$$

and is analytic in the cut complex plane $\mathbb{C}\setminus\{z\in\mathbb{C}:\operatorname{Im}(z)=0,\ \operatorname{Re}(z)\le a\}$. It is not restrictive to assume that $a_1<a_2<\ldots<a_n$. So $a=-a_1$. Note that each $F_j, j>1$, is analytic on $\mathbb{C}\setminus\{z\in\mathbb{C}:\operatorname{Im}(z)=0,\ \operatorname{Re}(z)\le -a_j\}$ and that F_0 is a polynomial. Assume that a linear combination of elements of $\mathcal{G}$ is identically zero. Then on a neighborhood of $z=-a_1$ all the functions except F_1 are analytic. Since the total sum is analytic as well (it is zero) we get that $F_1=0$. Thus $P^1_{n_1}=0$. Arguing similarly near a_2 we obtain that $P^2_{n_2}=0$ and so on. Hence all the functions are linearly independent as we wanted to prove. □

The principal skill to give an upper bound for $\mathcal{Z}(\mathcal{F})$ will be the Derivation-Division algorithm (see, for instance, [13, p. 119]). As we will see, all the functions that will appear when performing this procedure are linear combination of the two *basic* ones

$$P_n(x)(x+a)^\alpha \quad \text{and} \quad P_n(x)\frac{(x+a)^\alpha}{(x+b)^\beta},$$

with $P_n(x)$ a polynomial of degree n. Therefore, we need to know how their derivatives behave. This is the content of the following lemma. For any $m\ge 1$, we will denote the m-derivative operator as $\mathcal{D}^m=\frac{d^m}{dx^m}$. When $m=1, \mathcal{D}^1=\mathcal{D}$.

Lemma 2.2. *For any $n\ge 0$, $m\ge 1$,*

$$\begin{aligned}
\mathcal{D}^m\left(P_n(x)\,(x+a)^\alpha\right)&=\widetilde{P}_n(x)\,(x+a)^{\alpha-m}\\
\mathcal{D}^m\left(P_n(x)\,\frac{(x+a)^\alpha}{(x+b)^\beta}\right)&=\widetilde{P}_{n+m}(x)\,\frac{(x+a)^{\alpha-m}}{(x+b)^{\beta+m}},\\
\mathcal{D}^{n+1}\left(P_n(x)\,\frac{(x+a)^\alpha}{(x+b)^\alpha}\right)&=\widetilde{Q}_n(x)\,\frac{(x+a)^{\alpha-(n+1)}}{(x+b)^{\alpha+(n+1)}},
\end{aligned}$$

where P_k, $\widetilde{P}_k$ and $\widetilde{Q}_k$ are polynomials of degree at most k and $a\neq b$, α,β real constants.

Proof. The first equality is a simple computation. To prove the second one, it is convenient to write $P_n(x)=\sum_{k=0}^{n}c_k(x+a)^k$. Therefore, by linearity it suffices to study the derivatives of terms of the form $(x+a)^A(x+b)^B$ for some real numbers A and B. Hence the second equality follows straightforwardly from the expression

$$\mathcal{D}\left((x+a)^A(x+b)^B\right)=[A(b-a)+(A+B)(x+a)](x+a)^{A-1}(x+b)^{B-1}.$$

To prove the last assertion one has to see that many of the terms of the polynomial $\widetilde{P}_{2n+1}$, that one obtains applying the second equality of the lemma, indeed vanish. Consider the functions

$$\mathcal{S}_{\alpha,j}(x):=(x+a)^j\left(\frac{x+a}{x+b}\right)^\alpha,\quad j\in\mathbb{N}.$$

We claim that for $m \in \mathbb{N}$, its m-th derivative satisfies that

$$\mathcal{D}^m \mathcal{S}_{\alpha,j} = \sum_{i=0}^{m} \delta_{i,j}^m \mathcal{S}_{\alpha+m,j-m-i}, \tag{5}$$

with

$$\delta_{i,j}^m := (b-a)^i \binom{m}{i} \prod_{\ell=i}^{m-1} (j-\ell) \prod_{\ell=0}^{i-1} (\alpha+j-\ell),$$

where we denote $\prod_{\ell=i}^{i-1}(j-\ell) := 1$ and $\prod_{\ell=0}^{-1}(\alpha+j-\ell) := 1$. Notice that, in particular,

$$\prod_{\ell=i}^{m-1}(j-\ell) = (j-i)(j-(i+1))\cdots(j-(m-1)) = 0$$

if $j = 0, 1, \ldots, m-1$. Therefore it turns out that $\delta_{i,j}^m = 0$ if $i \le j$ and $m \ge j+1$, and so formula (5) can be simplified to

$$\mathcal{D}^m \mathcal{S}_{\alpha,j} = \begin{cases} \displaystyle\sum_{i=0}^{m} \delta_{i,j}^m \mathcal{S}_{\alpha+m,j-m-i}, & \text{if } m \le j, \\ \displaystyle\sum_{i=j+1}^{m} \delta_{i,j}^m \mathcal{S}_{\alpha+m,j-m-i}, & \text{if } m > j. \end{cases}$$

When $m = n+1$, we obtain that for $j \le n$,

$$\begin{aligned} \mathcal{D}^{n+1} \mathcal{S}_{\alpha,j} &= \sum_{i=j+1}^{n+1} \delta_{i,j}^{n+1} \mathcal{S}_{\alpha+n+1,j-n-1-i} = \sum_{i=j+1}^{n+1} \delta_{i,j}^{n+1} (x+a)^{j-n-1-i} \left(\frac{x+a}{x+b}\right)^{\alpha+n+1} \\ &= \sum_{i=j+1}^{n+1} \delta_{i,j}^{n+1} (x+a)^{n+1+j-i} \frac{(x+a)^{\alpha-n-1}}{(x+b)^{\alpha+n+1}} = \widetilde{P}_n^j(x) \frac{(x+a)^{\alpha-n-1}}{(x+b)^{\alpha+n+1}}, \end{aligned}$$

where $\widetilde{P}_n^j$ is a polynomial of degree n. From this result the third equality follows.

