LIMIT CYCLES FOR 3-MONOMIAL DIFFERENTIAL EQUATIONS

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Abstract. We study planar polynomial differential equations that in complex coordinates write as $\dot{z} = Az + Bz^k\bar{z}^l + Cz^m\bar{z}^n$. We prove that for each $p \in \mathbb{N}$ there are differential equations of this type having at least p limit cycles. Moreover, for the particular case $\dot{z} = Az + B\bar{z} + Cz^m\bar{z}^n$, which has homogeneous nonlinearities, we show examples with several limit cycles and give a condition that ensures uniqueness and hyperbolicity of the limit cycle.

1. Introduction and main results

The celebrated second part of the Hilbert's 16th problem [17] consists in determining a uniform upper bound on the number of limit cycles of all polynomial differential systems of degree N, see for instance [19, 21] and the references therein. This problem is still open even for the quadratic case, $N = 2$. Due to its extreme difficulty, usually people fix a subclass of polynomial differential equations, namely: quadratic, cubic, Kukles, Liénard, homogeneous nonlinearities,..., and then try to advance in the question restricted to the selected family. This paper goes in a similar direction, we consider a simple class of polynomial systems, but instead of fixing the degree, we fix a short number of monomials once the system is written in complex coordinates, and then we study its number of limit cycles.

To be more precise, consider two dimensional real differential systems,

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \quad (x,y) \in \mathbb{R}^2, \quad t \in \mathbb{R},$$

with P and Q polynomials. They can also be written as

$$\frac{dz}{dt} = \dot{z} = F(z,\bar{z}), \quad z \in \mathbb{C}, \quad t \in \mathbb{R},$$

where F is a complex polynomial. In this paper we face the question of the number of limit cycles for polynomial differential equations with three monomials that write as

$$\dot{z} = Az + Bz^k\bar{z}^l + Cz^m\bar{z}^n, \tag{1}$$

where $A, B, C \in \mathbb{C}$ and $k, l, m, n \in \mathbb{N} \cup \{0\}$. Our first result is:

Theorem A. *For any $p \in \mathbb{N}$ there is a differential equation of type (1) having at least p limit cycles.*

A celebrated family of differential equations of the form (1) is

$$\dot{z} = Az + Bz^2\bar{z} + C\bar{z}^{q-1}, \tag{2}$$

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with $q \geq 3$. It gives the *versal deformation* of a principal singular smooth systems having rotational invariance of $2\pi/q$ radians, see [3]. The cases $q = 3, 4$ are called *strong resonances* while the cases $q \geq 5$ are called *weak resonances.* The situation $q \neq 4$ is well understood and considered in several places, see for instance [8, 18, 25] for $q = 3$ and again [3] for $q \geq 5$.

The study of the limit cycles for case $q = 4$ turns out to be especially difficult and is considered in many works, see for instance [1, 3, 6, 9, 16, 20, 22, 26]. To know the number of limit cycles surrounding the origin, and eventually surrounding also the other 4 or 8 critical point that the system can posses is yet an open question. In [9, 22] it is proved that at least two limit cycles can exist surrounding the 9 critical points.

The problem of the number of limit cycles not surrounding the origin for (2) and $q = 4$ is totally solved. In [26] it is proved that either there are no limit cycles or that there are exactly four ($q = 4$) hyperbolic ones, each one of them surrounding exactly one of the critical points of index +1. It also remains to study the coexistence of these four limit cycles with other limit cycles that surround the origin.

Inspired by the presence of the four limit cycles non surrounding the origin for (2) and $q = 4$ and for the results presented in [21, Sec. 7], we consider a variation of (2) that allow us to prove that for each $p \geq 3$ there are systems in (1) with at least p limit cycles non surrounding the origin. More concretely, our proof of Theorem A relies on the study of the following subclass of (1),

$$\dot{z} = A\,z + B\,z^{p-1}\bar{z}^{p-2} + C\,\bar{z}^{p-1} = A\,z + B\,z|z|^{2(p-2)} + C\,\bar{z}^{p-1}, \tag{3}$$

with $p \geq 3$, which also has rotational invariance of $2\pi/p$ radians. We consider a Hamiltonian case in (3) and we perturb it without leaving the family. Then, studying the quotient of two Abelian integrals associated to the given perturbation, we prove the existence of at least one limit cycle surrounding a critical point that is not the origin. Afterwards, the rotational invariance property provides the existence of the p limit cycles. The main difficulty and difference with similar previous results is that one of the parameters of the differential equation is related with its degree.

Notice that (3) coincides with (2) only when $p = q = 3$. Therefore, for this case our proof gives three limit cycles surrounding three different critical points which posses $2\pi/3$ rotational symmetry. These limit cycles are not showed in [8, 9] but already appear in [25].

We remark that when $q \geq 5$ the existence of examples with q limit cycles non surrounding the origin of (2) is no more true, see Lemma 2.4 and Remark 2.5. For this reason, to prove our result we have used (3) instead of (2).

Theorem A shows that there is no upper bound for the number of limit cycles for general systems with three complex monomials. Hence, in the second part of the paper we fix another concrete subfamily with three monomials and we give conditions over its parameters in order to have uniqueness and hyperbolicity of its limit cycles.

This new family is also a subclass of the planar polynomial differential equations with homogeneous nonlinearities. These equations have also been extensively studied, see for instance [5, 7, 10, 15]. It can be seen that equations considered in the next result can have several limit cycles, see Subsection 4.2.

