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On the number of critical periods for planar polynomial systems of arbitrary degree

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ABSTRACT

We construct a class of planar systems of arbitrary degree n having a reversible center at the origin and such that the number of critical periods on its period annulus grows quadratically with n. As far as we know, the previous results on this subject gave systems having linear growth.

1. Introduction and main results

Given a smooth planar autonomous vector field having a continuum of periodic orbits we can parameterize them by a real number h, through a global transversal smooth section, and then introduce the *period function*, $T(h)$, as the smooth positive function which assigns to each orbit its minimal period. The isolated zeros of the derivative of this function are called *critical periods*. It is not difficult to prove that for a given continuum of periodic orbits, the number of critical periods depends neither on the transversal section, nor on its parametrization. It is well known that this function plays an

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important role in the study of many differential equations. For instance, in mathematical models in physics or ecology, see [9,11,18,20] and the references therein; in the study of the bifurcations of isolated periodic orbits from continua of them, see [5, pp. 369–370]; in the description of the dynamics of some integrable discrete dynamical systems, see [6]; or in the control of the number of solutions of some boundary value problems, see [1,2].

We also like the following point of view: when studying the celebrated Hilbert's 16th problem a primer approach in order to know what we can expect for the number of limit cycles of planar polynomial vector fields of degree n is the construction of examples having as many limit cycles as possible, or in other words to study lower bounds for the so-called Hilbert's numbers. As far as we know, if one only considers limit cycles surrounding one single critical point, then the best lower bounds for the number of limit cycles are $O(n^2)$, see for instance [13,17] and the references therein, while the best results concerning any configuration of limit cycles provide $O(n^2 \log n)$ lower bounds, see [4] and also [15,16]. It is worth to notice that most of the examples are obtained perturbing continua of periodic orbits.

On the other hand, to the best of our knowledge, the higher lower bound known for the number of critical periods of planar polynomial vector of degree n grows linearly with n, see [7,10]. Our main result is:

Theorem A. *There exist polynomial vector fields of degree n whose number of critical periods grows at least quadratically with n.*

Our proof gives the explicit expression $n^2/4 + 3n/2 - 4$ when n is even and a similar one when n is odd, see Theorem 7. As we will see it uses the strategy of perturbing a planar system having a continuum of periodic orbits (in fact a global isochronous center). Note that in our situation we have to consider the perturbation in such a way that the center structure remains. We have chosen to use reversible isochronous centers with reversible perturbations. Notice that as a consequence of our approach, all the critical periods that we obtain correspond to periodic orbits which only surround a critical point.

From our results it seems natural to state the following questions:

Question 1. Are there planar polynomial vector fields of degree n having $O(n^2 \log n)$ (or more) critical periods? Probably it will be necessary to consider continua of periodic orbits forming several nests.

Question 2. Is there some uniform bound (only depending on n) for the number of critical periods of polynomial vector fields of degree n?

It is clear that this second problem is a version of the second part of Hilbert's 16th problem, changing the number of limit cycles by the number of critical periods. It seems to us a very difficult one. A related result concerning the finiteness of critical periods for a given system was studied some years ago in [3].

2. Preliminary results

We start by recalling three well-known results of Lebesgue's theory adapted to our notation and interests. The reader interested in their proofs can consult any classical textbook on the subject, like for instance [14,19].

In all these results $\{f_m(\theta)\}_{m\in\mathbb{N}}$ denote a sequence of real valued measurable functions defined on a finite or infinite interval $\mathcal{I} \subset \mathbb{R}$.

Theorem 1 *(Lebesque's Monotonous Convergence Theorem). If for all $m \in \mathbb{N}$, $f_m(\theta) \geqslant 0$ and $\{f_m(\theta)\}_{m\in\mathbb{N}}$ is a monotonous increasing sequence, i.e.*

$$0 \leqslant f_1(\theta) \leqslant f_2(\theta) \leqslant \cdots \leqslant f_m(\theta) \leqslant f_{m+1}(\theta) \leqslant \cdots,$$

then

$$\lim_{m\to\infty}\int_{\mathcal{I}} f_m(\theta)\,d\theta = \int_{\mathcal{I}} \lim_{m\to\infty} f_m(\theta)\,d\theta,$$

even when some of these expressions is infinity.