So, to end the proof it suffices to prove the claim given in (5). We prove it inductively. First, a simple computation shows that, for $j \in \mathbb{N}$,

$$\mathcal{D}\mathcal{S}_{\alpha,j} = j\mathcal{S}_{\alpha+1,j-1} + (\alpha+j)(b-a)\mathcal{S}_{\alpha+1,j-2} = \delta_{0,j}^1 \mathcal{S}_{\alpha+1,j-1} + \delta_{1,j}^1 \mathcal{S}_{\alpha+1,j-2},$$

which corresponds to formula (5) for $m = 1$. So let us assume that it holds for m and let us prove it for $m+1$. Differentiating the function $\mathcal{D}^m \mathcal{S}_{\alpha,j}$ in (5) we obtain

$$\begin{aligned} &\mathcal{D}^{m+1} \mathcal{S}_{\alpha,j} = \sum_{i=0}^{m} \delta_{i,j}^m \mathcal{D}\mathcal{S}_{\alpha+m,j-m-i} = \\ &\quad \sum_{i=0}^{m} \delta_{i,j}^m (j-m-i)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i} + \sum_{i=0}^{m} \delta_{i,j}^m (\alpha+j-i)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i-1}. \end{aligned}$$

Using the definition of $\delta^m_{i,j}$, the first sum in the latter expression can be rewritten as

$$\sum_{i=0}^{m} \delta^m_{i,j}(j-m-i)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i} =$$
$$\sum_{i=0}^{m}\left((b-a)^i\binom{m}{i}\prod_{\ell=i}^{m-1}(j-\ell)\prod_{\ell=0}^{i-1}(\alpha+j-\ell)\right)(j-m-i)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i}$$

and the second one as

$$\sum_{i=0}^{m} \delta^m_{i,j}(\alpha+j-i)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i-1} =$$
$$\sum_{i=1}^{m+1}\left((b-a)^i\binom{m}{i-1}\prod_{\ell=i-1}^{m-1}(j-\ell)\prod_{\ell=0}^{i-1}(\alpha+j-\ell)\right)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i}.$$

Thus, we have

$$\begin{aligned}\mathcal{D}^{m+1}\mathcal{S}_{\alpha,j} = {}&(j-m)\prod_{\ell=0}^{m-1}(j-\ell)\prod_{\ell=0}^{-1}(\alpha+j-\ell)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)}+\\ &\sum_{i=1}^{m}(b-a)^i\left(\binom{m}{i-1}\prod_{\ell=i-1}^{m-1}(j-\ell)+\binom{m}{i}(j-m-i)\prod_{\ell=i}^{m-1}(j-\ell)\right)\times\\ &\prod_{\ell=0}^{i-1}(\alpha+j-\ell)\ \mathcal{S}_{\alpha+(m+1),j-(m+1)-i}+\\ &(b-a)^{m+1}\prod_{\ell=0}^{m}(\alpha+j-\ell)\mathcal{S}_{\alpha+(m+1),j-(m+1)-(m+1)}.\end{aligned}\tag{6}$$

Using that

$$\binom{m}{i-1}\prod_{\ell=i-1}^{m-1}(j-\ell)+\binom{m}{i}\prod_{\ell=i}^{m-1}(j-\ell)(j-m-i)=\prod_{\ell=i}^{m}(j-\ell)\binom{m+1}{i}$$

and having in mind the definition of $\delta^{m+1}_{i,j}$, formula (6) becomes

$$\mathcal{D}^{m+1}\mathcal{S}_{\alpha,j}=\sum_{i=0}^{m+1}\delta^{m+1}_{i,j}\mathcal{S}_{\alpha+(m+1),j-(m+1)-i},$$

which is formula (5) for $m+1$, proving the claim. □

As we will see, the third equality given in the above lemma, which deals with the case $\alpha=\beta$ and the number of derivatives being exactly one more than the degree of the polynomial P_n, will be a key point in the proof of Theorem 1.1.

Proof of Theorem 1.1. To simplify the proof, we will assume $n_0 = n$ in (3). The general case follows using a similar argument.

By Lemma 2.1 we know that the family $\mathcal{F}$ is formed by $m+1$ linearly independent functions and so $\mathcal{Z}(\mathcal{F}) \geq m$.

To prove that $\mathcal{Z}(\mathcal{F}) \leq m$ we will apply the Derivation-Division algorithm to the function F, defined by (3), in its interval of definition I. It is performed, step by step, as follows:

[Der_1] Let us define

$$F^{(0)}(x) = F(x) = P^0(x) + P^1(x)(x+a_1)^\alpha + \cdots + P^K(x)(x+a_K)^\alpha$$

and differentiate it $n+1$ times, one more than the degree of $P^0(x)$. Applying Lemma 2.2 it follows that the function obtained is of the form

$$P^{1,1}(x)(x+a_1)^{\alpha^{(1)}} + P^{2,1}(x)(x+a_2)^{\alpha^{(1)}} + \cdots + P^{K,1}(x)(x+a_K)^{\alpha^{(1)}}$$

with $\alpha^{(1)} = \alpha-(n+1)$ and $P^{1,1}(x), \ldots, P^{K,1}(x)$ being polynomials of degree n. For the general case, $n_0 \neq n$, this step of the argument is also valid with $\alpha^{(1)} = \alpha - (n_0+1)$. This should be taken into account in the computation of the following $\alpha^{(j)}$ and $\beta^{(j)}$ when $j \geq 2$.

[Div_1] Dividing the latter expression by $(x+a_1)^{\alpha^{(1)}}$, which does not vanish in I, we get

$$F^{(1)}(x) = P^{1,1}(x) + P^{2,1}(x)\left(\frac{x+a_2}{x+a_1}\right)^{\alpha^{(1)}} + \cdots + P^{K,1}(x)\left(\frac{x+a_K}{x+a_1}\right)^{\alpha^{(1)}}.$$

From Rolle's Theorem it follows that

$$\mathcal{Z}\left(F^{(0)}\right) \leq \mathcal{Z}\left(F^{(1)}\right) + (n+1).$$

[Der_2] Applying again $\mathcal{D}^{n+1}$ and having in mind Lemma 2.2 we obtain

$$P^{2,2}(x)\frac{(x+a_2)^{\alpha^{(2)}}}{(x+a_1)^{\beta^{(2)}}} + P^{3,2}(x)\frac{(x+a_3)^{\alpha^{(2)}}}{(x+a_1)^{\beta^{(2)}}} + \cdots + P^{K,2}(x)\frac{(x+a_K)^{\alpha^{(2)}}}{(x+a_1)^{\beta^{(2)}}},$$

where $\alpha^{(2)} = \alpha^{(1)} - (n+1) = \alpha - 2(n+1)$, $\beta^{(2)} = \alpha$ and again $P^{2,2}(x), \ldots, P^{K,2}(x)$ are polynomials of degree n.

