The focus–centre problem for a type of degenerate system

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Abstract. We consider differential systems in the plane defined by the sum of two homogeneous vector fields. We assume that the origin is a degenerate singular point for these differential systems. We characterize when the singular point is of focus–centre type in a generic case. The problem of its local stability is also considered. We compute the first generalized Lyapunov constant when some non-degeneracy conditions are assumed.

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1. Introduction

One of the classical problems in the qualitative theory of planar analytic differential systems is the study of the local phase portrait at the singularities to characterize when a singular point is of focus–centre type. Recall that a singular point is said to be of *focus–centre type* if it is either a focus or a centre. In what follows, this problem will be called the *focus–centre problem* or the *monodromy problem*. Of course, if the linear part of the singular point is non-degenerate (i.e. its determinant does not vanish) the characterization is well known. The problem has also been solved when the linear part is degenerate but not identically zero, see [2, 3]. Hence the main difficulties in solving the focus–centre problem appear when the singular point has an identically zero linear part. On the other hand, once we know that a singular point is of focus–centre type, one comes across another classical problem, usually called the *centre problem* or the *stability problem*, that is of distinguishing a centre from a focus. The Lyapunov–Poincaré theory was developed to solve this problem in the case where the singular point is non-degenerate, see [23, 28]. If the singular point has a nilpotent linear part, there are some results on the centre problem, see [27], but if the singular point has a zero linear part then there are very few results on the centre problem.

In this paper we study the focus–centre and centre problems for systems of the form

$$\begin{aligned} \dot{x} &= P(x, y) = P_m(x, y) + P_M(x, y) \\ \dot{y} &= Q(x, y) = Q_m(x, y) + Q_M(x, y) \end{aligned} \tag{1}$$

where P_k and Q_k are homogeneous polynomials of degree k, $k \in \{m, M\}$, $1 \leqslant m < M$, P and Q are coprime, and the dot denotes a derivative with respect to t. That is, these systems are *defined to be the sum of two homogeneous vector fields*. The case where (1) is a homogeneous system (i.e. either $P_m \equiv Q_m \equiv 0$ or $P_M \equiv Q_M \equiv 0$) is already well understood, see [7, 20] and also [14], and we do not consider it here, although all our results apply in this case.

The polynomial systems with a non-degenerate linear part and homogeneous nonlinearities are included in the family of systems (1). There are many papers dealing with the focus–centre problem and the centre problem for these systems, see [4, 9, 11, 16–18, 21, 30–32]. Also for the degenerate case, cyclicity is studied for some classes of systems in [12, 13]. Among other questions, the number of limit cycles for systems defined by the sum of two *quasi-homogeneous* vector fields is studied in [15]. These systems are a generalization of systems (1). Finally, the centre–focus problem is studied in [5, 6], for certain degenerate singular points without characteristic directions, see also section 3.1.

In section 2, we give necessary conditions in order that a system (1) has a focus or a centre at the origin and also sufficient conditions in a generic case.

We denote $P_k(\theta) = P_k(\cos\theta, \sin\theta)$ and $Q_k(\theta) = Q_k(\cos\theta, \sin\theta)$ with $k \in \{m, M\}$. Consider system (1), and take polar coordinates (R, θ), given by the change of variables $R^2 = x^2 + y^2$ and $\theta = \arctan(y/x)$. After a rescaling of time given by $\mathrm{d}s/\mathrm{d}t = R^{m-1}$, we have (again denoting the derivative with respect to s by a dot),

$$\begin{aligned} \dot{R} &= R\left[\cos\theta P_m(\theta) + \sin\theta Q_m(\theta) + R^{M-m}\left(\cos\theta P_M(\theta) + \sin\theta Q_M(\theta)\right)\right] \\ \dot{\theta} &= \cos\theta Q_m(\theta) - \sin\theta P_m(\theta) + R^{M-m}\left(\cos\theta Q_M(\theta) - \sin\theta P_M(\theta)\right). \end{aligned} \tag{2}$$

We say that $\theta = \theta_*$ is a *characteristic direction* for the origin of system (1) if $\cos\theta_*\, Q_m(\theta_*) - \sin\theta_*\, P_m(\theta_*) = 0$.

We introduce the following two conditions.

We say that system (1) satisfies condition (a) if there exists a neighbourhood $\mathcal{U}$ of the origin of system (1) such that $\Theta(x, y) = xQ(x, y) - yP(x, y) \neq 0$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\}$. If system (1) satisfies condition (a), we will denote the sign of $\Theta(x, y)$ for all $(x, y) \in \mathcal{U} \setminus \{(0, 0)\}$ by $\mathrm{sign}_{\bar{0}}(\Theta)$.

We say that system (1) satisfies condition (b) if either it has no characteristic directions, or else if all characteristic directions are isolated and $P_m(\theta_*) = Q_m(\theta_*) = 0$ for every characteristic direction θ_*.

Let $\theta_1, \theta_2, \ldots, \theta_k$ be the characteristic directions associated with system (1). For all $j = 1, \ldots, k$, we set $a_j = \cos\theta_j$, $b_j = \sin\theta_j$, $\alpha_j = \frac{\mathrm{d}}{\mathrm{d}z}\left(P_m^j(1, z)\right)\big|_{z=0}$ and $\beta_j = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(Q_m^j(1, z)\right)\big|_{z=0}$, where

$$P_m^j(1, z) = a_j P_m(a_j - b_j z, b_j + a_j z) + b_j Q_m(a_j - b_j z, b_j + a_j z)$$

and

$$Q_m^j(1, z) = -b_j P_m(a_j - b_j z, b_j + a_j z) + a_j Q_m(a_j - b_j z, b_j + a_j z).$$

A vector field $\mathcal{X} = (P_m(x, y) + P_M(x, y), Q_m(x, y) + Q_M(x, y))$ belongs to class $\mathcal{G}$ if either there are no characteristic directions, or else if for every characteristic direction θ_j, $j = 1, \ldots, k$ we have $\alpha_j^2 + \beta_j^2 \neq 0$. It is clear that this is a generic condition inside our family.

At the end of section 2.2 we prove our main result about the centre–focus problem. This result characterizes the monodromy condition for a generic set of systems (1).

Theorem A (Focus–centre condition). *Let $\mathcal{X}$ be the vector field associated with system (1). Then the following statements hold.*

(i) If the origin of (1) is a focus–centre then conditions (a) and (b) are satisfied, and m is odd. Furthermore, if system (1) has characteristic directions, then M is also odd.

(ii) Assume that $\mathcal{X} \in \mathcal{G}$. Then the origin of system (1) is a focus–centre if and only if the system satisfies conditions (a) and (b), and that for every characteristic direction θ_j, $\operatorname{sign}_{\bar{0}}(\Theta)\left((2+M-m)\,\alpha_j - 2\beta_j\right) \leqslant 0$ for all $j = 1, \ldots, k$.

Note that expressed in polar coordinates (R, θ), system (1) satisfies $(\dot{R}(R, \theta+\pi), \dot{\theta}(R, \theta+\pi)) = (\dot{R}(R, \theta), \dot{\theta}(R, \theta))$. Hence in theorem A we only need to check the genericity condition $\alpha_j^2 + \beta_j^2 \neq 0$, and the monodromy condition $\operatorname{sign}_{\bar{0}}(\Theta)\,((2+M-m)\alpha_j - 2\beta_j) \leqslant 0$ for those characteristic directions $\theta_j \in [0, \pi)$.

The main tool in the proof of this result is the blow-up technique (see [8] for a brief geometric description).

The problem to decide whether a degenerate singular point of focus–centre type is either a centre or a focus is very complicated in comparison to the case of non-degenerate singular points. If the degenerate singular point has no characteristic directions and system (1) satisfies the focus–centre condition then it is possible to obtain necessary conditions in order that the origin of system (1) be a centre, one may also study its stability using the method developed by Lyapunov and Poincaré in [23, 28], respectively. However, if there exist characteristic directions, then no general methods are known.

In [26], Medvedeva gives the first term V_1 of the return map for any monodromic singular point of an analytic system. As far as we know, this result is the latest in a series of papers on this subject [10, 25, 26]. To apply her result, it is necessary to do all the blow-ups to desingularize the point, in order to decide whether it is monodromic, and then to compute V_1. For the family of considered systems (1), our approach can be considered as a different and shorter method. Shorter because we present an algebraic condition to ensure that the singular point is monodromic. We also give an explicit expression for V_1, which is effective in the sense that it is not necessary to make the blow-ups to obtain it. Our approach uses a kind of generalized blow-up which is the key to shortening the desingularization process. At the end of this process one obtains that the monodromic points have elementary saddle nodes (see sections 2.2 or 3.3 for an explicit example). We note that Dumortier in [19] and also Medvedeva in [25, 26] prove that with the usual blow-up process, monodromic points just give rise to hyperbolic or elementary degenerate saddles at the end of the desingularization process.

In section 3 we deal with the problem of determining the stability of the origin of system (1) when it is monodromic.

In subsection 3.1 we give the first generalized Lyapunov constants if there are no characteristic directions, while in subsection 3.2, we investigate the general case. To state the main result of this section, we need the following definitions.

If the origin of system (1) is monodromic then we define the *return map* $\Pi(x)$, for $x > 0$ small enough, to be the first coordinate of the first cut with the positive x-axis, of the solution of (1) with initial condition $(x, 0)$.

Given a function f, continuous on $[0, 2\pi] \setminus \{\theta_1, \theta_1, \ldots, \theta_k\}$, we define the *Cauchy global principal value* of $\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta$, to be the following limit (if it exists):

$$\operatorname{GPV}\left\{\int_0^{2\pi} f(\theta)\,\mathrm{d}\theta\right\} := \lim_{\varepsilon \to 0} \int_{I_\varepsilon} f(\theta)\,\mathrm{d}\theta$$

where $I_\varepsilon = \left(\mathbb{R} \setminus \cup_{j=1}^{k} (\theta_j - \varepsilon, \theta_j + \varepsilon)\right) \bmod ([0, 2\pi))$.

Theorem B. *Let $\mathcal{X} \in \mathcal{G}$. Suppose that the origin of system (1) (associated with $\mathcal{X}$) is a focus–centre, and that $\beta_j - \alpha_j \neq 0$ for all $j = 1, \ldots, k$. Then,*

(i)
$$\operatorname{GPV}\int_0^{2\pi} \frac{\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta)}{\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)}\,\mathrm{d}\theta$$

exists.

(ii) The return map associated with the origin has the form $\Pi(x_0) = V_1 x_0 + \mathrm{o}(x_0)$, *where*

$$V_1 = \exp\left\{\mathrm{sign}_{\bar{0}}(\Theta)\,\mathrm{GPV}\int_0^{2\pi} \frac{\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta)}{\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)}\,\mathrm{d}\theta\right\}.$$

The number V_1 is called the *generalized first Lyapunov constant*. Note that if there are no characteristic directions, then

$$V_1 = \exp\left\{\mathrm{sign}_{\bar{0}}(\Theta)\,\int_0^{2\pi} \frac{\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta)}{\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)}\,\mathrm{d}\theta\right\}$$

see also section 3.1.

The condition that $\beta_j - \alpha_j \neq 0$ for all $j = 1, \ldots, k$, implies that at the end of a sequence of blow-ups at the origin (the process is described in subsection 2.2) all singular points appearing in the desingularized vector field are either hyperbolic saddles, or elementary degenerate saddle nodes (an elementary degenerate point is a singular point such that its linear part has non-vanishing trace and vanishing determinant) which are in some nice geometric display, so we can compute the transition map associated with their hyperbolic sectors. In fact, the proof of the above theorem uses the knowledge of the coefficient of the leading terms of the transition maps associated with a hyperbolic sector of a hyperbolic saddle or an elementary degenerate point. The study of this coefficient and the description of the transition map of the flow in a neighbourhood of a characteristic direction is carried out in subsection 3.2.1, and is a key point for the proof of the results of this paper.

If $\beta_j - \alpha_j = 0$, for some $j \in \{1, \ldots, k\}$, then elementary degenerate saddles appear at the end of the desingularization process. In this case, we have a partial result, that will be proved in subsection 3.2.3. In particular, we show that

Proposition C. *Let* $\mathcal{X} \in \mathcal{G}$. *Suppose that the origin of system (1) is a focus–centre. Suppose that there are only two opposite characteristic directions* θ_{i_1} *and* θ_{i_2} *(i.e.* $\theta_{i_2} = \theta_{i_1} + \pi$*) such that* $\beta_{i_k} - \alpha_{i_k} = 0$, *for* $k = 1, 2$. *Then the return map associated with the origin has the form* $\Pi(x_0) = x_0 + \mathrm{o}(x_0)$.

Let us remark that under the hypotheses of proposition C, the number

$$\mathrm{GPV}\int_0^{2\pi} \frac{\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta)}{\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)}\,\mathrm{d}\theta$$

is not always zero, as may be seen in example 2 of subsection 3.3. Hence under the hypotheses of proposition C, the expression for V_1 given in theorem B is not valid.

We think that the methods developed in this paper can be applied in a more general context, for instance, to monodromic singular points of an analytic vector field, in such a way that at the end of the desingularization process, the only kind of singular points that appear are hyperbolic saddles.

2. Focus–centre conditions for systems defined by the sum of two homogeneous vector fields

In the next two subsections we give the key points to prove theorem A.

2.1. Necessary conditions

Proposition 1. *If system (1) has a focus–centre at the origin, then system (2) satisfies conditions (a) and (b).*

Proof. Suppose that system (1) has a focus or a centre at the origin. After the change of variables $r = R^{M-m}$, system (2) becomes

$$\begin{aligned} \dot{r} &= a(\theta)\, r + b(\theta)\, r^2 \\ \dot{\theta} &= c(\theta) + d(\theta)\, r \end{aligned} \tag{3}$$

where

$$a(\theta) = (M-m)(\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta))$$

$$b(\theta) = (M-m)(\cos\theta\, P_M(\theta) + \sin\theta\, Q_M(\theta))$$

$$c(\theta) = \cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)$$

$$d(\theta) = \cos\theta\, Q_M(\theta) - \sin\theta\, P_M(\theta).$$

Observe that $\dot{\theta}$ for system (3) is non-vanishing for $r > 0$ small enough, if and only if $\dot{\theta}$ for system (2) is non-vanishing for $R > 0$ small enough. Therefore, we will study system (3).