Theorem 2 *(Lebesque's Dominated Convergence Theorem). If for all $m \in \mathbb{N}$, $|f_m(\theta)| \leqslant g(\theta)$ and g is integrable on $\mathcal{I}$, then*

$$\lim_{m\to\infty}\int_{\mathcal{I}} f_m(\theta)\,d\theta = \int_{\mathcal{I}} \lim_{m\to\infty} f_m(\theta)\,d\theta.$$

Theorem 3 *(Lesbesgue–Fubini's Integration Theorem, integrals and series version). If*

$$\text{either} \quad \int_{\mathcal{I}} \sum_{m=1}^{\infty} \left|f_m(\theta)\right| d\theta < \infty \quad \text{or} \quad \sum_{m=1}^{\infty} \int_{\mathcal{I}} \left|f_m(\theta)\right| d\theta < \infty,$$

then

$$\int_{\mathcal{I}} \sum_{m=1}^{\infty} f_m(\theta)\,d\theta = \sum_{m=1}^{\infty} \int_{\mathcal{I}} f_m(\theta)\,d\theta.$$

Next two results study some properties of the family of functions,

$$J(h) := J_{\alpha,p,q}(h) = \int_0^{2\pi} (h-\cos\theta)^{\alpha} \sin^{2p}\theta \cos^q\theta\,d\theta, \tag{1}$$

keeping $h \geqslant 1$, $\alpha < 0$ real and p and q non-negative integers. They will play a crucial role when studying the period function of the family of perturbed rigid centers considered in the next section.

Proposition 4. *Let $\alpha < 0$ and p and q be non-negative integers. Consider the function $J(h)$ defined in (1). Then:*

(i) *It is analytic for $h \in (1,\infty)$ and*

$$J^{(r)}(h) = r!\binom{\alpha}{r}\int_0^{2\pi} (h-\cos\theta)^{\alpha-r} \sin^{2p}\theta \cos^q\theta\,d\theta,$$

where $J^{(r)}(h)$ is the r-th order of derivative of $J(h)$.

(ii) *$J(1) < \infty$ if and only if $2\alpha + 2p > -1$.*

(iii) *It holds that*

$$\lim_{h\downarrow 1^+} J(h) = J(1), \tag{2}$$

also meaning that the limit is infinity when $2\alpha + 2p \leqslant -1$.

Proof. The proof of (i) is easy, so we omit it.

Notice that the convergence condition stated in (ii) holds because

$$(1-\cos\theta)^{\alpha}\sin^{2p}\theta\cos^{q}\theta \sim 2^{-\alpha}\theta^{2\alpha+2p} \quad \text{at } \theta = 0.$$

To prove item (iii) we write $J(h) = J_1(h) + J_2(h)$ where

$$J_1(h) = \int_{-\pi/2}^{\pi/2} j(h,\theta)\,d\theta \quad \text{and} \quad J_2(h) = \int_{\pi/2}^{3\pi/2} j(h,\theta)\,d\theta,$$

where $j(h,\theta) := (h-\cos\theta)^{\alpha}\sin^{2p}\theta\cos^{q}\theta$.

Concerning $J_1(h)$, note that for $\theta \in (-\pi/2, \pi/2)$, the family of positive functions $\{j(h,\theta)\}_{h>1}$ is increasing when $h \downarrow 1^+$ because $\alpha < 0$ and $\sin^{2p}\theta\cos^{q}\theta \geqslant 0$. Hence we can apply Theorem 1 obtaining

$$\lim_{h\downarrow 1^+} J_1(h) = \lim_{h\downarrow 1^+} \int_{-\pi/2}^{\pi/2} j(h,\theta)\,d\theta = \int_{-\pi/2}^{\pi/2} \lim_{h\downarrow 1^+} j(h,\theta)\,d\theta = J_1(1),$$

and the equality is valid even when $J_1(1)$ is divergent.

When $\theta \in (\pi/2, 3\pi/2)$, it holds that $|h-\cos\theta|^{\alpha} \leqslant 1$ and hence

$$\bigl|j(h,\theta)\bigr| = \bigl|(h-\cos\theta)^{\alpha}\sin^{2p}\theta\cos^{q}\theta\bigr| \leqslant \sin^{2p}\theta\bigl|\cos^{q}\theta\bigr|,$$

which is integrable on this interval. So, by Theorem 2,

$$\lim_{h\downarrow 1^+} J_2(h) = J_2(1) < \infty.$$

As a consequence the equality (2) holds, as we wanted to prove. □

Proposition 5. *Let $\alpha < 0$ and p and q be non-negative integers. Consider the function $J(h)$ defined in (1). Then for $h \in (1,\infty)$:*

$$J(h) = h^{\alpha}\sum_{r=0}^{\infty}\beta_r h^{-r},$$

where

$$\beta_r = \beta_r(\alpha,p,q) := (-1)^r\binom{\alpha}{r}\int_0^{2\pi}\sin^{2p}\theta\cos^{p+r}\theta\,d\theta.$$

Notice that $\beta_r = 0$ if and only if $p+r$ is odd.

