Nonlinearity **13** (2000) 699–729. Printed in the UK PII: S0951-7715(00)03777-4

The focus–centre problem for a type of degenerate system

Armengol Gasull†, Jaume Llibre†, Víctor Mañosa‡ and Francesc Mañosas†

† Departament de Matemàtiques, Universitat Autònoma de Barcelona Edifici Cc, 08193 Bellaterra, Spain

‡ Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, Colom 1, 08222 Terrassa, Spain

E-mail: gasull@mat.uab.es, jllibre@mat.uab.es, victor.manosa@upc.es and manyosas@mat.uab.es

Received 23 April 1999, in final form 10 February 2000
Recommended by A I Neishtadt

Abstract. We consider differential systems in the plane defined by the sum of two homogeneous vector fields. We assume that the origin is a degenerate singular point for these differential systems. We characterize when the singular point is of focus–centre type in a generic case. The problem of its local stability is also considered. We compute the first generalized Lyapunov constant when some non-degeneracy conditions are assumed.

AMS classification scheme numbers: 58F10, 34C23, 58F14

1. Introduction

One of the classical problems in the qualitative theory of planar analytic differential systems is the study of the local phase portrait at the singularities to characterize when a singular point is of focus–centre type. Recall that a singular point is said to be of *focus–centre type* if it is either a focus or a centre. In what follows, this problem will be called the *focus–centre problem* or the *monodromy problem*. Of course, if the linear part of the singular point is non-degenerate (i.e. its determinant does not vanish) the characterization is well known. The problem has also been solved when the linear part is degenerate but not identically zero, see [2, 3]. Hence the main difficulties in solving the focus–centre problem appear when the singular point has an identically zero linear part. On the other hand, once we know that a singular point is of focus–centre type, one comes across another classical problem, usually called the *centre problem* or the *stability problem*, that is of distinguishing a centre from a focus. The Lyapunov–Poincaré theory was developed to solve this problem in the case where the singular point is non-degenerate, see [23, 28]. If the singular point has a nilpotent linear part, there are some results on the centre problem, see [27], but if the singular point has a zero linear part then there are very few results on the centre problem.

In this paper we study the focus–centre and centre problems for systems of the form

$$\begin{aligned} \dot{x} &= P(x, y) = P_m(x, y) + P_M(x, y) \\ \dot{y} &= Q(x, y) = Q_m(x, y) + Q_M(x, y) \end{aligned} \tag{1}$$

where P_k and Q_k are homogeneous polynomials of degree k, $k \in \{m, M\}$, $1 \leqslant m < M$, P and Q are coprime, and the dot denotes a derivative with respect to t. That is, these systems are