Let $p_* = (0, \theta_*)$ be a singular point of system (3). So $c(\theta_*) = 0$, and θ_* is a characteristic direction. First we will see that $c(\theta)$ is not identically zero, that is p_* is an isolated singular point on $r = 0$. Suppose that $c(\theta) = 0$ for all $\theta \in [0, 2\pi)$, then after the reparametrization $\mathrm{d}\tau/\mathrm{d}s = r$ (again denoting the derivative with respect to τ by a dot), system (3) can be written as

$$\begin{aligned} \dot{r} &= a(\theta) + b(\theta)\, r \\ \dot{\theta} &= d(\theta). \end{aligned} \tag{4}$$

Note that $\dot{\theta}$ is not identically zero for $r > 0$ (otherwise the flow would be radial in contradiction with the fact that (4) has a focus–centre at the origin). Hence $d(\theta)$ is not identically zero. Note also that the fact that $P_m(\theta)$ and $Q_m(\theta) = 0$ are not both identically zero implies that $a(\theta)$ is also not identically zero. Then $r = 0$ is not invariant for system (4), and this implies that system (1) cannot have a focus or a centre at the origin because it would have an infinite number of orbits starting or ending at the origin. Hence, we have proved that p_* is isolated on $r = 0$. Furthermore, we want to stress that $c(\theta)$ does not change its sign in a neighbourhood of θ_*. We assume without loss of generality that $c(\theta) = \dot{\theta}|_{r=0} \geqslant 0$ for all $\theta \in [0, 2\pi)$.

The differential matrix of the vector field defined by (3) at the point p_*, is given by

$$\begin{pmatrix} a(\theta_*) & 0 \\ d(\theta_*) & c'(\theta_*) \end{pmatrix}.$$

Since $c(\theta) \geqslant 0$ for all $\theta \in [0, 2\pi)$, we have that $c'(\theta_*) = 0$. Note that if $a(\theta_*) \neq 0$ then p_* is an elementary degenerate singular point, and by the theorem of classification of this type of critical point (see [3]), p_* is either a topological saddle, or a topological node, or a saddle node with orbits starting or ending at p_* with $r > 0$. This last situation contradicts the assumption that the origin is a focus–centre. Hence $a(\theta_*) = c(\theta_*) = 0$, and consequently $Q_m(\theta_*) = P_m(\theta_*) = 0$. Therefore, it is already proved that system (1) satisfies condition (b).

Now we will see that the singular points $(0, \theta_*)$ of system (3), are isolated inside the set of points where $\dot{\theta} = 0$. Note first that $d(\theta_*) \neq 0$, because otherwise the ray $\theta = \theta_*$ would be invariant for system (3). The curve in $\mathbb{R}^2$ for which $\dot{\theta} = 0$ is given by

$$\left\{ r = -\frac{c(\theta)}{d(\theta)}; \theta \in [0, 2\pi) \text{ and } d(\theta) \neq 0 \right\}.$$

If $d(\theta_*) < 0$ then

$$\dot{\theta}|_{\theta=\theta_*, r>0} = r d(\theta_*) < 0.$$

Therefore, $\dot{\theta}$ is negative over the ray $\theta = \theta_*$. This fact, jointly with the fact that $\dot{\theta}|_{r=0} \geqslant 0$, prevents the existence of a return map in a neighbourhood of the origin, which contradicts the hypothesis that the origin is a monodromic point. Therefore, $d(\theta_*) > 0$, and consequently there exists $\varepsilon > 0$, such that $d(\theta) > 0$ when $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon)$. Then, since $c(\theta) \geqslant 0$ and we are looking for solutions $\dot{\theta} = 0$ with $r \geqslant 0$, the curve $\dot{\theta} = 0$ is not defined for $\theta \in (\theta_* - \varepsilon, \theta_* + \varepsilon) \setminus \{\theta_*\}$. Hence the point $(0, \theta_*)$ is isolated inside the set of points of the curve $\dot{\theta} = 0$ and so we have proved that $\dot{\theta}$ does not vanish in a punctured neighbourhood of the origin. In short, system (1) satisfies condition (a). □

Proposition 2. *If system (1) has a focus–centre at the origin, then m is odd. If we also assume that (1) has some characteristic direction, then M also has to be odd.*

Proof. Consider system (3), and suppose that m is even, then $c(\theta) = \cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)$ must have a root with odd multiplicity θ_* because $c(\theta)$ is a homogeneous trigonometric polynomial of odd degree $m + 1$ (remember that in the previous proposition we have seen that if the origin is a focus–centre then θ_* must be an isolated root of $c(\theta)$). So $c(\theta)$ changes its sign at θ_*, and therefore $\dot{\theta}|_{r=0} = c(\theta)$ changes its sign at $\theta = \theta_*$. This contradicts proposition 1. Hence, m must be odd.

Assume now that θ_* is some characteristic direction and M is even. Then from (2) we find that

$$\dot{\theta}|_{\{\theta=\theta_*\}} = R^{M-m} (\cos\theta\, Q_M(\theta) - \sin\theta\, P_M(\theta)) = -\dot{\theta}|_{\{\theta=\theta_*+\pi\}}.$$

This last equality is in contradiction with proposition 1 because it implies that condition (a) is not satisfied. □

2.2. Sufficient conditions in a generic case

Conditions (a) and (b) are not sufficient conditions to conclude that (1) has a focus–centre at the origin. To see this consider the following system:

$$\begin{aligned} \dot{x} &= y\left(\alpha x^2 + bxy + cy^2\right) \\ \dot{y} &= y^2(\alpha x + by) + x^5. \end{aligned} \tag{5}$$

A simple computation shows that in polar coordinates (R, θ), $\dot{\theta} = -c\sin^4\theta + R^2\cos^6\theta$; hence the characteristic directions are given by $\{y = 0\}$, and then condition (b) holds trivially. If we take $c < 0$, then condition (a) also holds. However, if we take $\alpha > 0$ the origin is not a monodromic point, since it has nodal sectors. This fact can be proved by using the blow-up technique, and can be checked from the computations done in lemma 3.

To obtain sufficient conditions first we will suppose that $\theta = 0$ is a characteristic direction and, by using the blowing-up method, we will establish sufficient conditions to ensure that there are no characteristic orbits approaching or leaving the origin in the direction $\theta = 0$. Since systems of type (1) are preserved under rotations we will use the results obtained in the case $\theta = 0$, to obtain sufficient conditions for the non-existence of characteristic orbits approaching or leaving any characteristic direction of the origin of a system of type (1).

Lemma 3. *Assume that $\theta = 0$ is a characteristic direction of the origin of system (1), and denote by*

$$\alpha = \lim_{z\to 0} \frac{P_m(1,z)}{z} = \frac{\mathrm{d}}{\mathrm{d}z} P_m(1,z)|_{z=0}$$

and

$$\beta = \lim_{z\to 0} \frac{Q_m(1,z)}{z^2} = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2} Q_m(1,z)|_{z=0}.$$

Then if $\alpha^2 + \beta^2 \neq 0$ there are no characteristic orbits approaching or tending to the origin in the direction $\theta = 0$ if and only if the following conditions hold:

(i) $\Theta(R,\theta) := \cos\theta Q_m(\theta) - \sin\theta P_m(\theta) + R^{M-m}(\cos\theta Q_M(\theta) - \sin\theta P_M(\theta)) \neq 0$, *for* $(R,\theta) \in \{(0,\bar{R}] \times (-\arctan(\delta), \arctan(\delta))\} \setminus \{(0,0)\}$, *for* $\delta > 0$ *and* $\bar{R} > 0$ *small enough,*

(ii) $P_m(\theta)|_{\theta=0} = 0$ *and* $Q_m(\theta)|_{\theta=0} = 0$ *and*

(iii) $\mathrm{sign}_\delta(\Theta)\,((2 + M - m)\alpha - 2\beta) \leqslant 0$

where $\mathrm{sign}_\delta(\Theta) = \mathrm{sign}\,(\Theta(R,\theta))$, *in* $\{(0,\bar{R}] \times (-\arctan(\delta), \arctan(\delta))\} \setminus \{(0,0)\}$.

Proof. To prove the above result we will use the blow-up technique. The successive blow-ups that we will perform in what follows are displayed in figures 3 and 4. The condition $\alpha^2 + \beta^2 \neq 0$ implies that the blow-ups described below are enough to completely desingularize the singularity.

Conditions (i) and (ii) are necessary to ensure that there are no characteristic orbits tending or leaving the origin in the direction $\theta = 0$. This can be proved by using the same arguments as in the proof of proposition 1. In the proof of the sufficiency of the three conditions, we will see that condition (iii) is also necessary.

Assume that (i)–(iii) hold and, without loss of generality, that $\mathrm{sign}_\delta(\Theta) = 1$. Hence condition (iii) can be written as $(2 + M - m)\alpha - 2\beta \leqslant 0$.

Consider system (1). Condition (ii) implies that $Q_m(1,0) = P_m(1,0) = 0$. Since we have taken $\mathrm{sign}_\delta(\Theta) = 1$, from condition (i) we have that $F(z) := Q_m(1,z) - zP_m(1,z) \geqslant 0$ for $z \in (-\delta,\delta)$. Hence the first non-vanishing derivative of F at the origin is of even order, this means that $F(0) = F'(0) = 0$ and $F''(0) = \beta - \alpha \geqslant 0$, and therefore $\mu_0\,(P_m(1,z)) \geqslant 1$ and $\mu_0\,(Q_m(1,z)) \geqslant 2$ (where μ_0 denotes the multiplicity at $z = 0$). Also we have that $Q_M(1,0) > 0$. Set $\gamma = Q_M(1,0)$.

We make the following change of variables $(u,z) = (x^{M-m}, y/x)$. This is not a global change of coordinates in $\mathbb{R}^2 \setminus (0,0)$, but it is a good change on $\{x > 0\}$. Anyway, the results obtained are also valid for $\{x < 0\}$ since equation (2) satisfies $(\dot{R}(R,\theta+\pi), \dot{\theta}(R,\theta+\pi)) = (\dot{R}(R,\theta), \dot{\theta}(R,\theta))$. After a rescaling we obtain

$$\begin{aligned} \dot{u} &= ku\,(P_m(1,z) + uP_M(1,z)) \\ \dot{z} &= Q_m(1,z) - zP_m(1,z) + u\,(Q_M(1,z) - zP_M(1,z)) \end{aligned} \tag{6}$$

where $k = M - m$. We concentrate our attention on the singularity given by $(u, z) = (0, 0)$ which is the singular point of (6) which comes from the direction $\{\theta = 0\}$. The differential matrix of (6) associated with $(u, z) = (0, 0)$ is given by

$$\begin{pmatrix} 0 & 0 \\ Q_M(1,0) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \gamma & 0 \end{pmatrix}.$$

This singularity is a nilpotent and, to desingularize it, we must continue the blow-up process.

Now consider system (6) and the change $(v, z) = (u/z, z)$. We obtain, after a reparametrization of the time,

$$\begin{aligned} \dot v &= v\left((1+k)P_m(1,z) - \frac{Q_m(1,z)}{z} + v\,(z(1+k)P_M(1,z) - Q_M(1,z))\right) \\ \dot z &= Q_m(1,z) - zP_m(1,z) + vz\,(Q_M(1,z) - zP_M(1,z)). \end{aligned} \tag{7}$$

Note that since $\mu_0(Q_m(1,z)) \geqslant 2$ the above equation is well defined. Since $Q_M(1,0) \neq 0$, the only singular point on $z = 0$ is $(v, z) = (0, 0)$ and its linear part is identically zero. Now we have to consider the other direction. Again taking the system (6), and the new change $(u, p) = (u, z/u)$. We obtain

$$\begin{aligned} \dot u &= ku\,(P_m(1,pu) + uP_M(1,pu)) \\ \dot p &= -p(1+k)P_m(1,pu) + \frac{Q_m(1,pu)}{u} - pu(1+k)P_M(1,pu) + Q_M(1,pu). \end{aligned} \tag{8}$$

Observe that there are no singular points on $u = 0$. Hence in this direction $(u, z) = (0, 0)$ has been desingularized.

Now consider system (7) and take the change of coordinates $(w, z) = (v/z, z)$. We obtain

$$\begin{aligned} \dot w &= w\left((2+k)\frac{P_m(1,z)}{z} - 2\frac{Q_m(1,z)}{z^2} + w\,(z(2+k)P_M(1,z) - 2Q_M(1,z))\right) \\ \dot z &= \frac{Q_m(1,z)}{z} - P_m(1,z) + wz\,(Q_M(1,z) - zP_M(1,z)). \end{aligned} \tag{9}$$

Note that the region $\{x \geqslant 0\}$ for system (1) has been transformed into the region $\{w \geqslant 0\}$. The dynamics in the region $\{w \leqslant 0\}$ is virtual, in the sense that it does not appear in coordinates (x, y). When $z = 0$ we have

$$\begin{aligned} \dot w|_{z=0} &= w\,((2+k)\alpha - 2\beta - 2\gamma w) \\ \dot z|_{z=0} &= 0. \end{aligned}$$

Then the singular points on $z = 0$ are $(w_1, 0) = (0, 0)$, and $(w_2, 0) = (((2+k)\alpha - 2\beta)/2\gamma, 0)$. If $(2+k)\alpha - 2\beta = 0$, there is just one singular point $(w_1, 0) = (0, 0)$.

Now we consider the other direction and we make the following change of coordinates $(v, q) = (v, z/v)$. We obtain the system

$$\begin{aligned} \dot v &= (1+k)P_m(1,qv) - \frac{Q_m(1,qv)}{qv} + v^2q(1+k)P_M(1,qv) - vQ_M(1,qv) \\ \dot q &= -q(2+k)\frac{P_m(1,qv)}{v} - vq^2(2+k)P_M(1,qv) + 2\frac{Q_m(1,qv)}{v^2} + 2qQ_M(1,qv). \end{aligned} \tag{10}$$

Note that the region $\{x \geqslant 0\}$ for system (1) has been transformed into the region $\{q \geqslant 0\}$. Again the dynamics in the region $\{q \leqslant 0\}$ is virtual. When $v = 0$ we have

$$\begin{aligned} \dot v|_{v=0} &= 0 \\ \dot q|_{v=0} &= q\,((-(2+k)\alpha + 2\beta)\,q + \gamma). \end{aligned}$$

Then the singular points on $v = 0$ are $(0, q_1) = (0, 0)$, and $(0, q_2) = (0, 2\gamma/[(2 + k)\alpha - 2\beta])$ (again if $(2 + k)\alpha - 2\beta = 0$, there is just one singular point $(0, q_1) = (0, 0)$).