[Div_2] Dividing this expression by $(x+a_2)^{\alpha^{(2)}}/(x+a_1)^{\beta^{(2)}}$, which has no zeroes in I, and applying the same argument as in [Div_1] one gets

$$F^{(2)}(x) = P^{2,2}(x) + P^{3,2}(x)\left(\frac{x+a_3}{x+a_2}\right)^{\alpha^{(2)}} + \cdots + P^{K,2}(x)\left(\frac{x+a_K}{x+a_2}\right)^{\alpha^{(2)}}$$

satisfying that

$$\mathcal{Z}\left(F^{(0)}\right) \leq \mathcal{Z}\left(F^{(1)}\right) + (n+1) \leq \mathcal{Z}\left(F^{(2)}\right) + 2(n+1).$$

$\vdots$

[Der_K] We reach the expression $P^{K,K}(x)\dfrac{(x+a_K)^{\alpha^{(K)}}}{(x+a_{K-1})^{\beta^{(K)}}}$ where

$$\begin{aligned}\alpha^{(K)} &= \alpha^{(K-1)} - (n+1) = \alpha - K(n+1),\\ \beta^{(K)} &= \alpha^{(K-1)} + (n+1) = \alpha - (K-1)(n+1).\end{aligned}$$

[Div_K] Finally, dividing by $(x+a_K)^{\alpha^{(K)}}/(x+a_{K-1})^{\beta^{(K)}}$ we obtain $F^{(K)}(x) = P^{K,K}(x)$, a polynomial of degree n. Now we have:

$$\mathcal{Z}\left(F^{(0)}\right) \leq \mathcal{Z}\left(F^{(K)}\right) + K(n+1) = \mathcal{Z}\left(P^{K,K}\right) + K(n+1).$$

This ends the proof of Theorem 1.1 because $\mathcal{Z}\big(P^{K,K}\big) \le n$ in I and $F^{(0)} = F$, and so

$$\mathcal{Z}(F) \le n + K(n+1) = m.$$

□

Remark 2.3. *As a consequence of the proof above we get that* Span $\mathcal{F}$ *belongs to the kernel of the differential operator*

$$\mathcal{D}^n\mathcal{D}_K \cdots \mathcal{D}^n\mathcal{D}_2\mathcal{D}^n\mathcal{D}_1\mathcal{D}^{n_0+1},$$

being

$$\mathcal{D}_1 f = \frac{d}{dx}\left(\frac{1}{(x+a_1)^{\alpha(1)}} f(x)\right) \quad \textit{and} \quad \mathcal{D}_m f = \frac{d}{dx}\left(\frac{(x+a_{m-1})^{\beta(m)}}{(x+a_m)^{\alpha(m)}} f(x)\right), \ m \ge 2.$$

The operator $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)$ *is known as the relative differentiation of f with respect to g. This point of view, derived from the classical Chebyshev theory* [12, 17], *could also provide a way of writing the proof of Theorem 1.1 which would be essentially equivalent to the one presented in this section. A short discussion about the relation between both approaches, using a concrete example, is presented in the next section.*

2.1. **Relation between our approach and the classical Chebyshev theory.** The proof of Theorem 1.1 is based on the Derivation-Division algorithm because, in its turn, appears to be useful as well to prove some results in Sections 3 and 4, devoted to the general framework of ET-systems and ET-systems with positive accuracy.

In [10, Theorem 1.2, Chap. XI] it is shown that, if there exist functions w_i having constant sign on a given closed interval $[a, b]$, the kernel of the differential operator $\mathcal{D}_n\mathcal{D}_{n-1}\cdots\mathcal{D}_1\mathcal{D}_0$, where $\mathcal{D}_i$ are the relative differentiations $\mathcal{D}_i(f) = \mathcal{D}(f/w_i)$, is an ET-system on this interval and a basis can be chosen in such a way that it is a complete ET-system. We recall that an ordered set of $\mathcal{C}^n[a, b]$ functions $[u_0, u_1, \ldots, u_n]$ defined on an interval $[a, b]$ is called a complete ET-system if $\{u_0, u_1, \ldots, u_k\}$ is an ET-system on $[a, b]$ for all $k = 0, \ldots, n$. This property is equivalent to the following one: an ordered set

$$\mathcal{F} = [u_0(x), u_1(x), \ldots, u_n(x)]$$

is a complete ET-system on a closed interval $[a, b]$ if and only if it can be written through the integral representation

$$\begin{aligned}
u_0(x) &= w_0(x),\\
u_1(x) &= w_0(x)\int_a^x w_1(s_1)\,ds_1,\\
u_2(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)\,ds_2\,ds_1,\\
&\ \ \vdots\\
u_n(x) &= w_0(x)\int_a^x w_1(s_1)\int_a^{s_1} w_2(s_2)\cdots\int_a^{s_{n-1}} w_n(s_n)\,ds_n\cdots ds_1,
\end{aligned} \tag{7}$$

with w_k non-vanishing $\mathcal{C}^{n-k}[a, b]$ functions for each $k = 0, \ldots, n$. The existence of such integral representation is also equivalent to the fact that $W(u_0, u_1, \ldots, u_k) \ne 0$, $k = 0, \ldots, n$ where W denotes the corresponding wronskians.

To clarify the relations between our approach in this paper and the above one we restrict our attention to a simple family. Consider

$$\mathcal{F}=\left\{\sqrt{x}, x\sqrt{x}, x^2\sqrt{x}, \sqrt{x+1}, x\sqrt{x+1}, x^2\sqrt{x+1}\right\}$$

on the interval $(0,\infty)$. Notice that Theorem 1.1 asserts that $\mathcal{F}$ is an ET-system.

To use the approach of [10] it is convenient to consider the equivalent family

$$\mathcal{F}'=\left\{1, x, x^2, \frac{\sqrt{x+1}}{\sqrt{x}}, x\frac{\sqrt{x+1}}{\sqrt{x}}, x^2\frac{\sqrt{x+1}}{\sqrt{x}}\right\}.$$

Although this result does not apply directly since the interval is open, similar ideas can be adapted. In particular the value a in (7) could be chosen at each step.

We consider the integral representation (7), choosing the positive weights functions $w_5 = w_4 = 1$, $w_3 = x^{-7/2}(1+x)^{-5/2}$ and $w_2 = w_1 = w_0 = 1$ suggested by the Division-Derivation algorithm performed in the proof of Theorem 1.1 (see Remark 2.3). The following functions are attained

$$\begin{aligned} u_0 &= 1, & u_3 &= -\frac{8}{15}\frac{\sqrt{1+x}}{\sqrt{x}}\left(1+12x+16x^2\right), \\ u_1 &= x, & u_4 &= \frac{8}{3}\frac{\sqrt{1+x}}{\sqrt{x}}\left(x+2x^2\right), \\ u_2 &= \frac{1}{2}x^2, & u_5 &= -\frac{4}{3}\frac{\sqrt{1+x}}{\sqrt{x}}x^2. \end{aligned}$$

So

$$\left[\sqrt{x}, x\sqrt{x}, x^2\sqrt{x}, \left(1+12x+16x^2\right)\sqrt{1+x}, \left(x+2x^2\right)\sqrt{1+x}, x^2\sqrt{1+x}\right]$$

is a complete ET-system on $(0,\infty)$, proving again that $\mathcal{F}$ is an ET-system on this interval. It is clear that this approach works, in general, for a system of type $\mathcal{F}$ and that the difficulty of its implementation is comparable to the one presented in this paper.