Theorem B. *Consider equation*

$$\dot z = A\,z + B\,\bar z + C\,z^m \bar z^n. \tag{4}$$

Then for $m=0$ it has no limit cycles. For $m \geq 2$, $\operatorname{Re}(A) \neq 0$ and

$$|B| \leq \frac{(m-1)|\operatorname{Re}(A)|}{m},$$

it has at most one limit cycle. Moreover if the limit cycle exists it is hyperbolic and stable (resp. unstable) if $\operatorname{sgn}(\operatorname{Re}(A)) > 0$ (resp. < 0) and it must surround the origin.

Our proof is based on showing that all possible limit cycles, all points of index $+1$ (except the origin) and all polycycles have the same stability. As we will see, our approach requires that $m \neq 1$.

It is easy to find equations (4) under the hypotheses of the theorem and having a limit cycle, see again Subsection 4.2. Also, as we will see in Remark 4.2, Theorem B gives in some cases sharp results about the number of limit cycles.

2. Preliminary results

We begin stating three technical results. Their proofs are straightforward.

Lemma 2.1. *Consider the polynomial differential equation with a critical point at the origin,*

$$\dot z = F(z,\bar z) = \sum_{j=1}^{N} F_j(z,\bar z),$$

where F_j are complex homogeneous polynomials of degree j. Then, its expression in polar coordinates, (r,θ), given by $z = r\,\mathrm{e}^{i\theta}$ is

$$\dot r = \sum_{j=1}^{N} \operatorname{Re}(S_j(\theta))\, r^j, \quad \dot\theta = \sum_{j=1}^{N} \operatorname{Im}(S_j(\theta))\, r^{j-1},$$

where $S_j(\theta) = (\bar z F_j(z,\bar z))\big|_{z=\mathrm{e}^{i\theta}}$.

Lemma 2.2. *Consider a Hamiltonian planar differential equation, with Hamiltonian function $H(r,\theta)$. Then its expression in polar coordinates is*

$$\dot r = -\frac{1}{r}\frac{\partial}{\partial\theta}H(r,\theta), \qquad \dot\theta = \frac{1}{r}\frac{\partial}{\partial r}H(r,\theta).$$

Lemma 2.3. *Let $\dot r = R(r,\theta)$, $\dot\theta = \Phi(r,\theta)$, be the expression in polar coordinates associated to a planar vector field X and let $\dot z = F(z,\bar z)$ be its expression in complex coordinates. Then*

$$\operatorname{div}(X) = \frac{1}{r}\frac{\partial}{\partial r}(rR(r,\theta)) + \frac{\partial}{\partial\theta}\Phi(r,\theta) = 2\operatorname{Re}\Big(\frac{\partial}{\partial z}F(z,\bar z)\Big).$$

Moreover

$$\det(\mathrm{d}\,X) = \Big|\frac{\partial}{\partial z}F(z,\bar z)\Big|^2 - \Big|\frac{\partial}{\partial \bar z}F(z,\bar z)\Big|^2,$$

where $\mathrm{d}\,X$ stands for the differential of X.

Next result gives the sum of the indices of all the critical points of a class of differential equations.

Lemma 2.4. *Consider the polynomial differential equation*

$$\dot z = F(z,\bar z) + Cz^m\bar z^n,$$

where $0 \neq C \in \mathbb{C}$ *and* $\deg(F) < m+n$. *Then, the sum of the indices of all its critical points is* $m-n$.

Proof. Taking the polar coordinates, $z = r\,\mathrm{e}^{i\theta}$, we get that

$$\frac{\left(F(z,\bar z)+Cz^m\bar z^n\right)\big|_{z=r\,\mathrm{e}^{i\theta},\bar z=r\,\mathrm{e}^{-i\theta}}}{r^{m+n}} = \frac{F(r\,\mathrm{e}^{i\theta}, r\,\mathrm{e}^{-i\theta})}{r^{m+n}} + C\,\mathrm{e}^{i(m-n)\theta}.$$

Therefore, for r big enough, the above right-hand side function, on the circle $|z| = r$, gives $|m-n|$ turns in clockwise (resp. counterclockwise) sense when $m-n \geq 0$ (resp. $m-n \leq 0$), proving the desired result. □

Remark 2.5. *By Lemma 2.3, the condition on differential equation* (2) *to be Hamiltonian is* $\mathrm{Re}(A) = \mathrm{Re}(B) = 0$. *By Lemma 2.4, for* $q \geq 5$ *the total sum of the indices of all its critical points in this case is* $1-q<0$. *In particular it can be seen that it has only one critical point of index* $+1$. *Therefore, for* $q \geq 5$, *Hamiltonian systems in* (2) *can not be taken as starting points to obtain, using small perturbations,* q *limit cycles with* $2\pi/q$ *rotational symmetry.*

We also use the expression of the first Lyapunov constant for a system written in complex variables, see [14].

Lemma 2.6. *Consider a system of class* $\mathcal{C}^4$ *with a weak focus at the origin. Choose a local system of coordinates and a constant change of the time scale such that in a neighborhood of the origin it writes as*

$$\dot z = F(z,\bar z) = iz + \sum_{m+n=2}^{3} f_{m,n}z^m\bar z^n + O_4(z,\bar z), \quad f_{m,n} \in \mathbb{C}. \tag{5}$$

Then its first Lyapunov constant is

$$V_3 = 2\pi\big(\mathrm{Re}(f_{2,1}) - \mathrm{Im}(f_{2,0}f_{1,1})\big).$$

Finally we present a result about the stability of some simple polycycles. It is a consequence of the results of [4, 24]. Recall that hyperbolic or semi-hyperbolic critical points are usually called elementary.