Note that the eigenvalues of the differential matrix of the vector field associated with system (9) at the point $(w_1, 0) = (0, 0)$ are $(2 + k)\alpha - 2\beta$ and $\beta - \alpha$. The eigenvalues of the differential matrix of the field associated with system (10) at the point $(0, q_1) = (0, 0)$ are $-\gamma$ and 2γ. Now we distinguish the following cases.

(a) $(2+k)\alpha - 2\beta \neq 0$ and $\beta - \alpha > 0$. Consider first the inequality $(2+k)\alpha - 2\beta < 0$. In this situation we obtain $(w_2, 0) \in \{w < 0\}$ and $(0, q_2) \in \{q < 0\}$, and hence these singular points of systems (9) and (10), respectively, are not relevant for studying the presence of characteristic orbits. Then the singular points $(w_1, 0)$ and $(0, q_1)$ are hyperbolic saddles with the separatrices in the coordinate axes, and these separatrices do not correspond to characteristic orbits tending to or leaving the origin in the system (1) in the direction $\theta = 0$.

When $(2+k)\alpha - 2\beta > 0$ the critical point $(w_1, 0)$ is a hyperbolic node and hence there are infinitely many orbits tending to the origin of system (1). Therefore, (iii) is a necessary condition.

(b) $\beta - \alpha = 0$. If $\beta - \alpha = 0$ then $\lambda = (2 + k)\alpha - 2\beta \neq 0$. Now we write system (9) in the following form:

$$\dot{w} = w(\Phi_1(z) + w\Phi_2(z))$$
$$\dot{z} = \Psi_1(z) + w\Psi_2(z).$$

Reparametrizing the system in order to apply the classification theorem of this type of critical point (see [4]) we obtain

$$\dot{z} = \frac{1}{\lambda}\,(\Psi_1(z) + w\Psi_2(z)) = X(z, w)$$
$$\dot{w} = \frac{w}{\lambda}\,(\Phi_1(z) + w\Phi_2(z)) = w + Y(z, w).$$

Note that $\Phi_1(0) = \lambda \neq 0$. Then the only solution of $w + Y(z, w) = 0$ passing through $(0, 0)$ is $w = 0$. Now observe that $F(z) = z\Psi_1(z) \geqslant 0$. Let $F^{(2n)}(z) > 0$ be the first non-vanishing derivative (note that we have seen above that the first non-vanishing derivative at zero of F is of even order). It is easy to prove by induction that

$$F^{(2n)}(z) = 2n\Psi_1^{(2n-1)}(z) + z\Psi^{(2n)}(z)|_{z=0} = 2n\Psi_1^{(2n-1)}(0) > 0.$$

Hence $\Psi_1^{(2n-1)}(0) > 0$ is the first non-vanishing derivative of Ψ_1. Therefore,

$$X(z, 0) = \frac{1}{\lambda(2n-1)!}\Psi_1^{(2n-1)}(0)z^{2n-1} + \cdots$$

and applying the classification theorem stated in [4] we have that if $\lambda > 0$ then the singular point is a node and if $\lambda < 0$ then the point is a topological saddle.

(c) $(2 + k)\alpha - 2\beta = 0$. Note that condition (c) implies that $\mu = \beta - \alpha = k\alpha/2 > 0$. Now we write system (9) in the following form:

$$\dot{w} = w\,(\Phi_1(z) + w\Phi_2(z))$$
$$\dot{z} = \Psi_1(z) + w\Psi_2(z) = \mu z + \cdots.$$

Doing a reparametrization as in case (b) we obtain

$$\dot{w} = \frac{w}{\mu}\,(\Phi_1(z) + w\Phi_2(z)) = X(w, z)$$
$$\dot{z} = \frac{1}{\mu}\,(\Psi_1(z) + w\Psi_2(z)) = z + Y(w, z).$$

Taking into account that $(1/\mu)(\Psi_1(z)+w\Psi_2(z)) = (z/\mu)(\psi_1(z)+w\psi_2(z))$ where $\psi_1(0) = \mu$, it follows that the only solution of $z+Y(z,w)=0$ passing through $(0,0)$ is $z=0$. Then $X(w,0) = (w/\mu)(\Phi_1(0)+w\Phi_2(0)) = (-2Q_M(1,0)/\mu)w^2 = -2(\gamma/\mu)w^2$, where $-2\gamma/\mu < 0$ because $\gamma > 0$ and $\mu > 0$. Applying again the classification theorem stated in [4] we have that the singular point is a saddle node, with the nodal sectors on $\{w<0\}$, the centre-manifold W^c in the w-axis, and the z-axis being the other separatrix. Hence there are no orbits tending to or leaving the singular point in $\{w>0\}$, and therefore there are no characteristic orbits of system (1) tending to or leaving the origin in the direction $\{\theta = 0\}$. □

We want to stress that in the case that $\mu_0(P_m(1,z)) > 1$ and $\mu_0(Q_m(1,z)) > 2$, i.e. when $\alpha^2+\beta^2=0$, it is possible to continue the desingularization process, but this is not done in this work.

Using lemma 3 we can give sufficient conditions for the origin of system (1) to be a singularity of focus–centre type. We denote by $\theta_1, \theta_2, \ldots, \theta_k$ the characteristic directions associated with system (1). For all $j=1,\ldots,k$ recall that

$$a_j = \cos\theta_j \qquad\qquad b_j = \sin\theta_j$$

$$\alpha_j = \frac{\mathrm{d}}{\mathrm{d}z}\left(P_m^j(1,z)\right)\Big|_{z=0} \qquad Q_m(1,z) = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(Q_m^j(z)\right)\Big|_{z=0}$$

where

$$P_m^j(1,z) = a_j P_m(a_j - b_j z, b_j + a_j z) + b_j Q_m(a_j - b_j z, b_j + a_j z)$$

and

$$Q_m^j(1,z) = -b_j P_m(a_j - b_j z, b_j + a_j z) + a_j Q_m(a_j - b_j z, b_j + a_j z).$$

Recall also that a vector field $\mathcal{X} = (P_m(x,y)+P_M(x,y), Q_m(x,y)+Q_M(x,y))$ belongs to the class $\mathcal{G}$ if for all $j=1,\ldots,k$ we have $\alpha_j^2+\beta_j^2 \neq 0$.

Proposition 4. *Let $\mathcal{X}$ be a vector field of class $\mathcal{G}$. If the system associated with X satisfies conditions (a) and (b), and* $\mathrm{sign}_{\bar{0}}(\Theta)\left((2+M-m)\,\alpha_j - 2\beta_j\right) \leqslant 0$, *for all $j=1,\ldots,k$, then the origin is a focus–centre.*

Proof. For every characteristic direction θ_j, after a rotation of angle $\varphi = -\theta_j$ system (1) is written (in the new coordinates (u,v)) as

$$\begin{aligned}
\dot u &= a_j P_m(a_j u - b_j v, b_j u + a_j v) + b_j Q_m(a_j u - b_j v, b_j u + a_j v)\\
&\qquad + a_j P_M(a_j u - b_j v, b_j u + a_j v) + b_j Q_M(a_j u - b_j v, b_j u + a_j v)\\
\dot v &= -b_j P_m(a_j u - b_j v, b_j u + a_j v) + a_j Q_m(a_j u - b_j v, b_j u + a_j v)\\
&\qquad - b_j P_M(a_j u - b_j v, b_j u + a_j v) + a_j Q_M(a_j u - b_j v, b_j u + a_j v)
\end{aligned} \tag{11}$$

where a_j and b_j have been defined above. This system has a characteristic direction in $v=0$. Applying lemma 3 to system (11) we have that it has no orbits starting or ending at the origin in the direction $\{v=0\}$. This direction corresponds to the direction $\{\theta=\theta_j\}$ for system (1). Hence the orbits in this neighbourhood rotate around the origin, and the origin is a monodromic singular point. □

Proof of theorem A. Follows straightforwardly from propositions 1 and 4. □

3. On the stability of the origin

3.1. Systems without characteristic directions

In the case where a critical point of an analytic system (degenerate or not) has no characteristic directions, the study of its stability can be done by using a straightforward generalization of the Lyapunov–Poincaré theory. The main difficulties that appear are of computational type (see, for instance, [5] or [6]). In this subsection we compute the first generalized Lyapunov constant for this type of critical point. The expression that we obtain is similar to that given in theorem B for family (1).

Consider an analytic system which in polar coordinates is written as

$$\begin{aligned} \dot{r} &= \sum_{i=k+1}^{\infty} a_i(\theta)\, r^i \\ \dot{\theta} &= \sum_{i=k}^{\infty} c_i(\theta)\, r^i \end{aligned} \tag{12}$$

where $c_k(\theta)$ does not vanish (i.e. the origin has no characteristic directions). It also can be written as

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = S(r,\theta) = \frac{\sum_{i=k+1}^{\infty} a_i(\theta)\, r^i}{\sum_{i=k}^{\infty} c_i(\theta)\, r^i} = \frac{a_{k+1}(\theta)}{c_k(\theta)} r + \mathrm{O}(r^2). \tag{13}$$

From this equation we easily obtain the following result.

Proposition 5. *If the origin of the system (12) has no characteristic directions, then the return map $\Pi(x)$ associated with it has the form*

$$\Pi(x_0) = V_1 x_0 + \mathrm{o}(x_0) \qquad \textit{where} \quad V_1 = \exp\left\{\mathrm{sign}\,(c_k(\theta)) \int_0^{2\pi} \frac{a_{k+1}(\theta)}{c_k(\theta)}\, \mathrm{d}\theta\right\}.$$

From this proposition and expression (2) of system (1) we have the following corollary.

Corollary 6. *Assume that the origin of system (1) has no characteristic directions. Then the return map associated with the origin has the form $\Pi(x_0) = V_1 x_0 + \mathrm{o}(x_0)$, where,*

$$V_1 = \exp\left\{\mathrm{sign}\,(\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)) \int_0^{2\pi} \frac{\cos\theta\, P_m(\theta) + \sin\theta\, Q_m(\theta)}{\cos\theta\, Q_m(\theta) - \sin\theta\, P_m(\theta)}\, \mathrm{d}\theta\right\}.$$

3.2. Systems with characteristic directions

As far as we know in the presence of characteristic directions a general method to decide the stability of the origin when it is monodromic is not known. The main difficulty in this case is that the return map is no longer differentiable. However, it has been proved that the leading term of this map is linear. This result due to Il'yashenko and stated without proof in [1], is proved in the work of Medvedeva [24]. We want to comment that from our approach we also re-obtain this result for the particular family (1).

3.2.1. Preliminary results. We start by obtaining an expression of the transition map of the flow associated with an isolated characteristic direction under some non-degeneracy conditions. These conditions ensure that the singular points appearing at the end of the blow-up process described in section 2.2 are either hyperbolic saddles or elementary degenerate saddle nodes.

Given $a_j = \cos\theta_j$ and $b_j = \sin\theta_j$, where θ_j is a characteristic direction of system (1) we introduce the following notation for $l \in \{m, M\}$:

$$P_l^j(u,v) = a_j P_l(a_j u - b_j v, b_j u + a_j v) + b_j Q_l(a_j u - b_j v, b_j u + a_j v)$$
$$Q_l^j(u,v) = -b_j P_l(a_j u - b_j v, b_j u + a_j v) + a_j Q_l(a_j u - b_j v, b_j u + a_j v).$$

For short we write $P_l^j(\theta) = P_l^j(\cos\theta, \sin\theta)$ and $Q_l^j(\theta) = Q_l^j(\cos\theta, \sin\theta)$. Recall that

$$\alpha_j = \frac{\mathrm{d}}{\mathrm{d}z} P_m^j(1,z)|_{z=0} \qquad \beta_j = \frac{\mathrm{d}^2}{\mathrm{d}z^2} Q_m^j(1,z)|_{z=0}.$$

Lemma 7. *Consider a system of type (1) having an isolated characteristic direction at $\{\theta = \theta_j\}$. Assume that the following conditions hold:*

(i) $\Theta_j(R,\theta) := \cos\theta Q_m^j(\theta) - \sin\theta P_m^j(\theta) + R^{M-m}\left(\cos\theta Q_M^j(\theta) - \sin\theta P_M^j(\theta)\right) \neq 0$, *for* $(R,\theta) \in \{(0,\bar{R}] \times [-\arctan(\varepsilon), \arctan(\varepsilon)]\} \setminus \{(0,0)\}$, *for* $\varepsilon > 0$ *and* $\bar{R} > 0$ *small enough. Denote by* s_j *the sign of* $\Theta_j(R,\theta)$ *in this set.*
(ii) $P_m^j(\theta)|_{\theta=0} = 0$ *and* $Q_m^j(\theta)|_{\theta=0} = 0$.
(iii) $\alpha_j^2 + \beta_j^2 \neq 0$ *and* $\beta - \alpha \neq 0$
(iv) $s_j\left((2+k)\alpha_j - 2\beta_j\right) \leqslant 0$, *where* $k = M - m$.

Let (R,θ) *denote a point represented in polar coordinates. Denote by* Δ_j^ε *the transition map of the flow for system (1) from* $\{\theta = \theta_j - s_j \arctan(\varepsilon)\}$ *to* $\{\theta = \theta_j + s_j \arctan(\varepsilon)\}$. *Then*

$$\Delta_j^\varepsilon(R_0) = D_j^\varepsilon R_0 + \mathrm{o}(R_0)$$

where

$$D_j^\varepsilon = \begin{cases} \exp\left\{\displaystyle\int_{-\varepsilon}^{\varepsilon} \frac{P_m^j(1,z)}{Q_m^j(1,z) - zP_m^j(1,z)} - \left(\frac{\alpha_j}{\beta_j - \alpha_j}\right)\frac{1}{z}\,\mathrm{d}z\right\} \\ \qquad \textit{if} \quad (2+k)\alpha_j - 2\beta_j \neq 0 \\ \exp\left\{\displaystyle\frac{1}{M-m}\int_{-\varepsilon}^{\varepsilon} \frac{(2+M-m)zP_m^j(1,z) - 2Q_m^j(1,z)}{zQ_m^j(1,z) - z^2P_m^j(1,z)}\,\mathrm{d}z\right\} \\ \qquad \textit{if} \quad (2+k)\alpha_j - 2\beta_j = 0. \end{cases}$$

Conditions (ii) and (iv) ensure that the integral appearing in the expression of D_j^ε is non-singular in a neighbourhood of $z = 0$. Condition (iii) also ensures that the blow-ups used in section 2.2 are enough to desingularize $\{\theta = \theta_j\}$. When $(2+k)\alpha_j - 2\beta_j \neq 0$ all the singular points arising from $\{\theta = \theta_j\}$ at the end of the blow-up process are hyperbolic saddles (see section 2.2). When $(2+k)\alpha_j - 2\beta_j = 0$ all the singular points arising from $\{\theta = \theta_j\}$ at the end of the desingularization process are hyperbolic saddles or elementary degenerate saddle nodes.

