J. Math. Anal. Appl. 387 (2012) 631–644

Contents lists available at SciVerse ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the Chebyshev property for a new family of functions [☆]

Armengol Gasull [a], J. Tomás Lázaro [b], Joan Torregrosa [a,*]

[a] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain
[b] Departament de Matemàtica Aplicada I, Universitat Politècnica de Catalunya, Av. Diagonal 647, 08028 Barcelona, Spain

A R T I C L E   I N F O

Article history:
Received 18 April 2011
Available online 14 September 2011
Submitted by D. Khavinson

Keywords:
Chebyshev system
Number of zeroes of real functions
Derivation–Division algorithm
Limit cycles of planar systems

A B S T R A C T

We analyze whether a given set of analytic functions is an Extended Chebyshev system. This family of functions appears studying the number of limit cycles bifurcating from some nonlinear vector field in the plane. Our approach is mainly based on the so called Derivation–Division algorithm. We prove that under some natural hypotheses our family is an Extended Chebyshev system and when some of them are not fulfilled then the set of functions is not necessarily an Extended Chebyshev system. One of these examples constitutes an Extended Chebyshev system with high accuracy.

1. Introduction

Given $m+1$ real, analytic and linearly independent functions $\mathcal{F}=\{f_0(x), f_1(x), \dots, f_m(x)\}$, defined on some open interval I, the problem of estimating the number of real zeroes of any non-zero function $F \in \operatorname{Span}\mathcal{F}$,

$$F(x)=\sum_{j=0}^{m}\lambda_j f_j(x), \quad \lambda_j \in \mathbb{R},$$

is of wide interest. We will denote by $\mathcal{Z}(F)$ the number of zeroes in I, counted with their multiplicities, of a function F and by

$$\mathcal{Z}(\mathcal{F})=\max_{F\in(\operatorname{Span}\mathcal{F})\setminus\{0\}}\mathcal{Z}(F),$$

whenever they exist. It is easy to see that $\mathcal{Z}(\mathcal{F})\geqslant m$. A set of functions $\mathcal{F}$ for which $\mathcal{Z}(\mathcal{F})=m$ is usually called an *Extended Chebyshev system* on I and denoted in short as an ET-system. The set of polynomials of degree m, $\{1, x, x^2, \dots, x^m\}$, on any open interval is a well-known example. Other nice examples are

$$\begin{aligned}
&\{1, \log x, x, x\log x, x^2, x^2\log x, \dots, x^n, x^n\log x\} \quad \text{on } (0,\infty),\\
&\{1, \cos x, \cos(2x), \dots, \cos(mx)\} \quad \text{on } (0,\pi),\\
&\{(x+a_0)^{-1}, (x+a_1)^{-1}, \dots, (x+a_m)^{-1}\} \quad \text{on } (a,\infty), \text{ where } a=\max_{j=0,\dots,m}(-a_j).
\end{aligned}$$

☆ The first and third authors are partially supported by the MICIIN/FEDER grant number MTM2008-03437 and by the Generalitat de Catalunya grant number 2009SGR410. The second author is partially supported by the MICIIN/FEDER grant number MTM2009-06973 and by the Generalitat de Catalunya grant number 2009SGR859.

* Corresponding author.
E-mail addresses: gasull@mat.uab.cat (A. Gasull), jose.tomas.lazaro@upc.edu (J.T. Lázaro), torre@mat.uab.cat (J. Torregrosa).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2011.09.019