Proposition 2.7. *Let* Γ *be a polycycle of an analytic vector field* X *with elementary corners* $u_1, u_2, \ldots, u_\ell$ *and such that* $\mathrm{div}(X(u_j)) < 0$ *(resp.* >0*) for all* j. *Then* Γ *is an attracting (resp. repelling) polycycle.*

Proof. It is well-known, see for instance [24], that when all the critical points at the corners are hyperbolic, with eigenvalues $-\lambda_j < 0 < \mu_j$, then Γ is stable (respectively, unstable) if $\rho(\Gamma) < 1$ (respectively, $\rho(\Gamma) > 1$), where

$$\rho(\Gamma) = \prod_{j=1}^{\ell}\frac{\mu_j}{\lambda_j}.$$

Fix for instance the case where, for $j = 1, \ldots, \ell$, $\mathrm{div}(X(u_j)) < 0$. Then, for all j, $\mu_j/\lambda_j < 1$ and the proposition follows.

In general, when either $\rho(\Gamma) = 1$ or there are semi-hyperbolic corners, the stability of the corresponding polycycles can be very hard to determine. In particular, for each semi-hyperbolic corner u, its associated local return Dulac map is flat (resp. vertical) when $\operatorname{div}(X(u)) < 0$ (resp. > 0). Recall that flat return maps have all their derivatives zero at the origin and that vertical maps are the inverse of flat maps. The really difficult situation appears when flat and vertical local return maps coexist. Fortunately, under our hypotheses all these maps are of the some type and the result easily follows. For more details see for instance [4]. □

3. Proof of Theorem A

Examples of equations (1) with one or two limit cycles can be easily constructed. For instance, for $p = 1$ simply consider

$$\dot{z} = (1+i)\,z - z^2\bar{z},$$

that has the circle $|z| = 1$ as limit cycle, because by Lemma 2.1 writes as $\dot{r} = r(1-r^2)$, $\dot{\theta} = 1$. Similarly, for $p = 2$, take

$$\dot{z} = (4+i)\,z - 5\,z^2\bar{z} + z^3\bar{z}^2,$$

which has the circles $|z| = 1$ and $|z| = 2$ as limit cycles. In fact it can be written as $\dot{r} = r(r^2-1)(r^2-4)$, $\dot{\theta} = 1$.

For $3 \le p \in \mathbb{N}$, consider the 2-parameter family of systems of the form (3),

$$\dot{z} = (a+i)\,z + (b+i)\,z|z|^{2(p-2)} - \frac{5i}{2}\,\bar{z}^{p-1}, \tag{6}$$

with $a, b \in \mathbb{R}$, $3 \le p \in \mathbb{N}$. When $a = b = 0$ the system is Hamiltonian, with Hamiltonian function

$$H(r,\theta) = \frac{r^2}{2} - \frac{5}{2p} r^p \cos(p\,\theta) + \frac{r^{2(p-1)}}{2(p-1)} - \tilde{\rho},$$

where

$$\tilde{\rho} = \frac{(p-2)(p-5)}{2p(p-1)}\,2^{\frac{2}{p-2}}.$$

Apart of the origin, the differential equation with $a = b = 0$ has p critical points of saddle type on the circle $r = (1/2)^{1/(p-2)}$ and p critical points of center type on the circle $r = 2^{1/(p-2)}$, see Figure 1. The value $\tilde{\rho}$ is chosen in such a way that H at the center points vanishes. The value of H at the saddle points is

$$\rho^* = \frac{(p-2)(4p-5)}{8p(p-1)} \left(\frac{1}{2}\right)^{\frac{2}{p-2}} - \tilde{\rho} < 0.$$

Therefore the periodic orbits surrounding each of the p centers can be parameterized as

$$H(r,\theta) = \rho, \quad \rho \in (\rho^*, 0).$$

By Lemmas 2.1 and 2.2, the expression of equation (6) in polar coordinates is

$$\begin{aligned} \dot{r} &= -\frac{1}{r}\frac{\partial H(r,\theta)}{\partial \theta} + a\,r + b\,r^{2p-3}, \\ \dot{\theta} &= \frac{1}{r}\frac{\partial H(r,\theta)}{\partial r}, \end{aligned} \tag{7}$$

or equivalently,

$$dH(r,\theta) - (a\,r^2 + b\,r^{2(p-1)})\,d\theta = 0. \tag{8}$$

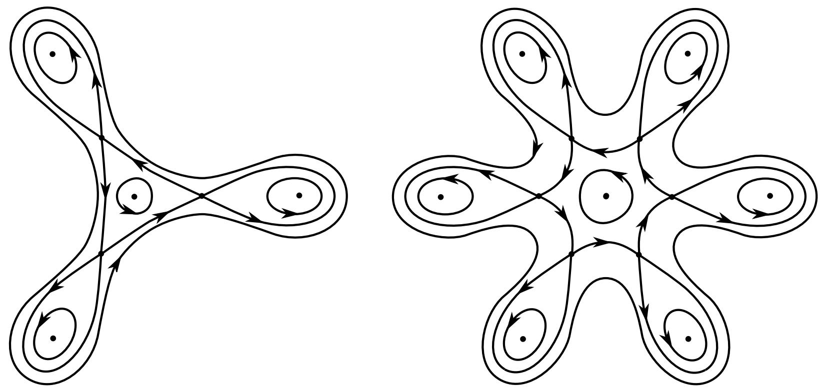

FIGURE 1. Centers of equation (6) when $a = b = 0$ for the cases $p = 3$ and $p = 6$.

