A NEW CHEBYSHEV FAMILY WITH APPLICATIONS TO ABEL EQUATIONS

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Abstract. We prove that a family of functions defined through some definite integrals forms an extended complete Chebyshev system. The key point of our proof consists of reducing the study of certain Wronskians to the Gram determinants of a suitable set of new functions. Our result is then applied to give upper bounds for the number of isolated periodic solutions of some perturbed Abel equations.

1. Introduction and main results

In this paper we introduce the family of analytic functions

$$I_{k,\alpha}(y) := \int_a^b \frac{g^k(t)}{(1-yg(t))^\alpha}dt, \tag{1}$$

for $k = 0, 1, \dots, n$, and prove that it is an *extended complete Chebyshev system* (for short, an ECT-system). In contrast to what is commonly done in other papers, no explicit integration of functions $I_{k,\alpha}$ is needed. In fact, our proof is based on the standard characterization of ECT-systems through the computation of certain Wronskians (see Theorem 2.1). The key point of our approach consists of showing that these Wronskians coincide with some Gram determinants, for a suitable new set of functions, associated to the usual inner product in $\mathcal{L}^2([a,b])$. Up to our knowledge, this is the first time that this kind of method has been used to prove that a given set of functions is an ECT-system.

We apply this result to determine upper bounds for the number of isolated 2π-periodic solutions which appear when one performs a first order analysis in ε of generalized Abel equations

$$\frac{dx}{dt} = \frac{\cos(t)}{q-1}\,x^q + \varepsilon P_n(\cos(t),\sin(t))\,x^p, \tag{2}$$

where $q,p \in \mathbb{N}\setminus\{0,1\}, q \neq p$, and P_n being a polynomial of degree n. Recall that the usual Abel equation corresponds to the values $\{q,p\} = \{2,3\}$. This problem is closely related to the Hilbert sixteenth problem for planar polynomial differential equations (see, for instance, [3, 4, 5, 7]). As it will be seen, our results improve the previous ones for equations (2) given in [1, 3, 7].

2010 *Mathematics Subject Classification.* Primary: 41A50. Secondary: 34C07, 34C23, 34C25 37C27.

Key words and phrases. Chebyshev system, Abel equation, Wronskian, integral Gram determinant, periodic solution, number of zeroes of real analytic functions.

Before stating our main theorems, it is convenient to introduce some notation. Thus, given $k \in \mathbb{N}$, $\alpha, a, b \in \mathbb{R}$ and any continuous non identically vanishing function $g(t)$ on $[a, b]$, we consider the new analytic function $I_{k,\alpha}(y)$ provided by formula (1) and defined on the open interval J given by the connected component of the set $\{y \in \mathbb{R} : 1 - yg(t) > 0 \text{ for all } t \in [a, b]\}$ which contains the origin. For instance, if we denote $m := \min_{t\in[a,b]} g(t) < 0$ and $M := \max_{t\in[a,b]} g(t) > 0$ then $J = (1/m, 1/M)$.

Our first result shows that, varying k, and for almost all α, the above set of functions constitutes an ECT-system (Section 2 contains a precise definition of such type of systems).

Theorem A. *For any $n \in \mathbb{N}$ and any $\alpha \in \mathbb{R} \setminus \mathbb{Z}^-$, the ordered set of functions $(I_{0,\alpha}, I_{1,\alpha}, \ldots, I_{n,\alpha})$, as defined in (1), is an ECT-system on J. When $\alpha \in \mathbb{Z}^-$ it is an ECT-system on J if and only if $n \le -\alpha$. In particular, the case where the set of functions is an ECT-system, any non-trivial function of the form*

$$\Phi_\alpha(y) := \sum_{k=0}^{n} a_k I_{k,\alpha}(y),$$

with $a_k \in \mathbb{R}$, has at most n zeros in J counting multiplicities.