On the other hand, the integral representation (7) can be used in a slightly different way: we seek for functions $w_i's$ such that from (7) give rise to the family $\mathcal{F}'$, that is, $u_0 = 1$, $u_1 = x$, $\dots$, $u_5 = x^2\sqrt{x+1}/\sqrt{x}$. Thus, one obtains

$$\begin{aligned} w_0 = {}& 1, \; w_3 = & -\frac{3}{16}\frac{8x^2+12x+5}{(1+x)^{5/2}\,x^{7/2}}, \\ w_1 = {}& 1, \; w_4 = & \frac{-(16x^2+20x+5)}{(8x^2+12x+5)^2}, \\ w_2 = {}& 2, \; w_5 = & \frac{-10(8x^2+12x+5)}{(5+16x^2+20x)^2}. \end{aligned}$$

It is easy to prove that these functions do not vanish at $(0,\infty)$ and, consequently, showing that

$$\left[\sqrt{x}, x\sqrt{x}, x^2\sqrt{x}, \sqrt{x+1}, x\sqrt{x+1}, x^2\sqrt{x+1}\right]$$

is a complete ET-system. Although this point of view leads to a stronger result, it is not always clear that, when applying it to a general family $\mathcal{F}$, the functions w_i are non-vanishing in the considered interval.

Summarising, the approach based on the Derivation-Division algorithm is essentially equivalent to the one derived from the classical theory of ET-systems. While the main difficulty in the former one is to choose, at each step, convenient division factors, the principal problem in the latter one is to find suitable weights for the relative differentiations which constitute a good differential operator.

An important advantage of the Derivation-Division viewpoint is that it can also be applied to prove that some families $\mathcal{G}$ are ET-system with positive accuracy. In Sections 3 and 4 some natural families $\mathcal{G}$, subsets of systems of the form $\mathcal{F}$, will be proved to be no ET-systems. Examples of this fact are the family considered in Theorem 4.1 and the results provided by Proposition 3.4, where the family

$$\left\{1, x, \sqrt{x+1}, x\sqrt{x+1}, \sqrt{x}, x\sqrt{x}, x^2\sqrt{x}\right\}$$

is proved to be an ET-system with accuracy 1.

3. Necessity of the conditions of Theorem 1.1

The main conditions in the theorem for proving that family $\mathcal{F}$ is an ET-system are that all the polynomials P^j, $j = 1, \ldots, n$, appearing in (3) have the same degree n and that all the exponents are equal to α. The aim of this section is to prove the necessity of such conditions. This will be done by introducing sets $\mathcal{G}$ of the form (1) which will be ET-systems with accuracy 1. The set in Proposition 3.1 presents different exponents and different degrees, the one in Proposition 3.3 has different exponents, but the same degrees, and the set in Proposition 3.4 has the same exponents and different degrees. All the examples correspond to $K \geq 2$. Note that for those cases, after a shift and a rescaling in x, it is not restrictive to assume $a_1 = 1$ and $a_2 = 0$.

Proposition 3.1. *The family* $\left\{1, x, x^2, \sqrt[3]{x+1}, \sqrt{x}, x\sqrt{x}\right\}$ *is an ET-system with accuracy* 1 *on* $(0, \infty)$.

Proof. Consider the function

$$F(x) = b_0 + b_1 x + b_2 x^2 + c_0\sqrt[3]{x+1} + (d_0 + d_1 x)\sqrt{x}.$$

It is not restrictive to assume $b_2 = 1$. Dividing $F(x)$ for the non zero function $\sqrt[3]{x+1}$ and then differentiating with respect to x we obtain

$$\frac{d}{dx}\left(\frac{F(x)}{\sqrt[3]{x+1}}\right) = -\frac{3d_0 + (9d_1 + d_0)x + 7d_1x^2 + \left((6b_1 - 2b_0) + (12 + 4b_1)x + 10x^2\right)\sqrt{x}}{6(\sqrt[3]{x+1})^4\sqrt{x}}.$$

Using Theorem 1.1 the previous function has at most 5 zeroes and so the function $F(x)$ has at most 6. This ensures that the family of the statement is an ET-system with accuracy at most 1. The proof ends, for instance, showing that there exist some values b_0, b_1, c_0, d_0, d_1 such that F has a zero of multiplicity 6 in $(0, \infty)$. This is the situation taking $x_0 = 15 - 12\sqrt{70}/7$

and

$$\begin{aligned}
b_0 &= -\frac{9}{7}(7507 - 900\sqrt{70}),\\
b_1 &= \frac{12}{7}(217 - 25\sqrt{70}),\\
c_0 &= \frac{144}{343}(973 - 118\sqrt{70})\sqrt[3]{19796 - 2352\sqrt{70}},\\
d_0 &= -\frac{150}{49}(67 - 8\sqrt{70})\sqrt{735 - 84\sqrt{70}},\\
d_1 &= -\frac{50}{49}\sqrt{735 - 84\sqrt{70}}.
\end{aligned}$$

These values have been obtained studying the non-linear system $F(x_0) = F'(x_0) = \cdots = F^{(5)}(x_0) = 0$, with unknowns $x_0, b_0, b_1, c_0, d_0, d_1$, and finally checking that for the values obtained, $F^{(6)}(x_0) = 49(34780 + 4157\sqrt{70})/256608 \neq 0$. □

Remark 3.2. *In the above proposition and in other proofs along this paper we show that the upper bound predicted with our approach is reached with a function having a zero with the highest multiplicity. Based on this function we can use a standard approach to obtain other examples with the maximum number of simple zeroes. For instance, in the example above it can also be proved the existence of functions inside the family with all the possible configurations of zeroes with upper bound* 6 *(taking into account their multiplicities). Provided we fix the previous values for b_i, c_i, d_i and x_0 we define the auxiliary perturbed function*

$$\widetilde{F}(x, \widetilde{b}, \widetilde{c}, \widetilde{d}) = b_0 + \widetilde{b}_0 + (b_1 + \widetilde{b}_1)x + x^2 + (c_0 + \widetilde{c}_0)\sqrt[3]{x+1} + (d_0 + \widetilde{d}_0 + (d_1 + \widetilde{d}_1)x)\sqrt{x},$$

where $x = x_0 + \widetilde{x}$ and $\widetilde{x}$, $\widetilde{b}_0$, $\widetilde{b}_1$, $\widetilde{c}_0$, $\widetilde{d}_0$ and $\widetilde{d}_1$ small enough. The existence of a versal unfolding of the unperturbed map is guaranteed by the fact that the Jacobian matrix of the map $G := (\widetilde{F}, \frac{\partial}{\partial \widetilde{x}}\widetilde{F}, \ldots, \frac{\partial^5}{\partial \widetilde{x}^5}\widetilde{F})$ with respect to $(\widetilde{x}, \widetilde{b}_0, \widetilde{b}_1, \widetilde{c}_0, \widetilde{d}_0, \widetilde{d}_1)$, evaluated at 0*, has a non-vanishing determinant. This can be checked by straightforward computations.*