Writing $a = \varepsilon\,\alpha$ and $b = \varepsilon\,\beta$, for $\alpha, \beta \in \mathbb{R}$ and ε small enough, the first order Melnikov function associated to (8) is

$$M(\rho) = \alpha\, I_2(\rho) + \beta\, I_{2p-2}(\rho),$$

where

$$I_j(\rho) = \int_{H=\rho} r^j\, d\theta, \quad j = 2, 2p-2, \text{ and } \rho \in (\rho^*, 0).$$

Due to the Z_p-equivariant symmetry and the properties of the Melnikov functions, if M has a simple zero $\widehat{\rho}$ in $(\rho^*, 0)$ then the differential equation (6) has a limit cycle tending, when ε goes to zero, to a closed periodic orbit in each of the p period annuli defined by the level set $\{H(r,\theta) = \widehat{\rho}\}$, see [12, 21]. By simplicity when we write $H(r,\theta) = \rho$ we only consider the connected component of this set corresponding to the center that, when $a = b = 0$, cuts the positive x-axis.

To study $M(\rho)$ we introduce the auxiliary analytic function

$$J(\rho) = \frac{I_{2p-2}(\rho)}{I_2(\rho)}, \quad \rho \in (\rho^*, 0).$$

Notice that $I_2(\rho) > 0$ because this function gives the double of the area surrounded by a connected component of the curve $H(r,\theta) = \rho$. Then

$$M(\rho) = I_2(\rho)\big(\alpha + \beta J(\rho)\big).$$

We claim that J is not a constant. Let us see how the theorem follows using this claim. Take $\widehat{\rho} \in (\rho^*, 0)$. Then choosing $\alpha = -J(\widehat{\rho})$ and $\beta = 1$ we have that $M(\widehat{\rho}) = 0$. Since J is not a constant and the function is analytic, this zero of M has a given finite multiplicity. If this zero is simple we are done. If not, by using again that $I_2(\rho) > 0$ and applying a result of [11, Lem. 4.5] we have that taking ν small enough, and with the suitable sign, the function $M(\rho) = (\nu - J(\widehat{\rho}))\, I_2(\rho) + I_{2p-2}(\rho)$ has a simple zero $\widehat{\rho}_\nu$, near $\rho = \widehat{\rho}$. Then, in any case, the result follows.

Let us prove the claim. We proceed by contradiction. If J was constant, we start computing its value. We first parameterize the oval $H(r,\theta) = \rho$ in polar coordinates, see Figure 2. For each $\theta \in [-\theta^*(\rho), \theta^*(\rho)]$ the values of r are $r_1(\theta,\rho)$ and $r_2(\theta,\rho)$, with $r_1(\theta,\rho) \le r_2(\theta,\rho)$. The equality between both r_j only holds for $\theta = \pm\theta^*(\rho)$.

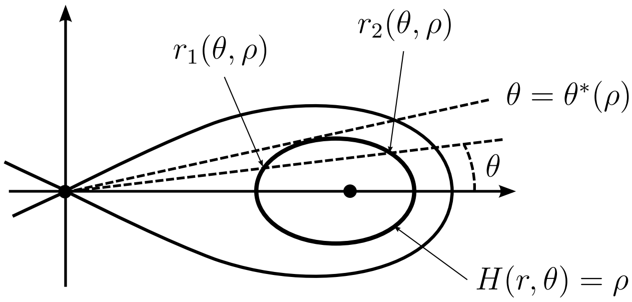

FIGURE 2. One of the ovals of the p centers and the definition of $\theta^*(\rho)$.

Then, by the mean value theorem for integrals,

$$\begin{aligned} I_2(\rho) &= \int_{H=\rho} r^2\,d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^2(\theta,\rho)-r_1^2(\theta,\rho)\right)\,d\theta \\ &= 2\left(r_2(\overline{\theta}(\rho),\rho)+r_1(\overline{\theta}(\rho),\rho)\right)\int_0^{\theta^*(\rho)} \left(r_2(\theta,\rho)-r_1(\theta,\rho)\right)\,d\theta, \end{aligned}$$

for some $\overline{\theta}(\rho)\in(0,\theta^*(\rho))$. Similarly,

$$\begin{aligned} I_{2p-2}(\rho) &= \int_{H=\rho} r^{2p-2}\,d\theta = 2\int_0^{\theta^*(\rho)} \left(r_2^{2p-2}(\theta,\rho)-r_1^{2p-2}(\theta,\rho)\right)\,d\theta \\ &= 2\sum_{j=0}^{2p-3} r_2^{2p-3-j}(\widehat{\theta}(\rho),\rho)\,r_1^{j}(\widehat{\theta}(\rho),\rho)\int_0^{\theta^*(\rho)} \left(r_2(\theta,\rho)-r_1(\theta,\rho)\right)\,d\theta, \end{aligned}$$

for some $\widehat{\theta}(\rho)\in(0,\theta^*(\rho))$. Moreover,

$$\lim_{\rho\to 0}\overline{\theta}(\rho)=\lim_{\rho\to 0}\widehat{\theta}(\rho)=0,$$

and

$$\lim_{\rho\to 0} r_j(\overline{\theta}(\rho),\rho)=\lim_{\rho\to 0} r_j(\widehat{\theta}(\rho),\rho)=2^{1/(p-2)},\quad j=1,2.$$