To prove lemma 7 we need to study the composition of the transition maps associated with three simpler situations: an absence of critical points, a hyperbolic sector of a hyperbolic saddle and a hyperbolic sector of a degenerate elementary singular point. We study the latter two situations in the following two lemmas.

Lemma 8. *Consider system*

$$\begin{aligned} \dot{x} &= -x(a + f(x,y)) = P(x,y) \\ \dot{y} &= y(b + g(x,y)) = Q(x,y) \end{aligned} \tag{14}$$

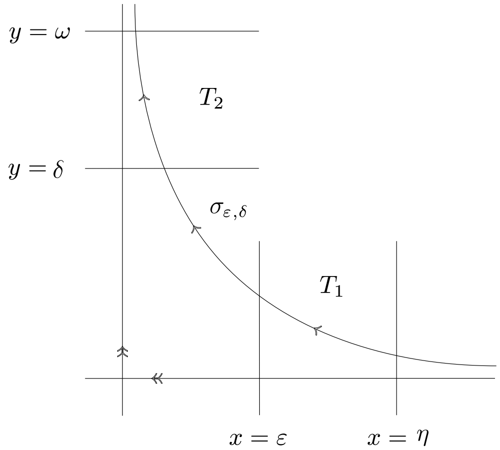

Figure 1. Transition maps in a neighbourhood of a hyperbolic saddle.

where f and g begin with first-order terms, and, a and b are positive. Let $\sigma_{\varepsilon,\delta}(y)$ be the transition map of the flow from $\{x = \varepsilon\}$ to $\{y = \delta\}$, where ε and δ are positive and small enough (see figure 1), then

$$\sigma_{\varepsilon,\delta}(y) = A(\varepsilon, \delta)y^{a/b} + \mathrm{o}(y^{a/b}) \qquad \textit{with} \quad A(\varepsilon, \delta) = \frac{\varepsilon}{\delta^{a/b}} \frac{\exp\{F(\delta)\}}{\exp\{(a/b)G(\varepsilon)\}}$$

where

$$F(\delta) = \int_0^\delta \left(\frac{P(x, y)}{xQ(x, y)} \right)\bigg|_{x=0} + \frac{a}{by}\, \mathrm{d}y$$

and

$$G(\varepsilon) = \int_0^\varepsilon \left(\frac{Q(x, y)}{yP(x, y)} \right)\bigg|_{y=0} + \frac{b}{ax}\, \mathrm{d}x.$$

Proof. It is well known (see section 3.4 and [22]) that $\sigma_{\varepsilon,\delta}(y) = A(\varepsilon, \delta)y^{a/b} + \mathrm{o}(y^{a/b})$, that is, $\sigma_{\varepsilon,\delta}$ is a semiregular map (see definition 12) with a leading term of order $y^{a/b}$. Let us denote by T_1 and T_2 the regular transition maps from $\{x = \eta\}$ to $\{x = \varepsilon\}$ and from $\{y = \delta\}$ to $\{y = \omega\}$, respectively, with $\eta > \varepsilon$ and $\omega > \delta$ (see figure 1). We can write them as

$$T_1(y) = C_1(\eta, \varepsilon)y + \cdots \qquad T_2(x) = C_2(\delta, \omega)x + \cdots$$

(we set $C_1 = C_1(\eta, \varepsilon)$ and $C_2 = C_2(\eta, \varepsilon)$). Using the first-order variational equations to compute C_1 and C_2, we obtain

$$C_1 = \exp\left\{ \int_\eta^\varepsilon \frac{\partial}{\partial y} \left(-\frac{y(b + g(x, y))}{x(a + f(x, y))} \right)\bigg|_{y=0} \mathrm{d}x \right\}$$

$$C_2 = \exp\left\{ \int_\delta^\omega \frac{\partial}{\partial x} \left(-\frac{x(a + f(x, y))}{y(b + g(x, y))} \right)\bigg|_{x=0} \mathrm{d}y \right\}.$$

A straightforward computation gives

$$C_1 = \left(\frac{\varepsilon}{\eta}\right)^{-b/a} \frac{\exp\{G(\varepsilon)\}}{\exp\{G(\eta)\}}$$

where $G(x) = \int_0^x \widetilde{G}(t)\,\mathrm{d}t$ and $\widetilde{G}(x) = (Q(x, y)/yP(x, y))|_{y=0} + b/ax$.

Analogously, we obtain

$$C_2 = \left(\frac{\delta}{\omega}\right)^{a/b} \frac{\exp\{F(\omega)\}}{\exp\{F(\delta)\}}$$

where $F(y) = \int_0^y \widetilde{F}(t)\,\mathrm{d}t$ and $\widetilde{F}(y) = (P(x, y)/xQ(x, y))|_{x=0} + a/by$.

Since $\sigma_{\varepsilon,\omega}(y) = T_2 \circ \sigma_{\varepsilon,\delta}(y)$ and $\sigma_{\eta,\delta}(y) = \sigma_{\varepsilon,\delta} \circ T_1(y)$ (see figure 1), by making computations we have that

$$A(\varepsilon, \omega) = C_2 A(\varepsilon, \delta) \qquad A(\eta, \delta) = C_1^{a/b} A(\varepsilon, \delta)$$

which gives

$$A(\varepsilon, \omega) = \left(\frac{\delta}{\omega}\right)^{a/b} \frac{\exp\{F(\omega)\}}{\exp\{F(\delta)\}} A(\varepsilon, \delta) \tag{15}$$

$$A(\eta, \delta) = \frac{\eta}{\varepsilon} \frac{\exp\{(a/b)G(\varepsilon)\}}{\exp\{(a/b)G(\eta)\}} A(\varepsilon, \delta). \tag{16}$$

Now we claim that $A(\varepsilon, \omega) = P(\varepsilon)Q(\omega)$ for some functions P and Q. From equation (15) we have

$$A(\varepsilon, \omega) = \frac{\exp\{F(\omega)\}}{\omega^{a/b}} \frac{\delta^{a/b}}{\exp\{F(\delta)\}} A(\varepsilon, \delta). \tag{17}$$

Set $Q(\omega) := \exp\{F(\omega)\}/\omega^{a/b}$. Then $A(\varepsilon, \omega) = Q(\omega)h(\varepsilon, \delta)$, where $h(\varepsilon, \delta) = (\delta^{a/b}/\exp\{F(\delta)\})\, A(\varepsilon, \delta)$. Since $A(\varepsilon, \omega)$ does not depend on δ we conclude that $h(\varepsilon, \delta)$ does not depend on δ, hence we can define $P(\varepsilon) := h(\varepsilon, \delta)$. From equation (16) we have

$$P(\eta)Q(\delta) = \frac{\exp\{(a/b)G(\varepsilon)\}}{\varepsilon} \frac{\omega}{\exp\{(a/b)G(\omega)\}} P(\varepsilon)Q(\delta).$$

Again it can be easily deduced that

$$P(\varepsilon) = K \frac{\varepsilon}{\exp\{(a/b)G(\varepsilon)\}}$$

where K is a constant. Hence

$$A(\varepsilon, \delta) = K \frac{\varepsilon}{\delta^{a/b}} \frac{\exp\{F(\delta)\}}{\exp\{(a/b)G(\varepsilon)\}}. \tag{18}$$

Now we prove that $K = 1$. By means of a local $\mathcal{C}^1$-smooth change of coordinates given by

$$\begin{aligned} u &= \varphi_1(x, y) = x(1 + \Phi_1(x, y)) \\ v &= \varphi_2(x, y) = y(1 + \Phi_2(x, y)) \end{aligned} \tag{19}$$

where Φ_1 and Φ_2 vanish at $(0, 0)$, system (14) is transformed into the system

$$\begin{aligned} \dot{u} &= -au \\ \dot{v} &= bv \end{aligned} \tag{20}$$

in a neighbourhood of the origin (see [29]). Since system (20) has a first integral given by $u^b v^a = h$, then the original system has the first integral:

$$H(x, y) = \varphi_1^b(x, y)\,\varphi_2^a(x, y) = h$$

and then, the integral curves of (14) are given by

$$x^b(1 + \Phi_1(x, y))^b\, y^a(1 + \Phi_2(x, y))^a = h.$$

Evaluating the integral in (ε, y) and (x, δ), two points on the integral curve $H(x, y) = h$, we have

$$x^b(1 + \Phi_1(x, \delta))^b\, \delta^a(1 + \Phi_2(x, \delta))^a = \varepsilon^b(1 + \Phi_1(\varepsilon, y))^b\, y^a(1 + \Phi_2(\varepsilon, y))^a. \tag{21}$$

From equation (18) we obtain

$$x = K\frac{\varepsilon}{\delta^{a/b}}\frac{\exp\{F(\delta)\}}{\exp\{(a/b)G(\varepsilon)\}}y^{a/b} + \mathrm{o}(y^{a/b}). \tag{22}$$

Substituting equation (22) into equation (21), using that $F(0) = G(0) = 0$, and $\Phi_1(0, 0) = \Phi_2(0, 0) = 0$, we have that (we omit the tedious but straightforward computations) $K^b = 1$. So we have proved that

$$A(\varepsilon, \delta) = \frac{\varepsilon}{\delta^{a/b}}\frac{\exp\{F(\delta)\}}{\exp\{(a/b)G(\varepsilon)\}}. \qquad \square$$

In the following lemma we study the transition map associated with a hyperbolic sector of a degenerate elementary singular point.

Lemma 9. *Consider the system*

$$\begin{aligned} \dot{x} &= -x(a + f(x, y)) = P(x, y) \\ \dot{y} &= yg(x, y) = Q(x, y) \end{aligned} \tag{23}$$

where f and g begin with first-order terms, and $g(0, y) = by^k + \mathrm{o}(y^k)$, a and b positive. Let $\sigma_{\varepsilon,\delta}(y)$ be the transition map of the flow from $\{x = \varepsilon\}$ to $\{y = \delta\}$ (where ε and δ are positive and small enough), and $\tau_{\delta,\varepsilon}(x)$ be the transition map from $\{y = \delta\}$ to $\{x = \varepsilon\}$ (see figure 2), then

$$\sigma_{\varepsilon,\delta}(y) = f_0\left(A(\varepsilon)y^k + \mathrm{o}(y^k)\right) \qquad \textit{where} \quad A(\varepsilon) = \frac{kb}{a}\exp\{-kF(\varepsilon)\}$$

being

$$F(\varepsilon) = \int_0^\varepsilon \left(\frac{Q(x, y)}{yP(x, y)}\right)\bigg|_{y=0} \mathrm{d}x$$

and

$$f_0 = \begin{cases} \exp\{-1/x\} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Moreover,

$$\tau_{\delta,\varepsilon}(x) = h\left(f_0^{-1}(x)\right)$$

where $h(\bar{x}) = B(\varepsilon)\bar{x}^{1/k} + \mathrm{o}(\bar{x}^{1/k})$ and $B(\varepsilon) = (kb/a)^{-1/k}\exp\{F(\varepsilon)\}$.

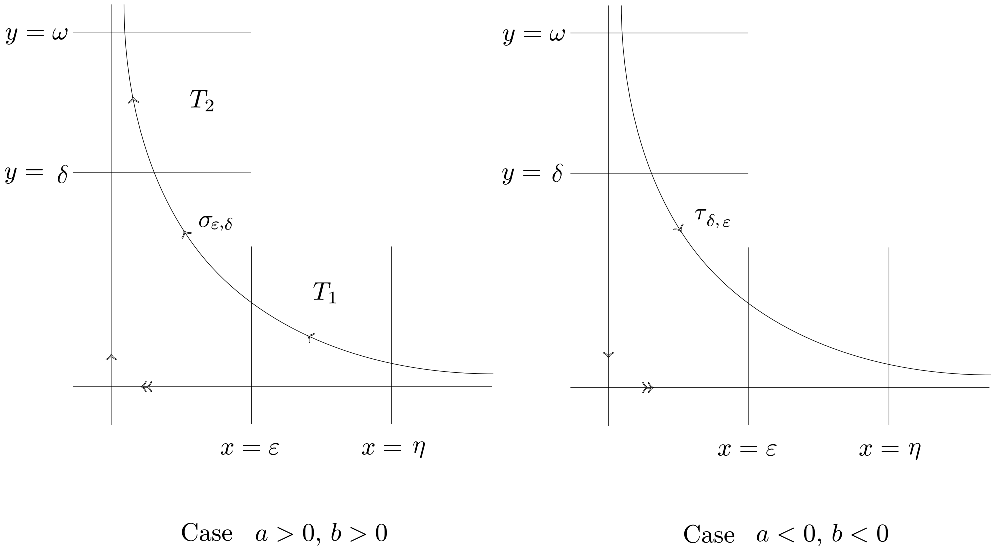

Figure 2. Transition maps in a hyperbolic sector of an elementary degenerate singular point.