Proposition 3.3. *The family $\{1, (x+1)^{\alpha_1}, x^{\alpha_2}\}$ is an ET-system with accuracy either* 0 *or* 1 *on $(0, \infty)$. Moreover, the accuracy is exactly* 1 *if and only if $\alpha_1 \neq \alpha_2$ and $x_0 := (\alpha_2 - 1)/(\alpha_1 - \alpha_2) > 0$.*

Proof. We will prove the first assertion checking that the function

$$F(x) = c_0 + c_1(x+1)^{\alpha_1} + c_2 x^{\alpha_2}$$

has at most three zeroes in the interval $(0, \infty)$. We will start assuming that $c_2 = 1$. Differentiating with respect to x, dividing by $(x+1)^{\alpha_1 - 1}$ and differentiating once more with respect to x we obtain

$$\frac{d}{dx}\left(\frac{1}{(x+1)^{\alpha_1-1}}F'(x)\right) = \frac{((\alpha_2 - \alpha_1)x + \alpha_2 - 1)\alpha_2}{x^{2-\alpha_2}(x+1)^{\alpha_1}}.$$

Since this last function has at most one zero and we have made two derivatives, by Rolle's Theorem, $F(x)$ has at most 3 zeroes and the family of the statement is an ET-system with accuracy at most 1.

When $\alpha_1 = \alpha_2$ or $x_0 \leq 0$ the family is an ET-system because the unique zero of $\widetilde{F}$, when exists, it is non-positive. The proof ends checking that when $\alpha_1 \neq \alpha_2$ and $x_0 > 0$

the function F has a zero of multiplicity 3 at $x = x_0$ when $c_0 = x_0^{\alpha_2-1}(\alpha_1-\alpha_2)^{2(\alpha_2-1)}$ and $c_1 = -c_0\alpha_2((\alpha_1-1)/(\alpha_1-\alpha_2))^{1-\alpha_1}$. □

Proposition 3.4. *The family* $\{1, x, \sqrt{x+1}, x\sqrt{x+1}, \sqrt{x}, x\sqrt{x}, x^2\sqrt{x}\}$ *is an ET-system with accuracy* 1 *on* $(0,\infty)$.

Proof. We can express any function of this family as

$$F(x) = b_0 + b_1x + (c_0 + c_1x)\sqrt{x+1} + \left(d_0 + d_1x + d_2x^2\right)\sqrt{x}.$$

It is not restrictive to consider $d_2 = 1$ because for $d_2 = 0$ Theorem 1.1 applies and, therefore, this family is already an ET-system. Now we perform the Derivation-Division algorithm step by step. First, we differentiate $F(x)$ two times with respect to x and we obtain

$$F_1(x) = F''(x) = \left(c_1 - \frac{1}{4}c_0 + \frac{3}{4}c_1x\right)\frac{1}{\left(\sqrt{x+1}\right)^3} + \left(-\frac{1}{4}d_0 + \frac{3}{4}d_1x + \frac{15}{4}x^2\right)\frac{1}{\left(\sqrt{x}\right)^3}.$$

Multiplying by $\left(\sqrt{x+1}\right)^3$ and differentiating two times more we obtain

$$F_2(x) = F_1''(x) = \frac{3}{16}\left(40x^4 + 20x^3 - 5x^2 + (3d_1 - 4d_0)x - 5d_0\right)\frac{1}{\sqrt{x+1}\left(\sqrt{x}\right)^7}.$$

Using the Descartes' rule (the first two coefficients of the numerator of the above expression have the same sign) $F_2(x)$ has at most three positive zeroes and so then, applying Rolle's Theorem, $F(x)$ has at most seven zeroes. The proof ends showing that, for instance, the function

$$F(x) = \frac{115100470551}{3020223088} + \frac{71810410560}{2439594757}x - \left(\frac{96706333051}{2537560854} - \frac{385291662549}{37064636504}x\right)\sqrt{x+1} + \left(\frac{8570393}{7872821176} + \frac{3625291351}{34312028120}x + x^2\right)\sqrt{x}$$

has exactly 7 simple zeroes which are located in the intervals

$$\left(\frac{1}{111}, \frac{1}{100}\right), \left(\frac{1}{50}, \frac{1}{33}\right), \left(\frac{1}{20}, \frac{1}{17}\right), \left(\frac{7}{100}, \frac{2}{25}\right), \left(\frac{1}{10}, \frac{1}{9}\right), \left(\frac{3}{25}, \frac{1}{8}\right), \left(\frac{3}{20}, \frac{4}{25}\right)$$

and are, approximately, 0.00964, 0.0249, 0.0507, 0.0733, 0.102, 0.123 and 0.150. □

4. Some more related families and ET-systems with high accuracy

As we have already said, when the polynomials $P^j(x), j > 1$ in expression (3),

$$F(x) = P^0(x) + \sum_{j=1}^{K} P^j(x)\,(x + a_j)^\alpha,$$

have not all the same degree, Theorem 1.1 also applies considering n as the maximum of the degrees of P^j. However, the associated upper bound is not necessarily attained. In this section we study two families. In the first one the predicted estimate is reached and moreover it provides examples of ET-systems with accuracy k, for $k = 1, 2, \ldots, 16$. In the second one we prove that this estimate is not reached.

Theorem 4.1. *Consider the family*

$$\widehat{\mathcal{F}} = \bigcup_{i=0}^{2k-1} \left\{\sqrt{x+a_i}\right\} \cup \bigcup_{i=k}^{2k-1} \left\{x\sqrt{x+a_i}\right\},$$

defined on the interval $I=(a,\infty)$, where a_i, $i=0,\dots,2k-1$, are different real numbers and $a=\max_{i=0,\dots,2k-1}\{-a_i\}$. Then $\mathcal{Z}(\widehat{\mathcal{F}}) \le 4k-1$. Moreover, for any k, $2\le k\le 16$, there exist a_i such that $\widehat{\mathcal{F}}$ is an ET-system with accuracy k on I.

Proof. The family $\widehat{\mathcal{F}}$ can be considered as a subfamily of the one introduced in Theorem 1.1 obtained by taking $K=2k$, $\alpha=1/2$, $n=1$ and with no pure polynomial part ($n_0=-1$). Therefore any non-zero function in $\widehat{\mathcal{F}}$ has at most $4k-1$ zeroes taking into account their multiplicities. This proves the first assertion. Concerning the second one, we take all $a_i>0$ and we seek for an element F in $\operatorname{Span}\widehat{\mathcal{F}}$ having a zero of multiplicity $4k-1$ at, for instance, $x=0$. Notice that once this function F is found, $\mathcal{Z}(F)=4k-1=(m-1)+k$, where $m=3k$ is the number of generators of $\widehat{\mathcal{F}}$ and so, for the corresponding values of $a_i, i=0,\dots,2k-1$, the family $\widehat{\mathcal{F}}$ is an ET-system with accuracy k as we wanted to prove.