Hence,

$$\begin{aligned} \lim_{\rho\to 0} J(\rho) &= \lim_{\rho\to 0}\frac{\sum\limits_{j=0}^{2p-3} r_2^{2p-3-j}(\widehat{\theta}(\rho),\rho)\,r_1^{j}(\widehat{\theta}(\rho),\rho)}{r_2(\overline{\theta}(\rho),\rho)+r_1(\overline{\theta}(\rho),\rho)} \\ &= \frac{(2p-2)\left(2^{1/(p-2)}\right)^{2p-3}}{2\cdot 2^{1/(p-2)}} = 4(p-1). \end{aligned}$$

So, assume to arrive to a contradiction, that $J(\rho)\equiv 4(p-1)$. Since,

$$J'(\rho)=\frac{I_{2p-2}'(\rho)I_2(\rho)-I_{2p-2}(\rho)I_2'(\rho)}{I_2^2(\rho)}\equiv 0,$$

it also holds that

$$\frac{I_{2p-2}'(\rho)}{I_2'(\rho)}\equiv 4(p-1). \tag{9}$$

By the Gelfand–Leray formula,

$$I'_{2p-2}(\rho) = \int_{H=\rho} (2p-2) r^{2p-3} \frac{\partial r}{\partial \rho}\, d\theta.$$

Since $H(r(\theta,\rho),\theta)) = \rho$ for all ρ, we get that

$$\frac{\partial H(r(\theta,\rho),\theta)}{\partial r} \frac{\partial r(\theta,\rho)}{\partial \rho} = 1.$$

Hence, using (7),

$$\frac{\partial r(\theta,\rho)}{\partial \rho} = \left(\frac{\partial H(r(\theta,\rho),\theta)}{\partial r} \right)^{-1} = \frac{1}{r(\theta,\rho)} \frac{dt}{d\theta}.$$

Therefore, we can parameterize the Abelian integrals using the variable t and we get that

$$I'_{2p-2}(\rho) = (2p-2) \int_0^{T(\rho)} r^{2p-4}(t)\, dt,$$

where $r(t)$ denotes the time parametrization of the periodic orbit contained in $H(r,\theta) = \rho$ (for shortness, we omit the dependence with respect to ρ) and $T(\rho)$ is its period. Similarly,

$$I'_2(\rho) = 2 \int_0^{T(\rho)} dt = 2\,T(\rho).$$

As a consequence we get that (9) is equivalent to

$$\frac{\displaystyle\int_0^{T(\rho)} r^{2p-4}(t)\, dt}{T(\rho)} = 4,$$

that can be written as

$$\int_0^{T(\rho)} G_\rho(t)\, dt = 0, \quad \text{where} \quad G_\rho(t) := r^{2p-4}(t) - \left(2^{1/(p-2)}\right)^{2p-4},$$

or by symmetry, as

$$K(\rho) := \int_0^{T(\rho)/2} G_\rho(t)\, dt = 0,$$

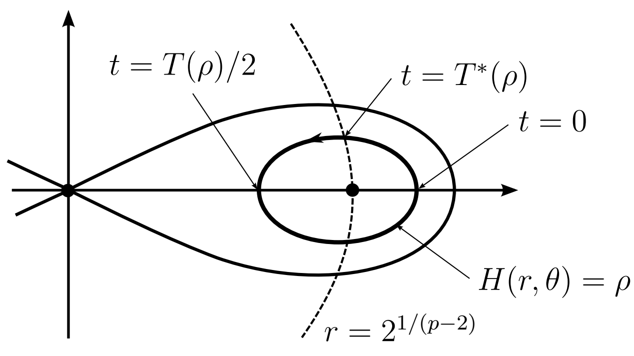

FIGURE 3. One of the p centers and the definition of $T^*(\rho)$.

Let us prove that the above equality is false for ρ near ρ^*. To do this, we introduce the time $t = T^*(\rho)$ at which the orbit in $H(r,\theta) = \rho$ cuts the circle $r = 2^{1/(p-2)}$, in the first quadrant, see Figure 3. Then we write $K(\rho) = K^+(\rho) + K^-(\rho)$, where

$$K^-(\rho) = \int_{T^*(\rho)}^{T(\rho)/2} G_\rho(t)\,dt, \quad K^+(\rho) = \int_0^{T^*(\rho)} G_\rho(t)\,dt,$$

where notice that the integrands are negative and positive, respectively. On one hand, it holds that for all $\rho \in [\rho^*, 0)$ the function $K^+(\rho)$ has an upper bound. On the other hand, for $\delta > 0$, small and fixed, it holds that for all $\rho \in [\rho^*, 0]$,

$$\min_{t\in[T^*(\rho)+\delta, T(\rho)/2]} \big|G_\rho(t)\big| = L_\delta > 0.$$

Hence,

$$\begin{aligned}|K^-(\rho)| &= \int_{T^*(\rho)}^{T^*(\rho)+\delta} \big|G_\rho(t)\big|\,dt + \int_{T^*(\rho)+\delta}^{T(\rho)/2} \big|G_\rho(t)\big|\,dt \\ &> \int_{T^*(\rho)}^{T^*(\rho)+\delta} \big|G_\rho(t)\big|\,dt + L_\delta\big(T(\rho)/2 - T^*(\rho) - \delta\big).\end{aligned}$$