Proof. It is well known that $\sigma_{\varepsilon,\delta}$ is a flat semiregular map (see section 3.4 and [22]) that can be written as $\sigma_{\varepsilon,\delta}(y) = f_0\big(A(\varepsilon,\delta)y^k + \mathrm{o}(y^k)\big)$. Consider now η and ω such that $\eta > \varepsilon$ and $\omega > \delta$. Let us denote by T_1 and T_2 the regular transition maps from $\{x=\eta\}$ to $\{x=\varepsilon\}$ and from $\{y=\delta\}$ to $\{y=\omega\}$, respectively. We can write them as

$$T_1(y) = C_1(\eta,\varepsilon)y + \cdots \qquad T_2(x) = C_2(\delta,\omega)x + \cdots$$

(we set $C_1 = C_1(\eta,\varepsilon)$ and $C_2 = C_2(\delta,\omega)$). Using the first-order variational equations we obtain

$$C_1 = \exp\left\{\int_\eta^\varepsilon \frac{\partial}{\partial y}\left(-\frac{yg(x,y)}{x(a+f(x,y))}\right)\bigg|_{y=0} \mathrm{d}x\right\}.$$

Since

$$\frac{\partial}{\partial y}\left(-\frac{yg(x,y)}{x(a+f(x,y))}\right)\bigg|_{y=0} = \frac{Q(x,y)}{yP(x,y)}\bigg|_{y=0}$$

we obtain

$$C_1 = \frac{\exp\{F(\varepsilon)\}}{\exp\{F(\eta)\}}. \tag{24}$$

Observe that $\sigma_{\eta,\delta} = \sigma_{\varepsilon,\delta} \circ T_1$ (see figure 2). So, we have that

$$f_0\big(A(\eta,\delta)y^k + \mathrm{o}(y^k)\big) = f_0\big(C_1^k A(\varepsilon,\delta)y^k + \mathrm{o}(y^k)\big)$$

and hence

$$A(\eta,\delta) = C_1(\eta,\varepsilon)^k A(\varepsilon,\delta). \tag{25}$$

On the other hand (see again figure 2), since $T_2 \circ \sigma_{\varepsilon,\delta} = \sigma_{\varepsilon,\omega}$, we obtain

$$T_2 \circ f_0\big(A(\varepsilon,\delta)y^k + \mathrm{o}(y^k)\big) = f_0\big(A(\varepsilon,\omega)y^k + \mathrm{o}(y^k)\big).$$

From the above equality we obtain that $f_0^{-1} \circ T_2 \circ f_0\big(A(\varepsilon,\delta)y^k + \mathrm{o}(y^k)\big) = A(\varepsilon,\omega)y^k + \mathrm{o}(y^k)$. Applying lemma 14 of the appendix, we obtain

$$A(\varepsilon,\delta)y^k + \mathrm{o}(y^k) = A(\varepsilon,\omega)y^k + \mathrm{o}(y^k)$$

and as a consequence $A(\varepsilon,\delta) = A(\varepsilon,\omega)$. Therefore, $A(\varepsilon,\delta)$ does not depend on the second argument. Then from equation (25), $A(\eta) = C_1(\eta,\varepsilon)^k A(\varepsilon)$, and from (24) we obtain

$$A(\varepsilon) = C\exp\{-kF(\varepsilon)\}. \tag{26}$$

It is well known (see again section 3.4 and [22]) that $\tau_{\delta,\varepsilon}(x) = h\big(f_0^{-1}(x)\big)$, where $h(\bar{x}) = B(\delta,\varepsilon)\bar{x}^{1/k} + \mathrm{o}(\bar{x}^{1/k})$. Since $\tau_{\delta,\varepsilon} \circ \delta_{\varepsilon,\delta}$ is the identity map, we obtain

$$B(\varepsilon,\delta)\left(\frac{-1}{\ln(f_0(A(\varepsilon)x^k + \mathrm{o}(x^k)))}\right)^{1/k} + \mathrm{o}(x) = B(\varepsilon,\delta)A(\varepsilon)^{1/k}x + \mathrm{o}(x) = x$$

which implies that $B(\varepsilon,\delta) = B(\varepsilon)$ and $B(\varepsilon) = A(\varepsilon)^{-1/k}$. Hence $B(\varepsilon) = C^{-1/k}\exp\{F(\varepsilon)\}$.

To finish the proof we only have to compute C. We claim that $C = kb/a$. Indeed, by means of a local $\mathcal{C}^\infty$ change of coordinates given by

$$\begin{aligned} u &= \varphi_1(x,y) = x(1+\Phi_1(x,y)) \\ v &= \varphi_2(x,y) = y(1+\Phi_2(x,y)) \end{aligned} \tag{27}$$

where Φ_1 and Φ_2 vanish at $(0,0)$, system (23) is transformed into Dulac's normal form

$$\begin{aligned} \dot{u} &= -au \\ \dot{v} &= v^{k+1}(b+cv^k) \end{aligned} \tag{28}$$

in a neighbourhood of the origin (see [22]). Its integral curves are given by

$$u\left(\frac{(b+cv^k)^{1/k}}{v}\right)^{ac/b^2}\exp\left\{-\frac{1}{(kb/a)\,v^k}\right\} = h.$$

It can be checked that the above expression can be written as

$$u\exp\left\{-\frac{1}{(kb/a)\,v^k + \mathrm{o}(v^k)}\right\} = h$$

that is,

$$uf_0\left(\left(\frac{kb}{a}\right)v^k + \mathrm{o}(v^k)\right) = h.$$

Hence the level curves of system (23) are

$$x(1+\Phi_1(x,y))\,f_0\left(\left(\frac{kb}{a}\right)y^k(1+\Phi_2(x,y))^k + \mathrm{o}(y^k(1+\Phi_2(x,y))^k)\right) = h.$$

Evaluating the integral curve of level h in (ε,y) and (x,δ), and using that by equation (26), $x = f_0\left(C\exp\{-kF(\varepsilon)\}y^k + \mathrm{o}(y^k)\right)$, a tedious computation gives that:

$$f_0\left(C\exp\{-kF(\varepsilon)\}y^k + \mathrm{o}(y^k)\right) = f_0\left(\left(\frac{kb}{a}\right)(1+\Phi_2(\varepsilon,0))y^k + \mathrm{o}(y^k)\right).$$

Hence

$$C\exp\{-kF(\varepsilon)\}y^k + o(y^k) = \frac{kb}{a}(1 + \Phi_2(\varepsilon, 0))y^k + o(y^k)$$

and since $F(0) = 0$ and $\Phi_2(0, 0) = 0$, we have that $C = kb/a$, as we wanted to prove. □

Proof of lemma 7. Without loss of generality we can assume that $s_j = 1$. Since the class of systems that we consider is closed by rotations, we prove the lemma assuming that $\theta_j = 0$. We put $\alpha = \alpha_j$, $\beta = \beta_j$, $P_m = P_m^j$, $Q_m = Q_m^j$ and $\Delta_0^\varepsilon = \Delta_j^\varepsilon$. The map Δ_0^ε is well defined since lemma 3 ensures that there are no orbits tending to or leaving the origin of system (1) in the direction $\{\theta = 0\}$. Following the steps of desingularization used in section 2.2, under the hypothesis assumed, the characteristic direction given by $\{\theta = 0\}$ corresponds to the singular point $(u_0, z_0) = (0, 0)$ of system (6) which unfolds at the end of the process of desingularization (systems (9) and (10)) in four singular points $\{p_1', p_2', p_2, p_1\}$ (see figure 4). A study of the blow-up gives that

$$\begin{pmatrix} -((2+k)\alpha - 2\beta) & 0 \\ 0 & -(\beta - \alpha) \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} (2+k)\alpha - 2\beta & 0 \\ 0 & \beta - \alpha \end{pmatrix}$$

are the differential matrices of the corresponding vector fields at the points p_1' and p_1, respectively. This follows from the fact that the vector field in a neighbourhood of p_1' is given by a time inversion of system (9), and the vector field in a neighbourhood of p_1 is given by system (9). Also we have that

$$\begin{pmatrix} \gamma & 0 \\ 0 & -2\gamma \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -\gamma & 0 \\ 0 & 2\gamma \end{pmatrix}$$

are the differential matrices of the corresponding vector fields associated with system (10) at the points p_2' and p_2, respectively, where $\gamma = Q_M(1, 0)$.

Assume now that the condition $((2 + M - m)\,\alpha - 2\beta) < 0$ is satisfied. We can write

$$\Delta_0^\varepsilon = \sigma_1 \circ \varphi_1 \circ \sigma_2 \circ \varphi_2 \circ \sigma_2' \circ \varphi_1' \circ \sigma_1' \tag{29}$$

where σ_1' is the transition map associated with the hyperbolic sector p_1' computed from $\{z = -\varepsilon\}$ to $\{w = \mu\}$ (where ε and μ are positive and small enough) in the coordinates of system (9). The map σ_1 is the transition map associated with the hyperbolic sector p_1 computed from $\{w = \mu\}$ to $\{z = \varepsilon\}$ in the coordinates of system (9). The map σ_2' is the transition map associated with the hyperbolic sector p_2' computed from $\{q = \mu\}$ to $\{v = -\delta\}$ (where δ is positive and small enough) in the coordinates of system (10). Lastly, σ_2 is the transition map associated with the hyperbolic sector p_2 computed from $\{v = \delta\}$ to $\{q = \mu\}$ in the coordinates of system (10).

All the above maps are not differentiable but semiregular maps (see definition 12 in section 3.4), whose leading term can be expressed as follows:

$$\sigma_1'(\bar{x}) = a\bar{x}^{\alpha_1} + o(\bar{x}^{\alpha_1})$$

$$\sigma_2'(\bar{x}) = b\bar{x}^{\beta_1} + o(\bar{x}^{\beta_1})$$

$$\sigma_2(\bar{x}) = c\bar{x}^{1/\beta_1} + o(\bar{x}^{1/\beta_1})$$

$$\sigma_1(\bar{x}) = \mathrm{d}\bar{x}^{1/\alpha_1} + o(\bar{x}^{1/\alpha_1})$$

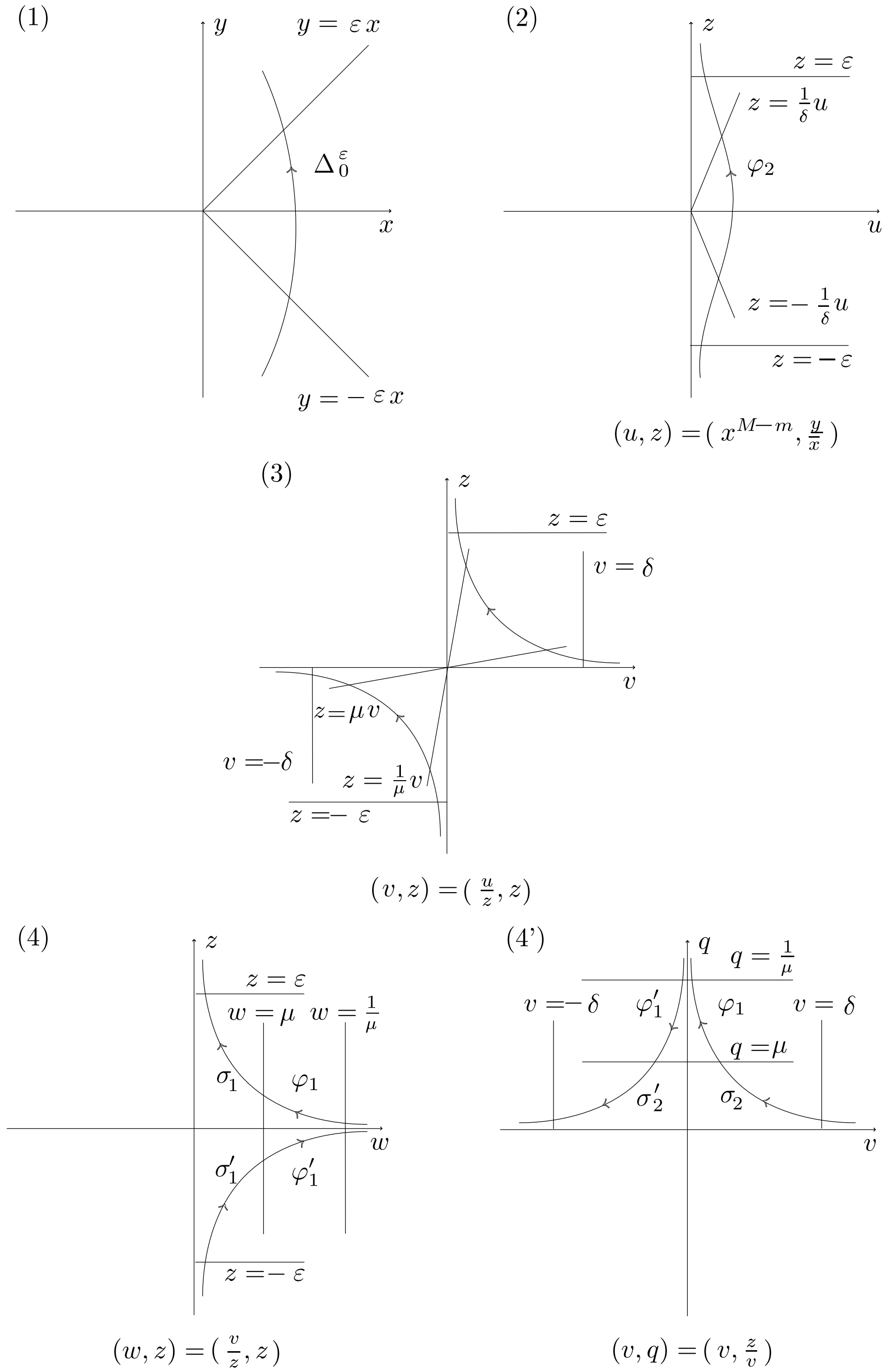

Figure 3. Blow-up of the characteristic direction $\{\theta = 0\}$ in local coordinates.