Proof. For $h > 1$,

$$(h-\cos\theta)^{\alpha} = h^{\alpha}\left(1-\frac{\cos\theta}{h}\right)^{\alpha} = h^{\alpha}\sum_{r=0}^{\infty}\binom{\alpha}{r}\left(\frac{-\cos\theta}{h}\right)^{r}.$$

Hence

$$J(h) = h^{\alpha} \int_0^{2\pi} \sum_{r=0}^{\infty} \binom{\alpha}{r} \left(\frac{-\cos\theta}{h} \right)^r \sin^{2p}\theta \cos^q \theta \, d\theta$$
$$= h^{\alpha} \int_0^{2\pi} \sum_{r=0}^{\infty} \binom{\alpha}{r} (-1)^r h^{-r} \sin^{2p}\theta \cos^{q+r}\theta \, d\theta = h^{\alpha} \sum_{r=0}^{\infty} \beta_r h^{-r},$$

where to justify the last equality we need to prove that the integration and the summation can be interchanged. This can be done by using Theorem 3, because for $h > 1$,

$$\int_0^{2\pi} \sum_{r=0}^{\infty} \left| \binom{\alpha}{r} (-1)^r h^{-r} \sin^{2p}\theta \cos^{q+r}\theta \right| d\theta$$
$$\leqslant \int_0^{2\pi} \sum_{r=0}^{\infty} \binom{\alpha}{r} (-1)^r h^{-r} \, d\theta = 2\pi \left(1 - \frac{1}{h} \right)^{\alpha} < \infty. \qquad \square$$

We finally state a technical result.

Lemma 6. *Let $\mathcal{U} \subset \mathbb{R}$ an open interval and let $f : \mathcal{U} \to \mathbb{R}$ be an analytic map having ℓ different zeros in $\mathcal{U}$ (not taking into account their multiplicities). Let $g : \mathcal{U} \to \mathbb{R}$ be an analytic map with constant sign. Then there exists $\alpha \in \mathbb{R}$ such that $f + \alpha g$ has at least ℓ simple zeros in $\mathcal{U}$.*

We only present an idea of the proof. More details are given in Lemma 4.5 of [8]. If $|\alpha|$ is small enough, the number of zeros of odd multiplicity never decreases and all of them become simple. The zeros of even multiplicity are divided between local maxima and local minima of f. Assume for instance that there are more or equal local maxima than local minima. By choosing adequately the sign of αg, and taking again $|\alpha|$ small enough, near each maximum there appear two simple zeros of $f + \alpha g$ and no zeros near the minima. In any case $f + \alpha g$ has at least ℓ simple zeros. Intuitively, the fact that g does not vanish allows to think on it like a constant.

3. Proof of Theorem A

We first prove the following result:

Theorem 7. *For each n even there is a polynomial system of degree n having a center such that the period function associated to its period annulus has at least $n^2/4 + 3n/2 - 4$ critical periods.*

Proof. Consider the following system

$$\begin{cases} \dot{x} = -y - xy\left(x^2 + y^2\right)^{n/2-1} + \varepsilon P(x, y), \\ \dot{y} = x - y^2\left(x^2 + y^2\right)^{n/2-1} + \varepsilon Q(x, y), \end{cases} \tag{3}$$

where P and Q are polynomials of degree n satisfying that $P(x, -y) = -P(x, y)$ and $Q(x, -y) = Q(x, y)$ and vanishing at the origin and ε a small parameter. Clearly (3) is a reversible system (invariant with respect to the transformation $(x, y, t) \to (x, -y, -t))$ and the origin is a center. In the polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, it writes as

$$\begin{cases} \dot r = -r^n \sin\theta + \varepsilon U(r, \theta), \\ \dot\theta = 1 + \varepsilon V(r, \theta), \end{cases} \tag{4}$$

where

$$U(r, \theta) = P(r\cos\theta, r\sin\theta)\cos\theta + Q(r\cos\theta, r\sin\theta)\sin\theta,$$
$$V(r, \theta) = \frac{Q(r\cos\theta, r\sin\theta)\cos\theta - P(r\cos\theta, r\sin\theta)\sin\theta}{r}.$$

It is clear that when $\varepsilon = 0$ it has a global rigid isochronous center at the origin. Moreover, also when $\varepsilon = 0$, it is easy to see that

$$r = R(r_0, \theta) := \left(r_0^{1-n} + (n-1)(1-\cos\theta)\right)^{\frac{1}{1-n}}$$

is the solution of the differential equation satisfying $R(r_0, 0) = r_0$.