To achieve this goal it will be convenient to consider first a function of the form

$$F(x)=\sum_{\ell=0}^{2k-1}(b_\ell+c_\ell x)\sqrt{x+a_\ell},$$

with $c_{2k-1}=1$, which observe that it is not necessarily in $\operatorname{Span}\widehat{\mathcal{F}}$. In fact $F\in\operatorname{Span}\widehat{\mathcal{F}}$ only when $c_\ell=0, \ell=0,1,\dots,k-1$.

The method that we use for obtaining a function $F\in\operatorname{Span}\widehat{\mathcal{F}}$ with the origin of multiplicity $4k-1$ is the following: first, for each fixed $a=(a_0,a_1,\dots,a_{2k-1})$, we consider the linear system of $4k-1$ equations

$$F(0)=F'(0)=\cdots=F^{(4k-2)}(0)=0, \tag{8}$$

with $4k-1$ unknowns $b_0,c_0,b_1,c_1,\dots,b_{2k-2},c_{2k-2},b_{2k-1}$. Let $b_i=b_i(a)$ and $c_i=c_i(a)$ denote the corresponding solution. If we are able to find a point $a^*\in\mathbb{R}^{2k}$ with all the components positive and such that all $c_i(a^*)=0, i=0,1,\dots,k-1$ then we are done.

Although the above procedure is quite simple, we arrive very fast to computational difficulties when we want to apply it for k bigger than 4. To clarify this approach we start studying with detail the case $k=2$. Solving the 7×7 linear system (8) we obtain b_0,b_1,b_2,b_3 and c_0,c_1 and c_2 in terms of $a=(a_0,a_1,a_2,a_3)$. We only show the expressions of $c_0=c_0(a)$ and $c_1=c_1(a)$,

$$c_0=\frac{\prod\limits_{\substack{i<j\\ i\neq 0, j\neq 0}}(a_i-a_j)^4\prod\limits_{\substack{i=0\\ i\neq 0}}^{3}(a_0-a_i)}{a_0^{\frac{11}{2}}a_1^9a_2^9a_3^9}E_0(a),\quad c_1=\frac{\prod\limits_{\substack{i<j\\ i\neq 1, j\neq 1}}(a_i-a_j)^4\prod\limits_{\substack{i=0\\ i\neq 1}}^{3}(a_1-a_i)}{a_1^{\frac{11}{2}}a_0^9a_2^9a_3^9}E_1(a),$$

where

$$
\begin{aligned}
E_0(a) &= -\frac{7\,(a_0^2(3a_0+a_1)+a_0(a_0-5a_1)(a_2+a_3)+(9a_1-5a_0)a_2a_3)}{1073741824},\\
E_1(a) &= \frac{7\,(a_1^2(3a_1+a_0)+a_1(a_1-5a_0)(a_2+a_3)+(9a_0-5a_1)a_2a_3)}{1073741824}.
\end{aligned} \tag{9}
$$

Defining as new variables the symmetric polynomials $S_1 = S_1(a_2, a_3) = a_2 + a_3$ and $S_2 = S_2(a_2, a_3) = a_2a_3$ we obtain that system $\{c_0(a) = 0,\ c_1(a) = 0\}$, for positive a_i, with $a_i \neq a_j, i \neq j$, is equivalent to

$$
\begin{aligned}
&a_0^2(3a_0+a_1)+a_0(a_0-5a_1)S_1+(9a_1-5a_0)S_2 = 0,\\
&a_1^2(3a_1+a_0)+a_1(a_1-5a_0)S_1+(9a_0-5a_1)S_2 = 0.
\end{aligned} \tag{10}
$$

The solution of this 2×2 new linear system is

$$
S_1 = -\frac{(a_0+a_1)(9a_0^2-2a_0a_1+9a_1^2)}{3a_0^2-22a_0a_1+3a_1^2}, \quad S_2 = -\frac{(5a_0^2+6a_0a_1+5a_1^2)a_0a_1}{3a_0^2-22a_0a_1+3a_1^2}.
$$

Then the values a_2 and a_3 such that $c_0 = c_1 = 0$ are the roots of the polynomial

$$
P(\lambda) = S_2 - S_1\lambda + \lambda^2
$$

whose discriminant is

$$
\frac{(3a_1+a_0)(3a_0+a_1)(27a_1^2+26a_0a_1+27a_0^2)(a_0-a_1)^2}{(3a_1^2-22a_0a_1+3a_0^2)^2}.
$$

Then, if $-3a_1^2+22a_0a_1-3a_0^2 > 0$ the roots of the above polynomial are both positive. These roots give explicit values for a_2 and a_3 in terms of a_0 and a_1 for which the origin is a zero of F of multiplicity $4k-1 = 7$, as we wanted to prove. Starting from this function F it can also be checked, as in Remark 3.2, that there are functions in $\widehat{\mathcal{F}}$, with all the possible configurations of zeroes with total multiplicity 7.

In general, we have obtained the following structure for $c_\ell = c_\ell(a)$, $\ell = 0, \ldots, k-1$.

$$
c_\ell = \frac{\displaystyle\prod_{\substack{0\le i<j\le 2k-1\\ i\neq\ell, j\neq\ell}} (a_i-a_j)^4 \prod_{i=0, i\neq\ell}^{2k-1} (a_\ell - a_i)}{\displaystyle a_\ell^{-4k+\frac{9}{2}} \prod_{i=0}^{2k-1} a_i^{8k-7}} E_\ell(a_0,\ldots,a_{2k-1}) \tag{11}
$$

where for each fixed $a_0, \ldots, a_{k-1}$ and each ℓ, the polynomial $E_\ell(a_0, \ldots, a_{2k-1})$ is a symmetric polynomial in the variables $(a_k, \ldots, a_{2k-1})$. So it can be expressed in terms of the elementary symmetric functions $S_0 = 1, S_1 = a_k + \cdots + a_{2k-1}, \ldots, S_k = a_k \cdots a_{2k-1}$, as

$$
E_\ell(a_0,\ldots,a_{2k-1}) = G_{\ell,0}(a_0,\ldots,a_{k-1})S_0 + G_{\ell,1}(a_0,\ldots,a_{k-1})S_1 + \cdots + G_{\ell,k}(a_0,\ldots,a_{k-1})S_k, \tag{12}
$$

for $\ell = 0, \ldots, k-1$. Note that this expression for any k, corresponds to the one given in (9) for $k = 2$.