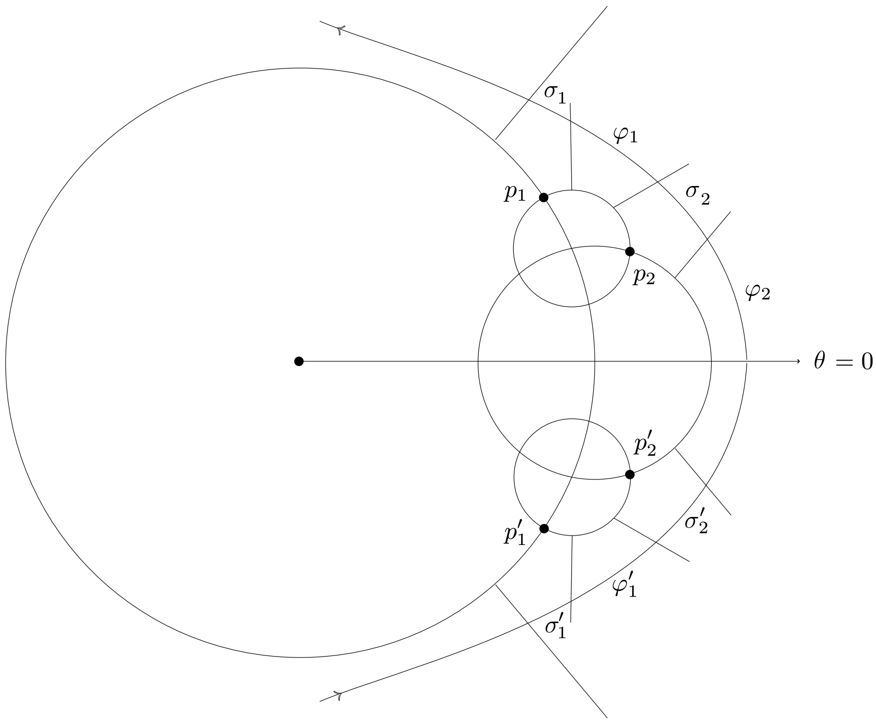

Figure 4. Blow-up of the characteristic direction $\{\theta = 0\}$.

where

$$\alpha_1 = -\frac{\beta - \alpha}{(2+k)\alpha - 2\beta}$$

$$\beta_1 = \frac{2\gamma}{\gamma} = 2$$

$$a = a(\varepsilon, \mu)$$

$$b = b(\mu, \delta)$$

$$c = c(\mu, \delta)$$

$$d = d(\mu, \delta)$$

are non-zero coefficients that will be computed below. In equation (29) φ_1' denotes the regular map from a neighbourhood of p_1' to a neighbourhood of p_2', φ_2 denotes the regular map from a neighbourhood of p_2' to a neighbourhood of p_2, and φ_1 denotes the regular map from a neighbourhood of p_2 to a neighbourhood of p_1. Since φ_1', φ_1 and φ_2 are regular maps, we can write them as

$$\varphi_1'(\bar{x}) = \delta_1'\bar{x} + \mathrm{o}(\bar{x})$$

$$\varphi_2(\bar{x}) = \delta_2\bar{x} + \mathrm{o}(\bar{x})$$

$$\varphi_1(\bar{x}) = \delta_1\bar{x} + \mathrm{o}(\bar{x})$$

where δ_1', δ_1 and δ_2 are positive coefficients, also depending on ε, δ and μ.

To complete the proof of the lemma we have to compute the leading term of the composition of these maps (see equation (29)). Firstly, we compute δ_2, δ_1' and δ_1', by integrating the first-order variational equations.

Computation of δ_2. To compute δ_2 we consider system (6). For this system the characteristic direction $\{\theta = 0\}$ corresponds to the singular point $(u, z) = (0, 0)$. We will compute the first polar blow-up obtaining the new system (30). The map φ_2 can be considered as the transition map from $\{z = -(1/\delta)u\}$ to $\{z = (1/\delta)u\}$, for $\delta > 0$ small enough, given above. Hence δ_2 can be obtained by integrating the first-order variational equations of the system (30) associated with the orbit $\{r = 0\}$ from $\{\theta_0 = \arctan(-1/\delta)\}$ to $\{\theta_f = \arctan(1/\delta)\}$. Next we explain the above procedure.

Taking the change of coordinates given by $(u, z) = (r\cos\theta, r\sin\theta)$, system (6) is written as

$$\begin{aligned} \dot{r} &= \gamma\cos\theta\sin\theta\, r + \mathrm{O}(r^2) \\ \dot{\theta} &= \gamma\cos^2\theta + \mathrm{O}(r). \end{aligned} \tag{30}$$

The first-order variational equation of system (30) associated with the orbit $\{r = 0\}$ is:

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}\theta} = \tan\theta\,\bar{r}.$$

By separation of variables and integrating the first term from $\bar{r}_0$ to $\bar{r}_f$ (note that $\varphi_2(r) = \bar{r}_f r + \mathrm{O}(r^2)$, and then $\bar{r}_f = \delta_2$), and the second term from θ_0 to θ_f, we have

$$\int_{\bar{r}_0}^{\bar{r}_f} \frac{\mathrm{d}\bar{r}}{\bar{r}} = \int_{\theta_0}^{\theta_f} \tan\theta\,\mathrm{d}\theta$$

which gives

$$\ln\left(\frac{\bar{r}_f}{\bar{r}_0}\right) = \ln\left(\frac{\cos\left(\arctan(-1/\delta)\right)}{\cos\left(\arctan(1/\delta)\right)}\right) = \ln(1) = 0$$

(note that $\bar{r}_0 = 1$), hence $\bar{r}_f = \delta_2 = 1$. This means that φ_2 is the identity at first order.

Computation of δ_1' and δ_1. To compute δ_1' and δ_1 we work with the coordinates (w, z) corresponding to system (9). The map φ_1' is the transition map from $\{w = \mu\}$ to $\{w = 1/\mu\}$, for $\mu > 0$ small enough, given above. Hence δ_1' can be obtained from the first-order variational equations of the system obtained after a time inversion of system (9), associated with the orbit $\{z = 0\}$. The map φ_1 is the transition map from $\{w = 1/\mu\}$ to $\{w = \mu\}$ and hence is the inverse of φ_1'. Therefore, $\delta_1' \cdot \delta_1 = 1$. A tedious computation gives

$$\delta_1' = \exp\left\{\int_{\mu}^{1/\mu} M(w)\,\mathrm{d}w\right\}$$

and hence

$$\delta_1 = \exp\left\{\int_{1/\mu}^{\mu} M(w)\,\mathrm{d}w\right\}$$

where

$$M(w) = \frac{(\beta - \alpha) + w\gamma}{w((2 + k)\alpha - 2\beta)}.$$

If we denote $\tau_1 = \sigma_2' \circ \varphi_1' \circ \sigma_1'$ and $\tau_2 = \sigma_1 \circ \varphi_1 \circ \sigma_2$, then $\Delta_0^\varepsilon = \tau_2 \circ \varphi_2 \circ \tau_1$. In what follows we will compute τ_1 and τ_2. We will use the following notation: $(a, b)^{(n)}$, where $n \in \{1, 2, 3, 4, 4'\}$, means a point expressed in the coordinates (x, y) of system (1) if $n = 1$, expressed in coordinates (u, z) of system (6) if $n = 2$, in coordinates (v, z) of (7) if $n = 3$, in coordinates (w, z) of system (9) if $n = 4$, and in coordinates (v, q) of (10) if $n = 4'$ (see again figure 3).

Computation of τ_1. We start with a point $(u_0, z_0)^{(2)} = (u_0, -\varepsilon)^{(2)}$. Following the notation introduced above we have

$$\begin{aligned}
\tau_1\left((u_0, -\varepsilon)^{(2)}\right) &= \tau_1\left(\left(-\frac{u_0}{\varepsilon}, -\varepsilon\right)^{(3)}\right) = \tau_1\left(\left(\frac{u_0}{\varepsilon^2}, -\varepsilon\right)^{(4)}\right) = \sigma_2' \circ \varphi_1' \circ \sigma_1'\left(\left(\frac{u_0}{\varepsilon^2}, -\varepsilon\right)^{(4)}\right) \\
&= \sigma_2' \circ \varphi_1'\left(\left(\mu, a\varepsilon^{-2\alpha_1} u_0^{\alpha_1} + \mathrm{o}(u_0^{\alpha_1})\right)^{(4)}\right) \\
&= \sigma_2'\left(\left(\frac{1}{\mu}, \delta_1' a\varepsilon^{-2\alpha_1} u_0^{\alpha_1} + \mathrm{o}(u_0^{\alpha_1})\right)^{(4)}\right) \\
&= \sigma_2'\left(\left(\frac{1}{\mu}\delta_1' a\varepsilon^{-2\alpha_1} u_0^{\alpha_1} + \mathrm{o}(u_0^{\alpha_1}), \delta_1' a\varepsilon^{-2\alpha_1} u_0^{\alpha_1} + \mathrm{o}(u_0^{\alpha_1})\right)^{(3)}\right) \\
&= \sigma_2'\left(\left(\frac{1}{\mu}\delta_1' a\varepsilon^{-2\alpha_1} u_0^{\alpha_1} + \mathrm{o}(u_0^{\alpha_1}), \mu\right)^{(4')}\right) \\
&= \left(-\delta, b\mu^{-\beta_1}(\delta_1')^{\beta_1} a^{\beta_1}\varepsilon^{-2\alpha_1\beta_1} u_0^{\alpha_1\beta_1} + o\left(u_0^{\alpha_1\beta_1}\right)\right)^{(4')} \\
&= \left(-\delta, -\delta b\mu^{-\beta_1}(\delta_1')^{\beta_1} a^{\beta_1}\varepsilon^{-2\alpha_1\beta_1} u_0^{\alpha_1\beta_1} + \mathrm{o}(u_0^{\alpha_1\beta_1})\right)^{(3)} \\
&= \left(\delta^2 b\mu^{-\beta_1}(\delta_1')^{\beta_1} a^{\beta_1}\varepsilon^{-2\alpha_1\beta_1} u_0^{\alpha_1\beta_1} + \mathrm{o}(u_0^{\alpha_1\beta_1}),\right. \\
&\qquad \left. -\delta b\mu^{-\beta_1}(\delta_1')^{\beta_1} a^{\beta_1}\varepsilon^{-2\alpha_1\beta_1} u_0^{\alpha_1\beta_1} + \mathrm{o}(u_0^{\alpha_1\beta_1})\right)^{(2)}.
\end{aligned}$$

Finally, we can write τ_1 as

$$\tau_1\left((u_0, -\varepsilon)^{(2)}\right) = (\delta A u_0^{\alpha_1\beta_1} + \mathrm{o}(u_0^{\alpha_1\beta_1}), -A u_0^{\alpha_1\beta_1} + \mathrm{o}(u_0^{\alpha_1\beta_1}))^{(2)}$$

where $A = \delta b\mu^{-\beta_1}(\delta_1')^{\beta_1} a^{\beta_1}\varepsilon^{-2\alpha_1\beta_1}$.

Computation of τ_2. We start with a point $(u_0, z_0)^{(2)} = (u_0, u_0/\delta)^{(2)}$. Following the notation introduced above we obtain

$$\begin{aligned}
\tau_2\left(\left(u_0, \frac{1}{\delta}u_0\right)^{(2)}\right) &= \tau_2\left(\left(\delta, \frac{1}{\delta}u_0\right)^{(3)}\right) = \tau_2\left(\left(\delta, \frac{1}{\delta^2}u_0\right)^{(4')}\right) \\
&= \sigma_1 \circ \varphi_1 \circ \sigma_2\left(\left(\delta, \frac{1}{\delta^2}u_0\right)^{(4')}\right) \\
&= \sigma_1 \circ \varphi_1\left((c\delta^{-2/\beta_1} u_0^{1/\beta_1} + \mathrm{o}(u_0^{1/\beta_1}), \mu)^{(4')}\right) \\
&= \sigma_1 \circ \varphi_1\left((c\delta^{-2/\beta_1} u_0^{1/\beta_1} + \mathrm{o}(u_0^{1/\beta_1}), \mu c\delta^{-2/\beta_1} u_0^{1/\beta_1} + \mathrm{o}(u_0^{1/\beta_1}))^{(3)}\right) \\
&= \sigma_1 \circ \varphi_1\left(\left(\frac{1}{\mu}, \mu c\delta^{-2/\beta_1} u_0^{1/\beta_1} + \mathrm{o}\left(u_0^{1/\beta_1}\right)\right)^{(4)}\right) \\
&= \sigma_1\left(\left(\mu, \delta_1\mu c\delta^{-2/\beta_1} u_0^{1/\beta_1} + \mathrm{o}\left(u_0^{1/\beta_1}\right)\right)^{(4)}\right)
\end{aligned}$$

$$= \left(d\delta_1^{1/\alpha_1}\mu^{1/\alpha_1}c^{1/\alpha_1}\delta^{-2/(\alpha_1\beta_1)}u_0^{1/(\alpha_1\beta_1)} + \mathrm{o}\left(u_0^{1/(\alpha_1\beta_1)}\right), \varepsilon\right)^{(4)}$$
$$= \left(\varepsilon d\delta_1^{1/\alpha_1}\mu^{1/\alpha_1}c^{1/\alpha_1}\delta^{-2/(\alpha_1\beta_1)}u_0^{1/(\alpha_1\beta_1)} + \mathrm{o}\left(u_0^{1/(\alpha_1\beta_1)}\right), \varepsilon\right)^{(3)}$$
$$= \left(\varepsilon^2 d\delta_1^{1/\alpha_1}\mu^{1/\alpha_1}c^{1/\alpha_1}\delta^{-2/(\alpha_1\beta_1)}u_0^{1/(\alpha_1\beta_1)} + \mathrm{o}\left(u_0^{1/(\alpha_1\beta_1)}\right), \varepsilon\right)^{(2)}.$$

Finally, we can write τ_2 as

$$\tau_2\left(\left(u_0, \frac{1}{\delta}u_0\right)^{(2)}\right) = \left(Bu_0^{1/(\alpha_1\beta_1)} + \mathrm{o}\left(u_0^{1/(\alpha_1\beta_1)}\right), \varepsilon\right)^{(2)}$$

where $B = \varepsilon^2 d\delta_1^{1/\alpha_1}\mu^{1/\alpha_1}c^{1/\alpha_1}\delta^{-2/(\alpha_1\beta_1)}$.

Computation of Δ_0^ε. Set $D = B\delta^{1/(\alpha_1\beta_1)}A^{1/(\alpha_1\beta_1)}$. Then we have that

$$\Delta_0^\varepsilon\left((u_0, -\varepsilon)^{(2)}\right) = \tau_2 \circ \varphi_2 \circ \tau_1\left((u_0, -\varepsilon)^{(2)}\right) = (Du_0 + \mathrm{o}(u_0), \varepsilon)^{(2)}.$$

Hence, since $(u, z)^{(2)} = (\sqrt[k]{u}, z\sqrt[k]{u})^{(1)}$, (where $k = M - m$) we obtain

$$\Delta_0^\varepsilon\left((\sqrt[k]{u_0}, -\varepsilon\sqrt[k]{u_0})^{(1)}\right) = (\sqrt[k]{Du_0}, \varepsilon\sqrt[k]{Du_0})^{(1)}.$$

Therefore, in the usual polar coordinates, the transition from the ray $\{\theta = -\arctan(\varepsilon)\}$ to the ray $\{\theta = +\arctan(\varepsilon)\}$ can be written as

$$\Delta_0^\varepsilon(R_0) = \sqrt[k]{D}R_0 + \mathrm{o}(R_0) \tag{31}$$

(and then $D_0^\varepsilon = \sqrt[k]{D}$). A simple calculation gives that $D = a^{1/\alpha_1}b^{1/(\alpha_1\beta_1)}c^{1/\alpha_1}d(\delta_1\delta_1')^{1/\alpha_1}$, Since $\delta_1\delta_1' = 1$, we obtain

$$D = a^{1/\alpha_1}b^{1/(\alpha_1\beta_1)}c^{1/\alpha_1}d. \tag{32}$$

Finally, to end the proof of the lemma we have to compute a, b, c and d. To do these computations we use lemma 8.