Let $T(r_0, \varepsilon)$ be the period of the orbit of (4) starting at $(x, y) = (r_0, 0)$. By using the results of [12] we have that

$$T(r_0, \varepsilon) = 2\pi + \widetilde{I}(r_0)\varepsilon + O\left(\varepsilon^2\right), \tag{5}$$

where

$$\widetilde{I}(r_0) = -\int_0^{2\pi} V(r, \theta)|_{r=R(r_0,\theta)}\, d\theta$$
$$= \int_0^{2\pi} \frac{P(r\cos\theta, r\sin\theta)\sin\theta - Q(r\cos\theta, r\sin\theta)\cos\theta}{r}\bigg|_{r=R(r_0,\theta)} d\theta,$$

and the remainder term in (5) and its derivative with respect to r_0, divided by ε^2, are bounded when r_0 takes values on a compact set and $|\varepsilon|$ is small enough. Hence we have that

$$\frac{\partial T(r_0, \varepsilon)}{\partial r_0} = \widetilde{I}'(r_0)\varepsilon + O\left(\varepsilon^2\right). \tag{6}$$

We claim that there exist P and Q as above such that $\widetilde{I}'(r_0)$ has at least $n^2/4 + 3n/2 - 4$ simple zeros. By using this claim and the Implicit Function Theorem, it is clear from (6) that our theorem follows. Let us prove the claim.

By introducing the new parameter $h = \Phi(r_0) := \frac{r_0^{1-n}}{n-1} + 1$ it follows that $r = R(r_0, \theta)$ can be written as

$$r = (n-1)^{\frac{1}{1-n}} (h - \cos\theta)^{\frac{1}{1-n}}. \tag{7}$$

Notice that the origin corresponds to h equals to infinity and the infinity goes to $h = 1$.

For simplicity, instead of studying $\widetilde{I}(r_0)$ and its derivative we will consider the same question for $I(h) := \widetilde{I}(\Phi^{-1}(h))$. After some computations we get that

$$I(h)=\int_0^{2\pi}\sum_{k=1}^{n-1} r^k\left(b_{k,0}\cos^{k+2}\theta+b_{k,1}\sin^2\theta\cos^k\theta+\cdots+b_{k,l_k}\sin^{2l_k}\theta\cos^{k+2-2l_k}\theta\right)d\theta,$$

with r given in (7) and $l_k=[\frac{k+2}{2}]$. Notice that the constants $\{b_{k,m}\}$, varying k and m among the values given above, can be taken arbitrarily by choosing suitable P and Q. Introduce the family of functions

$$I_{k,m}(h)=\int_0^{2\pi}(h-\cos\theta)^{\frac{k}{1-n}}\sin^{2m}\theta\cos^{k+2-2m}\theta\,d\theta,$$

for $k=1,\dots,n-1$; $m=0,\dots,l_k$. Then

$$I(h)=\sum_{k=1}^{n-1}\sum_{m=0}^{l_k}a_{k,m}I_{k,m}(h),$$

where the $\{a_{k,m}\}_{k,m}$ can also be considered as free parameters. Note that this set is formed by

$$\sum_{k=1}^{n-1}\left(\left[\frac{k+2}{2}\right]+1\right)=2+2\left(3+4+\cdots+\left(\frac{n}{2}+1\right)\right)=\frac{n^2}{4}+\frac{3n}{2}-2$$

functions. We shall show that all of them are linear independent. In other words, if $I(h)\equiv 0$, then $a_{k,m}=0$ for all k and m.

The linear independence holds by proving the following two assertions.

First assertion: If $I(h)\equiv 0$ then

$$I_k(h):=a_{k,0}I_{k,0}(h)+a_{k,1}I_{k,1}(h)+\cdots+a_{k,l_k}I_{k,l_k}(h)\equiv 0,$$

for each $k=1,\dots,n-1$.