It was proved in [7] that when $g(t) = \sin(t)$ and $[a, b] = [0, 2\pi]$, the function Φ_1 had n zeros in a neighbourhood of $y = 0$. In [3], this result was extended to any Φ_α, for $\alpha \in \mathbb{Q}^+$. Some of these local results were subsequently improved in [1]. More precisely, the functions Φ_1 and $\Phi_{-1/2}$ were explicitly computed and their global number of zeros in $J = (-1, 1)$ was studied. Indeed, the following expressions were achieved for them:

$$\Phi_1(y) = \Psi_1(y) \left(P_{2n}(y) + Q_{2n}(y)\sqrt{1 - y^2}\right), \tag{3}$$

and

$$\Phi_{-1/2}(y^2) = \Psi_{-1/2}(y^2) \left(P_n(r)\mathcal{K}(r) + Q_n(r)\mathcal{E}(r)\right), \tag{4}$$

P_j and Q_j being suitable polynomials of degree j, $r = 2\,y^2/(1 + y^2)$, $\Psi_1(y)$ and $\Psi_{-1/2}(y)$ being certain non-vanishing functions and $\mathcal{K}(r)$ and $\mathcal{E}(r)$ being some concrete elliptic functions (see [2]). Having in mind expressions (3) and (4), the authors proved that $\Phi_1(y)$ had at most n zeros, counting multiplicity, in $(-1, 1)$ and that this upper bound was sharp. Moreover, they obtained that the function $\Phi_{-1/2}(y^2)$ could have at most $4n + 2$ zeros in J and provided examples having at least $2n$ zeros.

Theorem A asserts that, for any g and α as in the statement, the upper bound, n, for the number of zeros of $\Phi_\alpha(y)$ in the whole interval J is sharp. Notice that for $\Phi_\alpha(y^2)$ the upper bound is $2n$.

Concerning Abel equations, it will be seen that if $x = \varphi(t, \rho, \varepsilon)$ is the solution of equation (2) starting at $x = \rho$, then:

$$\varphi(2\pi, \rho, \varepsilon) = \rho + \varepsilon\rho^p\Phi_\alpha(\rho^{q-1}) + O(\varepsilon^2), \tag{5}$$

where Φ_α is the function introduced in Theorem A for $g(t) = \sin(t)$, $\alpha = (p-q)/(q-1)$ and suitable real constants $a_0, a_1, \ldots, a_n$. This connection between Abel equations and the functions Φ_α has been, in fact, our main motivation to prove Theorem A. In this sense, it is well-known that simple zeros in $(-1,1) \setminus \{0\}$ of $\Phi_\alpha(\rho^{q-1})$, give rise to initial conditions for isolated 2π-periodic solutions of (2) which tend to these zeros as ε goes to 0. We call these 2π-periodic solutions, *periodic solutions obtained by a first order analysis*. Thus, we have:

Theorem B. *The maximum number of 2π-periodic solutions of the generalized Abel equation* (2)*, obtained by a first order analysis, is n when q is even and $2n$ when q is odd. Moreover in both cases these upper bounds are sharp.*

2. Preliminary results and proof of Theorem A

Let $f_0, f_1, \ldots, f_n$ be functions defined on an open interval J of $\mathbb{R}$. It is said that $(f_0, f_1, \ldots, f_n)$ is an extended complete Chebyshev system (ECT-system) on J if, for all $k = 0, 1, \ldots, n$, any nontrivial linear combination $a_0 f_0(y) + a_1 f_1(y) + \cdots + a_k f_k(y)$ has at most k isolated zeros on J counted with multiplicities. Here "T" stands for Tchebycheff, which is one of the transcriptions of the Russian name Chebyshev.

A very useful characterization of ECT-systems is given in the following theorem, see [6, 8]:

Theorem 2.1. *Let $f_0, f_1, \ldots, f_n$ be analytic functions defined on an open interval J of $\mathbb{R}$. Then $(f_0, f_1, \ldots, f_n)$ is an ECT-system on J if and only if for each $k = 0, 1, \ldots, n$, and all $y \in J$, the Wronskian*

$$W(f_0(y), f_1(y), \ldots, f_k(y)) := \begin{vmatrix} f_0(y) & f_1(y) & \cdots & f_k(y) \\ f_0'(y) & f_1'(y) & \cdots & f_k'(y) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(k)}(y) & f_1^{(k)}(y) & \cdots & f_k^{(k)}(y) \end{vmatrix}$$

is different from zero.

The following well-known result of linear algebra will be, as well, a key point in our argument.