Computation of a and d. Consider system (9). Applying lemma 8 we obtain that

$$a = \frac{-\varepsilon}{\mu^{\alpha_1}}\frac{\exp\{G(\mu)\}}{\exp\{\alpha_1 F(-\varepsilon)\}} \qquad \text{and} \qquad d = \frac{\mu}{\varepsilon^{1/\alpha_1}}\frac{\exp\{F(\varepsilon)\}}{\exp\{(1/\alpha_1)G(\mu)\}}$$

where

$$\alpha_1 = -\frac{\beta - \alpha}{(2+k)\alpha - 2\beta}$$

$$G(\mu) = \int_0^\mu \left(\frac{Q_4(w, z)}{zP_4(w, z)}\right)\Bigg|_{z=0} + \frac{\alpha_1}{w}\,\mathrm{d}w$$

and

$$F(\varepsilon) = \int_0^\varepsilon \left(\frac{P_4(w, z)}{wQ_4(w, z)}\right)\Bigg|_{w=0} + \frac{1}{\alpha_1 z}\,\mathrm{d}z.$$

Computation of b and c. Consider system (10). Again, applying lemma 8 we have

$$b = \frac{\mu}{(-\delta)^{\beta_1}} \frac{\exp\{J(-\delta)\}}{\exp\{\beta_1 H(\mu)\}} \qquad \text{and} \qquad c = \frac{\delta}{\mu^{1/\beta_1}} \frac{\exp\{H(\mu)\}}{\exp\{(1/\beta_1) J(\delta)\}}$$

where $\beta_1 = 2$,

$$J(\delta) = \int_0^\delta \left(\frac{Q_{4'}(v, q)}{q P_{4'}(v, q)} \right) \bigg|_{q=0} + \frac{\beta_1}{v} \, \mathrm{d}v = \int_0^\delta \left(\frac{-2}{v} + \frac{\beta_1}{v} \right) \mathrm{d}v = 0$$

and

$$H(\mu) = \int_0^\mu \left(\frac{P_{4'}(v, q)}{v Q_{4'}(v, q)} \right) \bigg|_{v=0} + \frac{1}{\beta_1 q} \, \mathrm{d}q.$$

Taking into account equation (32), a computation gives

$$D = a^{1/\alpha_1} b^{1/(\alpha_1 \beta_1)} c^{1/\alpha_1} d = \exp\{F(\varepsilon) - F(-\varepsilon)\}. \tag{33}$$

Finally, we obtain the expression of F. Note that

$$\begin{aligned} \left(\frac{P_4(w, z)}{w Q_4(w, z)} \right) \bigg|_{w=0} + \frac{1}{\alpha_1 z} &= -\frac{2}{z} + k \frac{P_m(1, z)}{Q_m(1, z) - z P_m(1, z)} + \frac{1}{\alpha_1 z} \\ &= k \frac{P_m(1, z)}{Q_m(1, z) - z P_m(1, z)} - k \frac{\alpha}{\beta - \alpha} \frac{1}{z}. \end{aligned}$$

Therefore,

$$F(\varepsilon) - F(-\varepsilon) = (M - m) \int_{-\varepsilon}^{\varepsilon} \frac{P_m(1, z)}{Q_m(1, z) - z P_m(1, z)} - \frac{\alpha}{\beta - \alpha} \frac{1}{z} \, \mathrm{d}z$$

and from equation (33) we obtain that

$$D = \exp\left\{ (M - m) \int_{-\varepsilon}^{\varepsilon} \frac{P_m(1, z)}{Q_m(1, z) - z P_m(1, z)} - \frac{\alpha}{\beta - \alpha} \frac{1}{z} \, \mathrm{d}z \right\}.$$

Then, from (31), we have that

$$D_0^\varepsilon = \exp\left\{ \int_{-\varepsilon}^{\varepsilon} \frac{P_m(1, z)}{Q_m(1, z) - z P_m(1, z)} - \frac{\alpha}{\beta - \alpha} \frac{1}{z} \, \mathrm{d}z \right\}.$$

Assume now that $((2 + M - m)\alpha - 2\beta) = 0$. In this case p_1' and p_1 are elementary degenerate saddle nodes with the hyperbolic sectors located in the region $\{w > 0\}$ in coordinates (w, z) of system (9), see figure 3. Note that σ_1' is again the transition map associated with the hyperbolic sector p_1' computed from $\{z = -\varepsilon\}$ to $\{w = \mu\}$ (where ε and μ are positive and small enough) in the coordinates of system (9), while σ_1 is the transition map associated with the hyperbolic sector p_1 computed from $\{w = \mu\}$ to $\{z = \varepsilon\}$ in the coordinates of system (9). From lemma 9 we have that

$$\sigma_1'(\bar{x}) = f_0\big(a x^\lambda + \mathrm{o}(x^\lambda)\big)$$

$$\sigma_1(\bar{x}) = h\big(f_0^{-1}(x)\big)$$

where $h(x) = b x^{1/\lambda} + \mathrm{o}(x^{1/\lambda})$, a and b are non-vanishing coefficients that will be computed below. The value λ is not relevant for our purposes.

Let φ denote the semiregular transition map from a neighbourhood of p_1' to a neighbourhood of p_1. From lemma 14 (see section 3.4), we have

$$\Delta_0^{\varepsilon}((u_0, -\varepsilon)^{(2)}) = \left(b\left(\frac{-1}{\ln(\varphi \circ f_0(au_0^{\lambda} + \mathrm{o}(u_0^{\lambda})))}\right)^{1/\lambda} + \mathrm{o}(u_0), \varepsilon\right)^{(2)}$$
$$= \left(ba^{1/\lambda}u_0 + \mathrm{o}(u_0), \varepsilon\right)^{(2)}.$$

In the usual polar coordinates, the transition from the ray $\{\theta = -\arctan(\varepsilon)\}$ to the ray $\{\theta = \arctan(\varepsilon)\}$ can be written as

$$\Delta_0^{\varepsilon}(R_0) = \sqrt[k]{D}u_0 + \mathrm{o}(u_0)$$

where $D = ba^{1/\lambda}$. Applying again lemma 9 we have that

$$a = C\exp\{-\lambda M(-\varepsilon)\}$$
$$b = C^{-1/\lambda}\exp\{M(\varepsilon)\}$$

where

$$M(x) = \int_0^x \left(\frac{P_4(w,z)}{wQ_4(w,z)}\right)\bigg|_{w=0} \mathrm{d}z = \int_0^x \frac{(2+M-m)zP_m(1,z) - 2Q_m(1,z)}{zQ_m(1,z) - z^2P_m(1,z)}\,\mathrm{d}z.$$

Hence

$$D_0^{\varepsilon} = \exp\left\{\frac{1}{M-m}\int_{-\varepsilon}^{\varepsilon} \frac{(2+M-m)zP_m(1,z) - 2Q_m(1,z)}{zQ_m(1,z) - z^2P_m(1,z)}\,\mathrm{d}z\right\}. \qquad \square$$

3.2.2. Proof of theorem B. Let $\theta_1, \ldots, \theta_k$ be the characteristic directions of system (1). Observe that is not restrictive to assume that $\{\theta = 0\}$ is not a characteristic direction. Let $\varepsilon > 0$ be small enough such that S_ε and I_ε are well defined. Integrating the first-order variational equations of system (2) associated with the orbit $\{R = 0\}$, we have that the transition map T_j^{ε} from $\{\theta = \theta_j + \varepsilon\}$ to $\{\theta = \theta_{j+1} - \varepsilon\}$ (which is regular) is given by

$$T_j^{\varepsilon}(R_0) = \exp\left\{\int_{\theta_j+\varepsilon}^{\theta_{j+1}-\varepsilon} \frac{\cos\theta P_m(\theta) + \sin\theta Q_m(\theta)}{\cos\theta Q_m(\theta) - \sin\theta P_m(\theta)}\,\mathrm{d}\theta\right\} R_0 + \mathrm{o}(R_0).$$

Also let T_0^{ε} be the regular transition map from $\{\theta = 0\}$ to $\{\theta = \theta_1 - \varepsilon\}$, and T_k^{ε} be the regular transition from $\{\theta = \theta_k + \varepsilon\}$ to $\{\theta = 2\pi\}$.

Set $\bar{\varepsilon} = \tan(\varepsilon)$. Since $\Pi = T_k^{\varepsilon} \circ \Delta_k^{\bar{\varepsilon}} \circ T_{k-1}^{\varepsilon} \cdots T_2^{\varepsilon} \circ \Delta_2^{\bar{\varepsilon}} \circ T_1^{\varepsilon} \circ \Delta_1^{\bar{\varepsilon}} \circ T_0^{\varepsilon}$, is a composition of regular and semiregular maps with non-vanishing linear leading terms, by lemma 14 of section 3.4, we can write $\Pi(x) = V_1 x + \mathrm{o}(x)$, where V_1 is the product of the principal terms of the maps Δ_j^{ε} and T_j^{ε}, for $j = 1, \ldots, k$. Therefore, for all $\varepsilon > 0$ small enough, we obtain

$$V_1 = \left(\prod_{j=1}^{k} D_j^{\bar{\varepsilon}}\right)\exp\left\{\int_{I_\varepsilon} \frac{\cos\theta P_m(\theta) + \sin\theta Q_m(\theta)}{\cos\theta Q_m(\theta) - \sin\theta P_m(\theta)}\,\mathrm{d}\theta\right\}. \tag{34}$$

By lemma 7 the integrals appearing in each $D_j^{\bar{\varepsilon}}$ are non-singular, hence $\lim_{\varepsilon\to 0} D_j^{\bar{\varepsilon}} = 1$. By taking $\varepsilon \to 0$ in equation (34), we have that the GPV exists and

$$V_1 = \exp\left\{\mathrm{GPV}\int_0^{2\pi} \frac{\cos\theta P_m(\theta) + \sin\theta Q_m(\theta)}{\cos\theta Q_m(\theta) - \sin\theta P_m(\theta)}\,\mathrm{d}\theta\right\}. \qquad \square$$

3.2.3. Proof of proposition C. Firstly, we outline the main difficulties in computing V_1 in the case that more than two opposite characteristic directions satisfying $(2+k)\alpha_j - 2\beta_j < 0$ and $\beta_j - \alpha_j = 0$ appear in the desingularization process. Unfortunately, in that case we do not have an analogue of lemma 7, and we have to consider the problem from a point of view different from that used in the proof of theorem B.

Let $\{j_1, j_2, \dots, j_n, j_{n+1}, \dots, j_{2n}\}$ denote the indices of the characteristic directions for which $(2+k)\alpha_{j_i} - 2\beta_{j_i} < 0$ and $\beta_{j_i} - \alpha_{j_i} = 0$. For the sake of simplicity we will use the notation $\bar{\theta}_i = \theta_{j_i}$ (note that for each $k = 1, \dots, n$, $\bar{\theta}_k \in [0, \pi)$ and $\bar{\theta}_{k+n} = \bar{\theta}_k + \pi$). As can be seen in section 2.2, each $\bar{\theta}_i$ at the end of the desingularization process unfolds into four singular points $\{p'_{1,i}, p'_{2,i}, p_{2,i}, p_{1,i}\}$ such that $p'_{2,i}$ and $p_{2,i}$ are hyperbolic saddles and $p'_{1,i}$ and $p_{1,i}$ are elementary degenerate saddles.

Let ρ_i denote the transition map of the flow from a neighbourhood of $p_{1,i}$ to a neighbourhood $p'_{1,i+1}$, which is semiregular or regular (depending on whether or not there are characteristic directions between $\bar{\theta}_i$ and $\bar{\theta}_{i+1}$) with a non-vanishing linear leading term. Let $T_i(\bar{x}) = t_i\bar{x} + \mathrm{o}(\bar{x})$ denote the transition map from a neighbourhood of $p'_{1,i}$ to a neighbourhood $p_{1,i}$. Denote by $(h'_i \circ f_0^{-1})$ the (vertical semiregular) map associated with the hyperbolic sector of $p'_{1,i}$, with

$$h'_i(\bar{x}) = b_i \bar{x}^{1/k_i} + \mathrm{o}(\bar{x}^{1/k_i}) \tag{35}$$

and by $(f_0 \circ h_i)$ the (flat semiregular) map associated with the hyperbolic sector of $p_{1,i}$, where

$$h_i(\bar{x}) = a_i \bar{x}^{k_i} + \mathrm{o}(\bar{x}^{k_i}). \tag{36}$$

Note that the fact that the systems of type (1) expressed in polar coordinates (R, θ) satisfy $(\dot{R}(R, \theta+\pi), \dot{\theta}(R, \theta+\pi)) = (\dot{R}(R, \theta), \dot{\theta}(R, \theta))$, implies that $k_i = k_{i+n}$, $a_i = a_{i+n}$ and $b_i = b_{i+n}$ for all $i \in \{1, \dots, n\}$. Note also that from lemma 14 we have (in the local coordinates of system (9))

$$h'_{i+1} \circ f_0^{-1} \circ \rho_i \circ f_0 \circ h_i(\bar{x}) = b_{i+1} a_i^{1/k_{i+1}} \bar{x} + \mathrm{o}(\bar{x}).$$

Taking this fact into account, and since

$$\begin{aligned}\Pi = T_1 \circ \left(h'_1 \circ f_0^{-1} \circ \rho_{2n} \circ f_0 \circ h_{2n}\right) \circ \cdots \circ T_3 \circ \left(h'_3 \circ f_0^{-1} \circ \rho_2 \circ f_0 \circ h_2\right) \\ \circ T_2 \circ \left(h'_2 \circ f_0^{-1} \circ \rho_1 \circ f_0 \circ h_1\right)\end{aligned}$$

applying inductively lemma 14 we have that

$$V_1 = \left(t_1 b_1 a_{2n}^{1/k_1} t_{2n} \cdots t_3 b_3 a_2^{1/k_3} t_2 b_2 a_1^{1/k_2}\right)^{1/(M-m)}.$$

Using the above remark we obtain

$$V_1 = \left(b_{n+1} a_n^{1/k_{n+1}} t_n \cdots t_3 b_3 a_2^{1/k_3} t_2 b_2 a_1^{1/k_2} t_1\right)^{2/(M-m)}. \tag{37}$$

As we will see below, each t_k can be easily computed, but to compute the number $\Pi_{j=1}^n b_{j+1} a_j^{1/k_{j+1}}$, it is necessary to obtain the normal form of the vector field in a neighbourhood of the points p'_{j+1} and p_j for each $j = 1, \dots, n$. This is the main difficulty in having an explicit expression of V_1 analogous to that which appears in theorem B, in the case under study. However, if $n = 1$, there are only two (opposite) characteristic directions, V_1 can be easily computed explicitly from equation (37). This is the situation stated in proposition C.