Notice that

$$\lim_{\rho\to\rho^*} (T(\rho)/2 - T^*(\rho) - \delta) = \infty,$$

and as a consequence

$$\lim_{\rho\to\rho^*} K^-(\rho) = -\infty.$$

This last result is in contradiction with the fact that $K^+(\rho) + K^-(\rho) = 0$ for all $\rho \in (\rho^*, 0)$. Therefore $J(\rho)$ is not a constant function and the claim follows. □

4. Equations with homogeneous nonlinearities

This section is devoted to prove Theorem B and to present several examples of differential equations of the form (4) with limit cycles.

4.1. **Proof of Theorem B.** When $m = 0$, then by Lemma 2.3, the divergence of the associated planar vector X of equation (4) is $\operatorname{div}(X) = \operatorname{Re}(A)$. Therefore, when $\operatorname{Re}(A) = 0$ the differential equation is Hamiltonian and if $\operatorname{Re}(A) \neq 0$ then $\operatorname{div}(X) \neq 0$ and Dulac criterion applies. In any case, the differential equation has no limit cycles.

Consider now that $m \geq 2$. We start proving that all existing limit cycles are hyperbolic and have the same stability.

It is well-known that the hyperbolicity and the stability of a given limit cycle, $z(t) = x(t) + iy(t)$, of period T of a vector field X is controlled by its characteristic exponent,

$$\sigma = \int_0^T \operatorname{div}(X)(x(t), y(t))\,dt,$$

see [2, Thm 17]. If the system is written in polar coordinates, using Lemma 2.3, then

$$\sigma = \int_0^T \left(\frac{1}{r}\frac{\partial(rR(r,\theta))}{\partial r} + \frac{\partial\Phi(r,\theta)}{\partial\theta}\right)(r(t),\theta(t))\,dt.$$

By Lemma 2.1, the expression in polar coordinates of (4) is

$$\dot r = \operatorname{Re}(A + S(\theta))\, r + \operatorname{Re}(U(\theta))\, r^{m+n},$$
$$\dot\theta = \operatorname{Im}(A + S(\theta)) + \operatorname{Im}(U(\theta))\, r^{m+n-1},$$

where

$$S(\theta) = B\, \mathrm{e}^{-2i\theta}, \quad U(\theta) = C\, \mathrm{e}^{(m-n-1)i\theta}.$$

Hence

$$\frac{1}{r}\frac{\partial(rR(r,\theta))}{\partial r} = 2\operatorname{Re}(A + S(\theta)) + (m+n+1)\operatorname{Re}(U(\theta))\, r^{m+n-1},$$

and

$$\begin{aligned}\frac{\partial\Phi(r,\theta)}{\partial\theta} &= \frac{\partial}{\partial\theta}\operatorname{Im}(A + S(\theta)) + \frac{\partial}{\partial\theta}\operatorname{Im}(U(\theta))\, r^{m+n-1}\\ &= -2\operatorname{Re}(S(\theta)) + (m-n-1)\operatorname{Re}(U(\theta))\, r^{m+n-1}.\end{aligned}$$

Therefore,

$$\sigma = \int_0^T 2\operatorname{Re}(A) + 2m\operatorname{Re}\big(U(\theta(t))\big)\, r^{m+n-1}(t)\, dt. \tag{10}$$

Using that $r = r(t), \theta = \theta(t)$ is a T-periodic orbit we get that

$$0 = \int_0^T \frac{\dot r(t)}{r(t)}\, dt = \int_0^T \operatorname{Re}\big(A + S(\theta(t))\big) + \operatorname{Re}\big(U(\theta(t))\big)\, r^{m+n-1}(t)\, dt,$$

and we can write (10) as

$$\begin{aligned}\sigma &= \int_0^T 2(1-m)\operatorname{Re}(A) - 2m\operatorname{Re}\big(S(\theta(t))\big)\, dt\\ &= 2m\int_0^T \frac{1-m}{m}\operatorname{Re}(A) - \operatorname{Re}\big(B\, \mathrm{e}^{-2i\theta(t)}\big)\, dt.\end{aligned}$$

Clearly, since $(1-m)\operatorname{Re}(A) \neq 0$, if $|B| \le (m-1)|\operatorname{Re}(A)|/m$, the above integrand does not change sign, and hence $\sigma \neq 0$ and $\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\operatorname{Re}(A))$, as we wanted to prove.