Unfortunately, for $k > 2$, we have not been able to obtain the general expressions of the polynomials $G_{\ell,j}$. Indeed, for $k > 4$, even for given values of $a_0, \ldots, a_{k-1}$ we neither have been able to obtain the corresponding values $G_{\ell,j}$. So, for $2 \le k \le 16$, we have decided to follow the next procedure to obtain our function F with a zero at the origin of the highest multiplicity.

For each ℓ, we fix $a_i=(i+1)^k$, $i=0,\ldots,k-1$ and we take M points $z_m=(a_{k,m},\ldots,a_{2k-1,m})$, $m=0,\ldots,M-1$, with positive rational entries and such that $a_{i,m}\neq a_{j,m}$ when $i\neq j$. Then for $a=(1,2^k,\ldots,k^k,a_{k,m},\ldots,a_{2k-1,m})$ we solve the system (8) to obtain the concrete values $c_\ell(1,2^k,\ldots,k^k,a_{k,m},\ldots,a_{2k-1,m})$ and therefore, by using (11), the corresponding ones $E_\ell(1,2^k,\ldots,k^k,a_{k,m},\ldots,a_{2k-1,m})=:e_{\ell,m}$. Since we also know $S_\ell(a_{k,m},\ldots,a_{2k-1,m})=:s_{\ell,m}$ we obtain that the values of $g_{\ell,0}:=G_{\ell,0}(1,2^k,\ldots,k^k),\ldots,g_{\ell,k}:=G_{\ell,k}(1,2^k,\ldots,k^k)$ satisfy the M equations

$$s_{0,m}g_{\ell,0}+s_{1,m}g_{\ell,1}+\cdots+s_{k,m}g_{\ell,k}=e_{\ell,m},\quad m=0,\ldots,M-1. \tag{13}$$

Although some of them can coincide, taking more points z_m, if necessary, for each ℓ we can obtain an over determined linear system (13) with $k+1$ unknowns $g_{\ell,0},\ldots,g_{\ell,k}$ and, for instance, $2k$ equations.

For example for $k=3$, taking the values

$$\begin{aligned}
z_0 &= (2/5,3/5,4/5), & z_1 &= (2/5,3/5,1/5), & z_2 &= (4/5,2/5,1/5),\\
z_3 &= (3/5,1/5,4/5), & z_4 &= (2/7,5/7,6/7), & z_5 &= (2/7,3/7,4/7),
\end{aligned}$$

we get the 6 linear equations

$$\begin{pmatrix} 1 & \frac{9}{5} & \frac{26}{25} & \frac{24}{125}\\ 1 & \frac{6}{5} & \frac{11}{25} & \frac{6}{125}\\ 1 & \frac{7}{5} & \frac{14}{25} & \frac{8}{125}\\ 1 & \frac{8}{5} & \frac{19}{25} & \frac{12}{125}\\ 1 & \frac{13}{7} & \frac{52}{49} & \frac{60}{343}\\ 1 & \frac{9}{7} & \frac{26}{49} & \frac{24}{343} \end{pmatrix} \begin{pmatrix} g_{\ell,0}\\ g_{\ell,1}\\ g_{\ell,2}\\ g_{\ell,3} \end{pmatrix} = \begin{pmatrix} e_{\ell,0}\\ e_{\ell,1}\\ e_{\ell,2}\\ e_{\ell,3}\\ e_{\ell,4}\\ e_{\ell,5} \end{pmatrix},$$

where

$$\begin{aligned}
e_{0,0} &= -219837158541/96714065569170333976494080000,\\
e_{0,1} &= -11436123699/604462909807314587353088000,\\
e_{0,2} &= -80903135313/2417851639229258349412352000,\\
e_{0,3} &= -16707794103/604462909807314587353088000,\\
e_{0,4} &= -82555087329/379119137031147709187856793 6,\\
e_{0,5} &= -2180333727/118474730322233659121205248
\end{aligned}$$

and we omit the values $e_{\ell,m}$, with $\ell\geq 1$ for the sake of brevity. Then solving the above linear system for $\ell=0$ we obtain that $(g_{0,0},g_{0,1},g_{0,2},g_{0,3})=(521,-885,1249,-1613)C$ where $C=42513471/1547425049106725343623905 28$. Similarly we obtain the remaining values $g_{\ell,m}$ for $\ell=1,2$.

Then the linear system equivalent to system (10), but for $k=3$ and $(a_0,a_1,a_2)=(1,8,27)$ is

$$\begin{pmatrix} 521 & -885 & 1249 & -1613\\ 143872 & -52032 & 10760 & -1877\\ 23521185 & -150903 & 21087 & -1769 \end{pmatrix} \begin{pmatrix} 1\\ S_1\\ S_2\\ S_3 \end{pmatrix} = \begin{pmatrix} 0\\0\\0 \end{pmatrix}.$$

Solving it we obtain the values S_1, S_2 and S_3 and construct the polynomial

$$H_{1,8,27}(\lambda) := \frac{123021703800}{376124569} - \frac{183145920859}{376124569}\lambda + \frac{34475941044}{376124569}\lambda^2 - \lambda^3, \tag{14}$$

that has three real roots. Therefore taking (a_3, a_4, a_5) as these roots, $(a_0, a_1, a_2) = (1, 8, 27)$ and b_ℓ and c_ℓ satisfying system (8) we obtain that $x = 0$ is a zero of multiplicity $4k-1 = 11$ of F, as desired. We remark that not all choices of (a_0, a_1, a_2) provide a polynomial $H_{a_0,a_1,a_2}(\lambda)$, like in (14), with three real positive roots.

The cases $4 \leq k \leq 16$ are treated with the same procedure and we get polynomials $H_{1,2^k,\ldots,k^k}(\lambda)$ with k real positive roots. We omit the details. $\square$

Remark 4.2. *It is natural to believe that Theorem 4.1 holds for any $k \geq 1$, but our approach only works for a given k. We have decided to stop at $k = 16$ because the computations take more than one hour of CPU time, but bigger k could also be treated.*

In fact the main goal of this result is to show that for a fixed α the hypothesis of Theorem 1.1 forcing that all the degrees of the polynomials P^j, $j \geq 1$, coincide is essential and moreover that, when it is not assumed, arbitrarily high accuracy may happen. In particular it shows that the control of the zeroes of functions in (1) is quite complicated.