Theorem 2.2. *Let $v_0, v_1, \ldots, v_n$ be elements of a vectorial space E endowed with an inner product $\langle\, , \rangle$. Then*

$$G(v_0, v_1, \ldots, v_n) := \begin{vmatrix} \langle v_0, v_0\rangle & \langle v_0, v_1\rangle & \cdots & \langle v_0, v_n\rangle \\ \langle v_1, v_0\rangle & \langle v_1, v_1\rangle & \cdots & \langle v_1, v_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle v_n, v_0\rangle & \langle v_n, v_1\rangle & \cdots & \langle v_n, v_n\rangle \end{vmatrix} \geq 0$$

and it is zero if and only if the vectors $v_0, v_1, \ldots, v_n$ are linearly dependent.

The determinant above is usually called the *Gram determinant*. We will use this result to E being the space of continuous functions on a closed interval $[a, b]$

and with inner product $\langle u, v\rangle = \int_a^b u(t)v(t)\,dt$. In this context G is also called the *integral Gram determinant* (see [9, pp. 45-48]).

Before proving Theorem A we need some preliminary results.

Lemma 2.3. *(i) For any $k \geq 0$ and $\ell \geq 1$, $I_{k,\beta}^{(\ell)}(y) = \prod_{j=0}^{\ell-1}(\beta+j)I_{k+\ell,\beta+\ell}(y)$.*
(ii) For any $k \geq 1$ and $m \leq k$,

$$I_{k,\beta}(y) = y^{-m}\Big(I_{k-m,\beta}(y) + \sum_{j=1}^{m} c_j(m) I_{k-m,\beta-j}(y)\Big),$$

where $c_j(m) = (-1)^j\binom{m}{j}$.

Proof. The functions $I_{k,\beta}(y)$ are analytic on J and

$$I_{k,\beta}^{(\ell)}(y) = \int_a^b \frac{\partial^\ell}{\partial y^\ell}\frac{g^k(t)}{(1-yg(t))^\beta}\,dt,$$

so the proof of statement (i) is straightforward.

We will prove statement (ii) by induction on m. The case $m=1$ follows multiplying by $(1-yg(t))$ the numerator and the denominator of the integrand of (1):

$$I_{k-1,\beta-1}(y) = \int_a^b \frac{g^{k-1}(t)}{(1-yg(t))^\beta}dt - y\int_a^b \frac{g^k(t)}{(1-yg(t))^\beta}dt = I_{k-1,\beta}(y) - yI_{k,\beta}(y).$$

Thus, let us assume that the expression of $I_{k,\beta}$ holds until m. Then, taking into account that $c_j(m) - c_{j-1}(m) = c_j(m+1)$, it follows that

$$\begin{aligned}
I_{k,\beta}(y) &= y^{-m}\Big(\sum_{j=0}^{m} c_j(m) I_{k-m,\beta-j}(y)\Big)\\
&= y^{-m}\Big(\sum_{j=0}^{m} c_j(m)\, y^{-1}(I_{k-m-1,\beta-j}(y) - I_{k-m-1,\beta-j-1}(y))\Big)\\
&= y^{-m-1}\Big(\sum_{j=0}^{m+1} c_j(m+1) I_{k-m-1,\beta-j}(y)\Big)
\end{aligned}$$

and, therefore, the assertion is proved for $m+1$. □

The following lemma relates a Wronskian with the determinant of a symmetric matrix which, at the end, will become a Gram determinant.

Lemma 2.4. *Let $I_{0,\alpha},\ldots,I_{n,\alpha}$ be the functions defined in* (1). *Then for $y \neq 0$,*

$$W_n := W(I_{0,\alpha}, I_{1,\alpha},\ldots,I_{n,\alpha}) = y^{-(1+n)n}D_n(\alpha)\begin{vmatrix} I_{0,\alpha-n} & I_{0,\alpha-n+1} & \cdots & I_{0,\alpha}\\ I_{0,\alpha-n+1} & I_{0,\alpha-n+2} & \cdots & I_{0,\alpha+1}\\ \vdots & \vdots & \ddots & \vdots\\ I_{0,\alpha} & I_{0,\alpha+1} & \cdots & I_{0,\alpha+n}\end{vmatrix}, \tag{6}$$

where $D_n(\alpha) = \prod_{j=0}^{n-1} (\alpha + j)^{n-j}$.