Second assertion: Given k, if $I_k(h)\equiv 0$ then

$$a_{k,0}=a_{k,1}=\cdots=a_{k,l_k}=0.$$

To prove the first assertion notice that by Proposition 5, for $h>1$, each

$$I_{k,m}(h)=h^{\frac{k}{1-n}}\sum_{r=0}^{\infty}\beta_r h^{-r},\tag{8}$$

for some $\beta_r=\beta_r(k,m)$ depending on k and m. So each $I_k(h)$ has the form

$$I_k(h)=h^{\frac{k}{1-n}}\sum_{r=0}^{\infty}\gamma_r h^{-r},$$

where γ_r depends on n,k and $a_{k,0},a_{k,1},\dots,a_{k,l_k}$.

Clearly for every different $k_1,k_2\in\{1,2,\dots,n-1\}$, $\frac{k_1}{1-n}-\frac{k_2}{1-n}=\frac{k_1-k_2}{1-n}$ can never be an integer. Therefore the relation that $I(h)\equiv 0$ implies that $I_k(h)\equiv 0$, for any $k=1,\dots,n-1$, as we wanted to show.

To prove the second assertion we could study the conditions obtained by imposing that $I_k(h) \equiv 0$, taking into account the values $\beta_r = \beta_r(k, m)$ given in (8), but it is much easier to study the values of $I_{k,m}(h)$ near $h = 1$. This will be done by using Proposition 4.

We consider two cases: (a) $2k > n - 1$, and (b) $2k < n - 1$.

Case (a): $2k > n - 1$. In this case, by Proposition 4, the first term, i.e. $I_{k,0}(h)$ is divergent at $h = 1$ because $-2 \leqslant \frac{2k}{1-n} < -1$. All the other $I_{k,m}(h)$, $m > 0$, are convergent because $\frac{2k}{1-n} + 2m \geqslant 0$. Therefore $a_{k,0} = 0$.

Now we drop the first term and show the vanishing of the second one. To prove this, we consider its first order derivative, $I'_k(h)$. Also by Proposition 4, for each m,

$$I'_{k,m}(h) = \frac{k}{1-n} \int_0^{\pi} (h - \cos\theta)^{\frac{k}{1-n} - 1} \sin^{2m}\theta \cos^{k+2-2m}\theta \, d\theta.$$

Again by Proposition 4, when h tends to 1^+ the new leading term is divergent and all the others are convergent. Therefore the second term $a_{k,1}$ of $I_k(h)$ also vanishes.

We go on this process by considering successive derivatives of $I_{k,m}$, obtaining that in each step we can make the leading term divergent and the others convergent. Thus finally we have proved that all the coefficients are zero, as we wanted to see.

Case (b): $2k < n - 1$. In this case, notice that by Proposition 4 all the terms are convergent because for all m, $\frac{2k}{1-n} + 2m > -1$. To study this situation we follow again the above procedure but starting with $I'_k(h)$ instead of $I_k(h)$. Thus the second assertion follows.

So $I(h)$ is an arbitrary linear combination of $n^2/4 + 3n/2 - 2$ linearly independent functions defined on $(1, \infty)$. It is easy to prove that given any $n^2/4 + 3n/2 - 3$ values on $(1, \infty)$ we can choose suitable values $a_{k,m}$ such that $I(h)$ vanishes at these points and it is not identically zero. By Rolle's Theorem we know that $I'(h)$ has at least $n^2/4 + 3n/2 - 4$ zeros in the same interval. Finally notice that

$$I'(h) = \sum_{k=1}^{n-1} \sum_{m=0}^{l_k} a_{k,m} I'_{k,m}(h) \tag{9}$$

and that it is easy to see that there always exist some $\tilde{k}$, $\tilde{m}$ such that $I'_{\tilde{k},\tilde{m}}(h) > 0$ on this interval. Therefore by using Lemma 6 with $f = I'$ and $g = I'_{\tilde{k},\tilde{m}}$ we can modify (9) in such a way that it has at least $n^2/4 + 3n/2 - 4$ simple zeros in $(1, \infty)$. Hence the claim follows, and the proof is finished. □

Remark 8. Since the coefficients $a_{k,m}$ in (9) are free, the $n^2/4 + 3n/2 - 4$ zeros of $I'(h)$ obtained in Theorem A can be placed in any given compact in $(1, \infty)$. As a consequence the periodic orbits of (3) giving rise to the critical periods can be located on any given annular region of the plane, which surrounds the origin.

Proof of Theorem A. When n is even the proof is clearly a consequence of Theorem 7. When n is odd, we can consider the same unperturbed reversible isochronous system of degree $n - 1$ given in (3) when $\varepsilon = 0$, but with the perturbation of degree n. The result will be similar and of course quadratic in n with the same dominant term $n^2/4$. □

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