Proof of proposition C. Without loss of generality we can take $\bar{\theta}_1 = 0$. From equation (37) we have

$$V_1 = \left(b_2 a_1^{1/k_2} t_1\right)^{2/(M-m)}$$

but as noted above $b_2 = b_1$ and $k_2 = k_1$. To compute a_1, b_1 and δ_1 we will use the coordinates of (w, z) of system (9). We claim that $b_1 a_1^{1/k_1} = 1$.

We use the same notation as in the proof of lemma 9 in section 3.2.1, and we use the same transversal sections. From this lemma we have that $a_1 = a_1(\delta) = C \exp\{-k_1 F(\delta)\}$ and $b_1 = b_1(\delta) = C^{1/k_1} \exp\{F(\delta)\}$, where C is a constant and

$$F(w) = \int \left(\frac{Q_4(w,z)}{z P_4(w,z)} \right) \Big|_{z=0} \mathrm{d}w = \int \frac{\gamma}{k_1\alpha - 2\gamma w} \mathrm{d}w = -\tfrac{1}{2} \ln |k_1\alpha - 2\gamma w| + K$$

where, as usual, $\gamma = Q_M(1, 0)$, and K is a constant. Hence

$$a_1(\delta) = C \left(\sqrt{k_1\alpha - 2\gamma\delta}\right)^{k_1} \qquad \text{and} \qquad b_1(\delta) = C^{-1/k_1} \left(\sqrt{k_1\alpha - 2\gamma\delta}\right)^{-1}$$

and then $b_1(\delta)\, a_1(\delta)^{1/k_1} = 1$. Hence the claim is proved.

Computation of t_1. An easy computation shows that

$$\begin{aligned}
T_1\big((\mu, z_o)^{(4)}\big) &= \varphi_1 \circ \sigma_2 \circ \varphi_2 \circ \sigma_2' \circ \varphi_1'\big((\mu, z_o)^{(4)}\big) = \varphi_1 \circ \sigma_2 \circ \varphi_2 \circ \sigma_2'\left(\left(\frac{1}{\mu}, \delta_1' z_o\right)^{(4)}\right) \\
&= \varphi_1 \circ \sigma_2 \circ \varphi_2 \circ \sigma_2'\left(\left(\frac{1}{\mu}\delta_1' z_o, \delta_1' z_o\right)^{(3)}\right) \\
&= \varphi_1 \circ \sigma_2 \circ \varphi_2 \circ \sigma_2'\left(\left(\frac{1}{\mu}\delta_1' z_o, \mu\right)^{(4')}\right) \\
&= \varphi_1 \circ \sigma_2 \circ \varphi_2\big((-\delta, b\mu^{-\beta_1}(\delta_1' z_o)^{\beta_1})^{(4')}\big) \\
&= \varphi_1 \circ \sigma_2\big((\delta, b\mu^{-\beta_1}(\delta_1' z_o)^{\beta_1})^{(4')}\big) = \varphi_1\big((cb^{1/\beta_1}\mu^{-1}\delta_1' z_o, \mu)^{(4')}\big) \\
&= \varphi_1\big((cb^{1/\beta_1}\mu^{-1}\delta_1' z_o, cb^{1/\beta_1}\delta_1' z_o)^{(3)}\big) = \varphi_1\left(\left(\frac{1}{\mu}, cb^{1/\beta_1}\delta_1' z_o\right)^{(4)}\right) \\
&= (\mu, cb^{1/\beta_1}\delta_1\delta_1' z_o)^{(4)} = (\mu, cb^{1/\beta_1} z_o)^{(4)}.
\end{aligned}$$

The values b and c are computed in the proof of lemma 7 (in section 3.2.1), and they satisfy $cb^{1/\beta_1} = -1$.

Hence, setting $x_0 = \sqrt[k]{\mu z_0^2}$, and $y_0 = z_0 \sqrt[k]{\mu z_0^2}$, we have

$$T_1\left((x_0, y_0)^{(1)}\right) = (\mu, -z_0)^{(4)} T_1\left((\mu, z_0)^{(4)}\right) = (\mu, -z_0)^{(4)} = (x_0, -y_0)^{(1)}$$

which implies that $t_1 = 1$. Therefore, $V_1 = 1$. □

Note that all the coefficients t_i that appear in equation (37) can be computed in the same way as in the proof of proposition C, giving that $t_i = 1$ for $i = 1, \ldots, 2n$. Thus, we obtain the following result.

Proposition 10. *Assume that $\mathcal{X}$ belongs to the class $\mathcal{G}$. Suppose that the origin of system (1) (associated with $\mathcal{X}$) is a focus–centre, and there exist $\{\bar\theta_1, \ldots, \bar\theta_{2n}\}$ characteristic directions such that $\beta_{i_k} - \alpha_{i_k} = 0$ for $k = 1, \ldots, 2n$. Then the return map associated with the origin has the form $\Pi(x_0) = V_1 x_0 + \mathrm{o}(x_0)$, where*

$$V_1 = \Pi_{i=1}^{n} \left(b_{i+1} a_i^{1/k_{i+1}} \right)^{2/(M-m)}$$

and a_i and b_{i+1} are defined in expressions (35) and (36).

3.3. Examples

Example 1. *Consider the system*

$$\begin{aligned} \dot x &= P(x, y) = y(x^2 + xy - y^2) \\ \dot y &= Q(x, y) = y^2(2x + y) + x^5 \end{aligned} \tag{38}$$

which can be written in polar coordinates (R, θ) as

$$\begin{aligned} \dot R &= R(\cos^3\theta \sin\theta + \cos^2\theta \sin^2\theta + \cos\theta \sin^3\theta + \sin^4\theta + R^2 \sin\theta \cos^5\theta) \\ \dot\theta &= \sin^2\theta + R^2\cos^6\theta. \end{aligned} \tag{39}$$

Its origin has two characteristic directions given by $\{\theta_1 = 0\}$ and $\{\theta_2 = \pi\}$. As explained in the introduction we only need to verify the focus–centre conditions for $\{\theta_1 = 0\}$. Condition (a) is fulfilled because $xQ(x, y) - yP(x, y) = y^2(x^2 + y^2) + x^6 > 0$, except at the origin. Also condition (b) is verified trivially because $P_3(\theta_i) = Q_3(\theta_i) = 0$ for $i = 1, 2$, and finally

$$\alpha_1 = \frac{\mathrm{d}}{\mathrm{d}z} P_m(1, z)|_{z=0} = \frac{\mathrm{d}}{\mathrm{d}z} \left(z(1 + z - z^2) \right) \Big|_{z=0} = 1$$

$$\beta_1 = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} Q_m(1, z)|_{z=0} = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z^2} \left(z^2(2 + z) \right) \Big|_{z=0} = 2.$$

Note that $\alpha_2 = \alpha_1$, $\beta_2 = \beta_1$, hence $\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = 5 \neq 0$, and then the system belongs to class $\mathcal{G}$. Since $(2 + M - m)\alpha_1 - 2\beta_1 = (2 + M - m)\alpha_2 - 2\beta_2 = 0$, by theorem A, system (38) is a focus–centre. Recall that each characteristic direction θ_i with $i = 1, 2$, at the end of the desingularization process unfolds into four singular points $\{p_1', p_2', p_2, p_1\}$ where p_1' and p_1 are elementary degenerate saddle nodes, and p_2' and p_2 are hyperbolic saddles. From theorem B, the Poincaré return map can be written as $\Pi(x_0) = V_1 x_0 + \mathrm{o}(x_0)$, where

$$\begin{aligned} V_1 &= \exp\left\{ \mathrm{GPV} \int_0^{2\pi} \frac{\cos^3\theta \sin\theta + \cos^2\theta \sin^2\theta + \cos\theta \sin^3\theta + \sin^4\theta}{\sin^2\theta} \, \mathrm{d}\theta \right\} \\ &= \exp\left\{ \mathrm{GPV} \int_0^{2\pi} \left(\frac{\cos^3\theta}{\sin\theta} + \cos\theta \sin\theta + 1 \right) \mathrm{d}\theta \right\} = \exp\{2\pi\}. \end{aligned}$$

Hence the origin is a repulsive focus.

Example 2. *Consider the system*

$$\begin{aligned} \dot x &= P(x, y) = y(\alpha x^2 + bxy + cy^2) \\ \dot y &= Q(x, y) = y^2(\alpha x + by) + x^5 \end{aligned} \tag{40}$$

with $\alpha < 0$ and $c < 0$. In polar coordinates (R, θ), system (40) is written as

$$\begin{aligned} \dot R &= R(\alpha \cos^3\theta \sin\theta + (c + \alpha)\cos\theta \sin^3\theta + b\sin^2\theta + R^2 \sin\theta \cos^5\theta) \\ \dot\theta &= -c\sin^4\theta + R^2\cos^6\theta. \end{aligned} \tag{41}$$

Its origin has two characteristic directions given by $\{\theta_1 = 0\}$ *and* $\{\theta_2 = \pi\}$. *As in the previous example we just need to verify the focus–centre conditions for* $\{\theta_1 = 0\}$. *Condition (a) is verified since* $xQ(x, y) - yP(x, y) = -cy^4 + x^6 > 0$, *except at the origin. Also condition (b) is trivially verified since* $P_3(\theta_i) = Q_3(\theta_i) = 0$ *for* $i = 1, 2$, *and finally as* $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \alpha < 0$, $\alpha_1^2 + \beta_1^2 = \alpha_2^2 + \beta_2^2 = \alpha \neq 0$, *hence the system is of class* $\mathcal{G}$. *Since* $(2 + M - m)\alpha_1 - 2\beta_1 = (2 + M - m)\alpha_2 - 2\beta_2 = 2\alpha < 0$, *by theorem A, system (40) is a focus–centre.*

From proposition C, the Poincaré return map can be written as $\Pi(x_0) = x_0 + \mathrm{o}(x_0)$. *On the other hand, an easy computation gives*

$$\mathrm{GPV} \int_0^{2\pi} \frac{\alpha \cos^3\theta \sin\theta + (c + \alpha)\cos\theta \sin^3\theta + b \sin^2\theta}{-c \sin^4\theta} \, \mathrm{d}\theta$$
$$= \lim_{\varepsilon \to 0^+} \frac{-4b}{c} \frac{\cos\varepsilon}{\sin\varepsilon} = \mathrm{sign}(b)\,\infty.$$

Hence we have shown that in the hypothesis of proposition C, the expression of V_1 *given in theorem B is not valid for studying the stability of the origin.*

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Appendix

It is well known, from the Bendixon–Dumortier theory (see [22] for instance), that by means of finitely many blow-ups a real analytic vector field given in a neighbourhood of a real isolated singular point on the plane can be carried into an analytic field of directions given in a neighbourhood of a union of glued-in projective lines and having only finitely many singular points, each of them elementary and different from a focus or a centre (a brief geometric description of the method can be found in [8]). This result enables us to turn the singular monodromic point into a polycycle having at the vertices singular points each of them elementary (hyperbolic or degenerate elementary), and having the same associated return map. A polycycle having all its singular points elementary (that is, its linearizations have at least one non-zero eigenvalue) is called elementary.

We briefly recall some concepts and results (see again [22] for more details).

Definition 11. *A Dulac series is a formal series of the form*

$$D(x) = cx^{\mu_0} + \sum_{j=1}^{\infty} P_j(\ln(x))x^{\mu_j}$$

where $c > 0$, $0 < \mu_0 < \cdots < \mu_j < \cdots$, $\mu_j \to \infty$, *and the* P_j *are polynomials.*

Definition 12. *A germ of a mapping* $f : (\mathbb{R}^+, 0) \longrightarrow (\mathbb{R}^+, 0)$ *is said to be semiregular if it can be expanded into an asymptotic Dulac series, that is for any* N *there exists a partial sum* S *of the above series such that* $f(x) - S(x) = \mathrm{o}(x^N)$. *The coefficient* c *of the above series is called the principal term.*

Semiregular mappings are relevant because the transition map of a hyperbolic sector associated with a hyperbolic saddle is not in general differentiable (regular), but it is semiregular (see lemma 2 in section 0.2 of [22]). Moreover, it can be proved that the germs of semiregular mappings form a group. In particular, the composition of semiregular mappings is also semiregular. As usual we denote

$$f_0 = \begin{cases} \exp\{-1/x\} & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Definition 13.

(i) *A germ of a mapping* $f : (\mathbb{R}^+, 0) \longrightarrow (\mathbb{R}^+, 0)$ *is said to be flat semiregular if* $f_0^{-1} \circ f$ *is semiregular.*

(ii) *A germ of a mapping* $f : (\mathbb{R}^+, 0) \longrightarrow (\mathbb{R}^+, 0)$ *is said to be vertical semiregular if its inverse is the germ of a flat semiregular germ.*

The transition map of a hyperbolic sector associated with a degenerate elementary singular point is either a flat semiregular map or a vertical semiregular map.

The following are well known results (see again [22]).

Lemma 14. *Let* m, m', h *be semiregular maps such that* $h(x) = cx^{\mu} + \mathrm{o}(x^{\mu}), m(x) = ax^{\lambda} + \mathrm{o}(x^{\lambda})$*, and* $m'(x) = bx^{1/\lambda} + \mathrm{o}(x^{1/\lambda})$*. Set* $\sigma = f_0 \circ m$*, and* $\sigma' = m' \circ f_0^{-1}$*. Then*

(i) $h \circ m(x) = ca^{\mu}x^{\lambda\mu} + \mathrm{o}(x^{\lambda\mu})$*;*

(ii) $\sigma' \circ h \circ \sigma(x) = (ba^{1/\lambda}/\mu^{1/\lambda})x + \mathrm{o}(x)$.

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