Proposition 4.3. *The family $\left\{\sqrt{x+a}, \sqrt{x+b}, \ldots, x^n\sqrt{x+b}\right\}$ is an ET-system on the interval $I = (\max\{-a,-b\}, \infty)$.*

Proof. Any function of the family can be expressed as

$$F(x) = P_0^1(x)\sqrt{x+a} + P_n^2(x)\sqrt{x+b}$$

where $P_0^1(x)$ and $P_n^2(x)$ are polynomials of degree 0 and n, respectively. To prove the statement it suffices to see that $\mathcal{Z}(F) \leq n+1$ in I. That $\mathcal{Z}(F) \geq n+1$ follows from Lemma 2.1. We will differentiate the function $G(x) = F(x)/\sqrt{x+a}$ and we will show that the polynomial, of degree $n+1$, obtained using Lemma 2.2, has only n zeroes in I. Therefore, F will have at most $n+1$ zeroes. Notice that this upper bound is neither the $n+2$ predicted by the degree of the polynomial obtained by the Derivation-Division procedure nor the $2n+1$ that we would obtain applying Theorem 1.1.

We start considering $a < b$. It is not restrictive to assume $a = 0$ and so we focus our interest in the positive zeroes. Then,

$$\begin{aligned} G'(x) =& \frac{d}{dx}\left(P_0^1(x) + P_n^2(x)\frac{\sqrt{x+b}}{\sqrt{x}}\right) = \frac{d}{dx}\left(\sum_{i=0}^{n} c_i x^i \frac{\sqrt{x+b}}{\sqrt{x}}\right) = \\ & \sum_{j=0}^{n+1} d_j x^j \frac{1}{\sqrt{x^3}\sqrt{x+b}} = Q_{n+1}(x)\frac{1}{\sqrt{x^3}\sqrt{x+b}} \end{aligned}$$

where $Q_{n+1}(x) = d_{n+1}x^{n+1} + \cdots + d_1 x + d_0$ with

$$d_j = \begin{cases} nc_n & \text{if } j = n+1, \\ (j-1)c_{j-1} + \left(j - \frac{1}{2}\right) c_j b & \text{if } 1 \leq j \leq n, \\ -\frac{1}{2}c_0 b & \text{if } j = 0. \end{cases} \tag{15}$$

Since we are only interested on the number of zeroes of Q_{n+1} it is also non restrictive to assume $c_n > 0$. Using Descartes' rule, a necessary condition for the polynomial $Q_{n+1}(x)$ to have $n+1$

positive zeroes is that the number of sign changes of its coefficients is exactly $n+1$, that is $d_j d_{j+1} < 0$ for $j = 0, \dots, n$. From (15) this implies that $c_j c_{j+1} < 0$ for $j = 2, \dots, n-1$ which will lead us to a contradiction. Let us do it for n even. The odd case follows analogously. Indeed $\operatorname{sgn}(d_j) = \operatorname{sgn}(c_{j-1})$, for $j = n, \dots, 2$ and $\operatorname{sgn}(d_1) = \operatorname{sgn}(c_1)$. In particular this implies $d_2 < 0$ and $c_1 < 0$. Then $d_1 = c_1 b/2 < 0$ which is contradictory with the condition $d_1 d_2 < 0$. Thus we have, at least, two consecutive coefficients d_j with the same sign and therefore the polynomial $Q_{n+1}(x)$ has, at most, n positive zeroes.

For the other case, $a > b$ (which is not symmetric to the case $a < b$ because P_0^1 and P_n^2 have different degrees), we can assume that $b = 0$ as well. It evolves as above, except that relation (15) now reads as follows:

$$d_j = \begin{cases} n c_n & \text{if } j = n+1, \\ (j-1)c_{j-1} + \left(j + \frac{1}{2}\right) c_j a & \text{if } 1 \le j \le n, \\ \frac{1}{2} c_0 a & \text{if } j = 0. \end{cases}$$

□

5. Some applications to nonlinear planar vector fields

Problems where families of functions (1) appear could be found in the literature. A first example of this relates to the paper [9]. In that work, among other results, the authors provide an upper bound for the maximum number of limit cycles that could have a family of vector fields F_λ possessing a generic algebraic polycycle of four hyperbolic equilibrium points q_j, $j = 1, \dots, 4$. To do it they consider the displacement function $\pi_\lambda - \mathrm{id}$, where π_λ is the first-return Poincaré map associated to a transversal section σ. This map π_λ is obtained as the composition of the corresponding Poincaré maps in a vicinity of the hyperbolic points. The problem of seeking zeroes of this displacement function is reduced to the one of studying the number of zeroes (counting multiplicity) of functions of type

$$\Delta_\lambda(x) = (((x^{r_1} + a_1)^{r_2} + a_2)^{r_3} + a_3)^{r_4} + a_4 - x(1 + \cdots)$$

where $a_j = a_j(\lambda)$ are constants. The dots denote an analytic function decreasing to 0 at infinity faster than any power of $1/x$ and $r_j(\lambda) = |\mu_+/\mu_-|$ stand for the ratios between the two (real) eigenvalues of the differential $DF_\lambda(q_j)$, for $j = 1, \dots, 4$. Their zeroes are close to the ones of

$$(((x^{r_1} + a_1)^{r_2} + a_2)^{r_3} + a_3)^{r_4} + a_4 - x.$$

For instance, the case $r_1 = 1/3$, $r_2 = 3$ and $r_3 = r_4 = 2$ leads to the study the zeroes of

$$P_2(x) + P_1(x)x^{1/3} + Q_1(x)x^{2/3} + P_0(x)(a_4 - x)^{1/2},$$

$P_n(x)$, $Q_n(x)$ denoting polynomials of degree n. In a similar way, the situation for a polycycle with three hyperbolic points of ratios $r_1 = 1$, $r_2 = 1/s$ and $r_3 = s$, can be reduced to the problem of bounding the number of zeroes of functions of type

$$a_2 + (x + a_1)^{1/s} + (a_3 - x)^{1/s}.$$

The above functions can be studied as the functions of family (2).

A second type of examples where these results can be applied is given in [7]. This paper deals with systems of the form

$$\begin{cases} \dot{x} &= -yG(x,y) + \varepsilon\, P(x,y), \\ \dot{y} &= \quad xG(x,y) + \varepsilon\, Q(x,y), \end{cases} \tag{16}$$

with

$$G(x,y) = \prod_{j=1}^{K_1}(x - a_j) \prod_{\ell=1}^{K_2}(y - b_\ell),$$

$P(x,y)$, $Q(x,y)$ polynomials of degree n and a_j and b_ℓ real numbers. This kind of differential equations corresponds to perturbations of systems having a center at the origin and a family of vertical and/or horizontal lines of equilibrium points. The maximum number of limit cycles appearing for $\varepsilon \neq 0$ is closely related to the maximum number of zeroes (taking into account their multiplicity) of some Abelian integrals. This problem is commonly referred as *the weakened Hilbert's 16th Problem.* In this case, these Abelian integrals turn out to be of the form

$$P_{n_0}(x) + P_n(x)(x + a_1)^{-1/2} + \cdots + P_n(x)(x + a_K)^{-1/2}$$

for whom Theorem 1.1 directly applies. In particular system (16) includes the vector fields studied in [1, 2, 6, 11, 15].

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