Proof. Using the expression for the derivatives provided by Lemma 2.3(i) we can write

$$W_n = D_n(\alpha) \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I_{1,\alpha+1} & I_{2,\alpha+1} & \cdots & I_{n+1,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{n,\alpha+n} & I_{n+1,\alpha+n} & \cdots & I_{2n,\alpha+n} \end{vmatrix}. \tag{7}$$

If we denote the i-row of the previous determinant by $R_i = [I_{i,\alpha+i}, \dots, I_{i+n,\alpha+i}]$ for $i = 0, \dots, n$, and use Lemma 2.3(ii) we get

$$R_i = y^{-i}\Big(\widehat{R}_i + \sum_{j=1}^{i} c_j(i)\widehat{R}_{i-j}\Big),$$

where $\widehat{R}_i = [I_{0,\alpha+i}, \dots, I_{n,\alpha+i}]$. Then, from the elementary properties of the determinants we obtain that

$$W_n = y^{-\frac{(1+n)n}{2}} D_n(\alpha) \begin{vmatrix} I_{0,\alpha} & I_{1,\alpha} & \cdots & I_{n,\alpha} \\ I_{0,\alpha+1} & I_{1,\alpha+1} & \cdots & I_{n,\alpha+1} \\ \vdots & \vdots & \ddots & \vdots \\ I_{0,\alpha+n} & I_{1,\alpha+n} & \cdots & I_{n,\alpha+n} \end{vmatrix}.$$

Applying again Lemma 2.3(ii), but this time to the columns of the determinant, the desired result is achieved. □

Next result will be the key point in our proof of Theorem A.

Proposition 2.5. *Let W_n be the Wronskian defined in Lemma 2.4. Then, if α is a negative integer and $n > -\alpha$ then $W_n = 0$. Otherwise, W_n does not vanish on the interval J and* $\operatorname{sgn}(W_n) = \operatorname{sgn}(D_n(\alpha))$.

Proof. If α is a negative integer and $n > -\alpha$ it is clear that $D_n(\alpha) = 0$ and, from equality (7), it follows that $W_n = 0$. So, assume that $D_n(\alpha) \neq 0$ and consider the auxiliary functions $f_i(t) = (1 - yg(t))^{(n-\alpha)/2-i}$, for $i = 0, 1, \dots, n$, which are well defined on J since they satisfy $1 - yg(t) > 0$ on this set. Notice that

$$\langle f_i, f_j \rangle = \int_a^b (1 - yg(t))^{n-\alpha-i-j}\, dt = I_{0,\alpha-n+i+j}(y).$$

Hence, using the equivalent expression (6) of the Wronskian for $y \neq 0$,

$$W_n = y^{-(1+n)n} D_n(\alpha) G(f_0, f_1, \dots, f_n), \tag{8}$$

where $G(f_0, f_1, \dots, f_n)$ is the integral Gram determinant. From Theorem 2.2, it is non-negative and vanishes if and only if the functions f_i are linearly dependent. The independence of the functions $f_i(t) = (1 - yg(t))^{-(\alpha+n)/2}(1 - yg(t))^{n-i}$ follows from the fact that $g(t) \not\equiv 0$. Therefore, the sign of W_n on the set $J \setminus \{0\}$ is the sign of $D_n(\alpha)$ because the Gram determinant in (8) is always positive and $y^{(1+n)n} > 0$.

It can be seen in the expression of W_n in (7) that the determinant appearing there is also positive when it is evaluated at $y = 0$ since it can also be written as the new integral Gram determinant,

$$\begin{vmatrix} I_{0,\alpha}(0) & I_{1,\alpha}(0) & \cdots & I_{n,\alpha}(0) \\ I_{1,\alpha+1}(0) & I_{2,\alpha+1}(0) & \cdots & I_{n+1,\alpha+1}(0) \\ \vdots & \vdots & \ddots & \vdots \\ I_{n,\alpha+n}(0) & I_{n+1,\alpha+n}(0) & \cdots & I_{2n,\alpha+n}(0) \end{vmatrix} = G(1, g, g^2, \ldots, g^n) > 0.$$