Let us continue studying the critical points of the differential equation. Using Lemma 2.3 the divergence of X at the origin is $2\operatorname{Re}(A)$. At any other critical point, (r^*, θ^*) with $r^* \neq 0$, it holds that

$$\operatorname{Re}\big(A + S(\theta^*)\big) + \operatorname{Re}\big(U(\theta^*)\big)\big(r^*\big)^{m+n-1} = 0.$$

Hence, using again Lemma 2.3 and similar computations that in the study of the stability of the periodic orbits we get that

$$\operatorname{div}\big(X(r^*, \theta^*)\big) = 2m\Big(\frac{1-m}{m}\operatorname{Re}(A) - \operatorname{Re}\big(B\, \mathrm{e}^{-2i\theta^*}\big)\Big).$$

Therefore, under the hypotheses of the statement,

$$\operatorname{sgn}\big(\operatorname{div}\big(X(r^*, \theta^*)\big)\big) = -\operatorname{sgn}(\operatorname{Re}(A)).$$

Moreover, by Lemma 2.3, at the origin the determinant of the differential of X is $\det((\mathrm{d}\,X)_{(0,0)}) = |A|^2 - |B|^2$ which is positive because

$$|A|^2 - |B|^2 > |A|^2 - \Big(\frac{m}{m-1}\Big)^2 |B|^2 \ge |A|^2 - \big(\operatorname{Re}(A)\big)^2 \ge 0.$$

Notice that, since under our hypotheses, the divergence does not vanish at the critical points, all them are elementary. Therefore they are of node, foci, saddle or saddle-node type. Moreover the stability of the node, foci and nodal sector of the saddle-node points is controlled by the sign of $\mathrm{Re}(A)$. In particular the origin is a node or a focus.

Finally we study the stability of the polycycles. Their corners are formed by hyperbolic saddles or semi-hyperbolic saddles or saddle-nodes. Since the divergence at them has always the sign of $-\mathrm{Re}(A)$, by Proposition 2.7 we know that all the polycycles are stable (resp. unstable) when $\mathrm{Re}(A) > 0$ (resp. $\mathrm{Re}(A) < 0$).

From now on we fix $\mathrm{Re}(A) > 0$. The case $\mathrm{Re}(A) < 0$ can be studied similarly. Therefore we know that:

(a) Focus and node points different from the origin are attractor.
(b) The origin is an unstable focus or node.
(c) Saddle-node points have the nodal sector of attracting type.
(d) All periodic orbits are hyperbolic and attractive limit cycles.
(e) All polycycles are attractors.

Let us prove the uniqueness of the limit cycle. Consider a periodic orbit γ of the differential equation and denote by $\mathcal{D}$ the bounded region that it surrounds. First, we prove that the origin must be in $\mathcal{D}$.

Assume, to arrive to a contradiction, that the origin is not in $\mathcal{D}$. Then all periodic orbits and polycycles contained in $\mathcal{D}$ are also attractors. The same holds with the critical points of index $+1$ and with the nodal sectors of the saddle-node points. In short, there are only a measure zero set of points in $\mathcal{D}$ that can have α-limit (the unstable manifolds of the saddle and saddle-node points). This result is in contradiction with the Poincaré–Bendixson theory. Hence all periodic orbits must surround the origin, and eventually other critical points.

To end the proof let us show that there is at most one limit cycle surrounding the origin. Assume that there were two, γ_1 and γ_2. Then we can consider the annular bounded region $\mathcal{G}$ with boundary $\gamma_1 \cup \gamma_2$. Arguing as in the previous paragraph but in the region $\mathcal{G}$ we would arrive again to a contradiction. So the theorem is proved. □

4.2. **Systems with homogeneous nonlinearities and limit cycles.** In this section we give several equations of the form (4) with limit cycles.

Consider first equation (4) with $A = a + i$ and $B = 0$, that is

$$\dot{z} = (a+i)z + Cz^{n+1}\bar{z}^n, \quad n \geq 1.$$

Because it is in normal form, the stability of the origin is given by the sign of $\mathrm{Re}(a+i) = a$ when $a \neq 0$. When $a = 0$ the next non-zero Lyapunov quantity is given by $\mathrm{Re}(C)$. Therefore, when $a\,\mathrm{Re}(C) < 0$ and $|a|$ is small enough, a hyperbolic limit cycle is born from the origin by an Andronov–Hopf like bifurcation. Hence, for these values of a and C, and $|B|$ small enough, the differential equation

$$\dot{z} = (a+i)\,z + B\,\bar{z} + C\,z^{n+1}\bar{z}^n,$$

has exactly one limit cycle surrounding the origin and is under the hypotheses of Theorem B.

We continue with a second example of application of Theorem B.

Lemma 4.1. *Consider differential equation*

$$\dot{z} = \big(\varepsilon\,(1-\lambda) + i\big)\, z - \varepsilon\,(1+\lambda)\,\bar{z} - \frac{1}{2}\, i\, z^2, \tag{11}$$

with $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ a small parameter. Then the following holds:

(i) When $\lambda \in (0,1)$ and ε is small enough, it has a limit cycle surrounding the origin.

(ii) When $\lambda \in (-1/3, 0)$ and ε is small enough, it has a limit cycle surrounding the critical point that when $\varepsilon = 0$ is at $z = 2$.

(iii) When $\lambda \in [-3, -1/3]$, it is under the hypotheses of Theorem B.

