Periodic points of a Landen transformation

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January 11, 2018

Abstract

We prove the existence of 3-periodic orbits in a dynamical system associated to a Landen transformation previously studied by Boros, Chamberland and Moll, disproving a conjecture on the dynamics of this planar map introduced by the latter author. To this end we present a systematic methodology to determine and locate analytically isolated periodic points of algebraic maps. This approach can be useful to study other discrete dynamical systems with algebraic nature. Complementary results on the dynamics of the map associated with the Landen transformation are also presented.

Mathematics Subject Classification 2010: 37C25, 33E05.

Keywords: Landen transformation, Periodic points, Poincaré-Miranda theorem.

1 Introduction

Given a definite integral depending on several parameters, a *Landen transformation* is a map on these parameters that leaves invariant the integral. In [1, 2], G. Boros and V. Moll introduced the dynamical system given by

$$\left\{\begin{array}{l} a_{n+1}=\dfrac{5a_n+5b_n+a_nb_n+9}{(a_n+b_n+2)^{4/3}}, \quad b_{n+1}=\dfrac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}}, \\[4ex] c_{n+1}=\dfrac{d_n+e_n+c_n}{(a_n+b_n+2)^{2/3}}, \quad d_{n+1}=\dfrac{(b_n+3)c_n+(a_n+3)e_n+2d_n}{a_n+b_n+2}, \quad e_{n+1}=\dfrac{c_n+e_n}{(a_n+b_n+2)^{1/3}}, \end{array}\right.$$

as a Landen transformation associated to the integral

$$I(a,b,c,d,e)=\int_0^{\infty}\frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\mathrm{d}x, \tag{1}$$