Thus W_n is well defined on the whole J, does not vanish and its sign coincides with the one of $D_n(\alpha)$, as we wanted to prove. $\square$

Remark 2.6. *Notice that, although for $\alpha = -m \in \mathbb{Z}^-$ the functions*

$$I_{k,-m} = \int_a^b g^k(t)(1 - y\, g(t))^m \, dt,$$

are well defined for all $y \in \mathbb{R}$, our result only proves that the set $(I_{0,-m}, I_{1,-m}, \ldots, I_{n,-m})$, for $n \leq -\alpha = m$, is an ECT-system on J. For instance, it is easy to see that the functions

$$a_0 I_{0,-2}(y) + a_1 I_{1,-2}(y),$$

which are polynomials of degree 2 in y, can have two zeros in $\mathbb{R}$. This shows that $(I_{0,-2}, I_{1,-2})$ is not a ECT-system on the whole $\mathbb{R}$.

Proof of Theorem A. From Theorem 2.1 we know that it is enough to show that, under our hypotheses and for any $k = 0, 1, \ldots, n$, the Wronskian of the functions $(I_{0,\alpha}, I_{1,\alpha}, \ldots, I_{k,\alpha})$ does not vanish on J. This is a direct consequence of Proposition 2.5. $\square$

3. Proof of Theorem B

First we prove that expression (5) holds. Following the computations of [1, 3] one can easily get that

$$\begin{aligned} \varphi(t, \rho, \varepsilon) =& \rho \left(\frac{1}{1 - \rho^{q-1} \sin(t)} \right)^{\frac{1}{q-1}} \\ &+ \varepsilon \left(\frac{\rho}{1 - \rho^{q-1} \sin(t)} \right)^p \int_0^t \frac{P_n(\cos(s), \sin(s))}{(1 - \rho^{q-1} \sin(s))^\alpha} \, ds + O(\varepsilon^2). \end{aligned}$$

Notice that since $\rho \in (-1, 1)$ the flow is well defined for all $t \in \mathbb{R}$. Then

$$\varphi(2\pi, \rho, \varepsilon) = \rho + \varepsilon \rho^p \int_0^{2\pi} \frac{P_n(\cos(t), \sin(t))}{(1 - \rho^{q-1} \sin(t))^\alpha} \, dt + O(\varepsilon^2).$$

Since $\cos^{2\ell}(t) = (1 - \sin^2(t))^\ell$ and $\cos^{2\ell+1}(t) = (1 - \sin^2(t))^\ell \cos(t)$ it turns out that

$$\int_0^{2\pi} \frac{P_n(\cos(t), \sin(t))}{(1 - \rho^{q-1} \sin(t))^\alpha} \, dt = \int_0^{2\pi} \frac{Q_n(\sin(t))}{(1 - \rho^{q-1} \sin(t))^\alpha} \, dt = \Phi_\alpha(\rho^{q-1}),$$

where Q_n is a new polynomial of degree n, that we can write as $Q_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$. Hence (5) follows.

The maximum number of isolated 2π-periodic solutions obtained by a first order analysis can be obtained studying the zeroes of $(\varphi(2\pi,\rho,\varepsilon)-\rho)/\varepsilon = \rho^p\Phi_\alpha(\rho^{q-1}) + O(\varepsilon)$. This number is controlled by the zeroes of $\Phi_\alpha(\rho^{q-1})$. Taking $y=\rho^{q-1}$ and $J=(-1,1)$ we know from Theorem A that the maximum number of zeros of $\Phi_\alpha(y)$ in J, counting multiplicities, is n and that this upper bound is sharp. Hence Theorem B follows taking into account that when q is odd $\Phi_\alpha(\rho^{q-1}) = \Phi_\alpha((-\rho)^{q-1})$. □

Acknowledgments

This work was elaborated when C. Li was visiting the Department of Mathematics of the Universitat Autònoma of Barcelona. He is very grateful for the hospitality.

The first and third authors are partially supported by the MICIIN/FEDER grant number MTM2008-03437 and by the Generalitat de Catalunya grant number 2009SGR410. The second author is partially supported by NSFC-10831003 and by AGAUR grant number 2009PIV00064.

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