Proof. Notice that for $\varepsilon = 0$ equation (11) is a holomorphic differential equation. It has two centers, located at $z = 0$ and $z = 2$. Doing similar computations to the ones appearing in [13, 23], we can easily obtain the Melnikov functions associated to the period annuli of both centers, named M_0 and M_2, respectively. They are:

$$\begin{aligned} M_0(\rho) &= 2\rho(\rho-1)(\rho-\lambda), \\ M_2(\rho) &= -2(2\lambda+1)\rho(\rho-1)(\rho-\lambda^*), \quad \text{with} \quad \lambda^* = -\frac{\lambda}{2\lambda+1}, \end{aligned}$$

where, in both cases $\rho \in (0,1)$, parameterizes the continua of periodic orbits surrounding each of the centers. In fact, the computations done in one of the cases can be obtained from the ones of the other case because the change of variables $w = 2 - z$ together with a constant rescaling of the time leaves invariant the differential equation (11) with $\varepsilon = 0$.

Therefore, since when $\lambda \in (0,1)$, $M_0(\lambda) = 0$ and $\rho = \lambda$ is a simple zero of M_0, statement (i) follows, see [12].

Similarly, when $\lambda \in (-1/3, 0)$, $\lambda^* \in (0,1)$, $M_2(\lambda^*) = 0$ and $M_2'(\lambda^*) \neq 0$ and item (ii) is proved.

Observe that for $\lambda > 0$, $\lambda\lambda^* < 0$. Hence, two limit cycles surrounding simultaneously each of them one of the centers cannot appear using this approach.

Finally, notice that for equation (11), the inequality in Theorem B,

$$|B| \le \frac{(m-1)|\operatorname{Re}(A)|}{m},$$

writes as $|1+\lambda| \le |1-\lambda|/2$, which is equivalent to $\lambda \in [-3, -1/3]$, as we wanted to prove. $\square$

Remark 4.2. *Notice that by Theorem B and Lemma 4.1(iii), when $\lambda \in [-3, -1/3]$ the differential equation* (11) *has no limit cycles surrounding the critical point that is not the origin. Note that $\lambda = -1/3$ is one of the boundaries of the interval and precisely, for any $-1/3 > \lambda > 0$, a limit cycle appears surrounding the critical point that is near $z = 2$. This fact shows that Theorem B sharply detects this phenomenon.*

Now we present a couple of examples with more than one limit cycle. These examples show the necessity of adding some hypotheses for proving uniqueness of limit cycles for equation (4).

Lemma 4.3. *There are equations of the form*

$$\dot{z} = A\, z + B\,\bar{z} + z^3, \tag{12}$$

having at least two limit cycles, each one of them surrounding a different critical point.

Proof. Consider the 1-parameter family,

$$\dot{z} = (-9+28\,\varepsilon^2+16\,\varepsilon^3+(4+8\,\varepsilon)i)\,z+(10+18\,\varepsilon+16\,\varepsilon^2+8\,\varepsilon^3)\,\bar{z}+(4+8\,\varepsilon)\,z^3. \quad (13)$$

It is constructed forcing that for all $\varepsilon \in \mathbb{R}$, the points $z_\varepsilon^\pm = \pm(1+(1/2+\varepsilon)i)$ are critical points of the equation. Moreover, using Lemma 2.3, we get that

$$\begin{aligned} \operatorname{div}(X)_{z_\varepsilon^+} &= 4\,\varepsilon(3+2\,\varepsilon)(1-2\,\varepsilon), \\ \det(\mathrm{d}\,X)_{z_\varepsilon^+} &= 4(39+280\,\varepsilon+456\,\varepsilon^2+224\,\varepsilon^3-16\,\varepsilon^4)(1+2\,\varepsilon). \end{aligned} \quad (14)$$

Hence, for $\varepsilon = 0$, the divergence of system (13) vanishes and $\det(\mathrm{d}\,X) > 0$ at z_0^+ and so the differential equation has a weak focus at $z = z_0^+$. Therefore, using an afine change of variables together with a constant rescaling of the time, we can write equation (13) with $\varepsilon = 0$ in the form (5). Applying Lemma 2.6 we get that the first Lyapunov quantity of the equation at the point $z = z_0^+$ is

$$V_3 = -\frac{30}{169}\sqrt{39}\,\pi,$$

and, as a consequence, it is a first order attracting weak focus. Hence, using (14), we know that for $\varepsilon > 0$, small enough, a hyperbolic attracting limit cycle is born for equation (13) via an Andronov–Hopf bifurcation and this limit cycle surrounds the point z_ε^+. Since the equation is invariant by the change of variables $z \to -z$, a second symmetric limit cycle appears surrounding z_ε^-, and the lemma follows. □

It is easy to see that Theorem B does not apply for equation (13) when ε is small, but it applies for other values of ε.

Finally, our numerical simulations also show that differential equation (12), with $A = 1+9i/4$, $B = 13/4+i/2$ has at least four limit cycles, see in Figure 4 its phase portrait on the Poincaré disc.

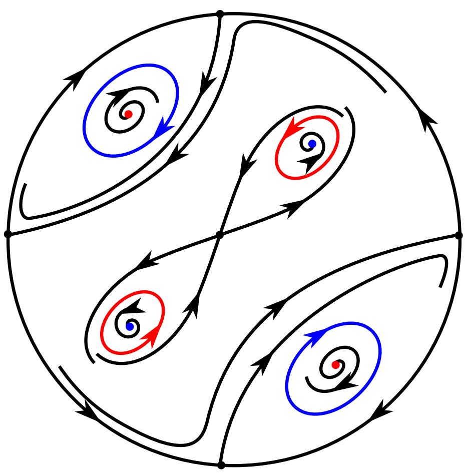

FIGURE 4. Phase portrait of equation (12) on the Poincaré disc.

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