Subseries and signed series

Armengol Gasull and Francesc Mañosas

Departament de Matemàtiques
Universitat Autònoma de Barcelona, Bellaterra, Barcelona, Spain
gasull@mat.uab.cat, manyosas@mat.uab.cat

Abstract. For any positive decreasing to zero sequence a_n such that $\sum a_n$ diverges we consider the related series $\sum k_n a_n$ and $\sum j_n a_n$. Here, k_n and j_n are real sequences such that $k_n \in \{0,1\}$ and $j_n \in \{-1,1\}$. We study their convergence and characterize it in terms of the density of 1's in the sequences k_n and j_n. We extend our results to series $\sum m_n a_n$, with $m_n \in \{-1,0,1\}$ and apply them to study some associated random series.

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1 Introduction and main results

Given a divergent series $\sum a_n$, with $a_n > 0$, decreasing and with limit zero, we study properties of its subsums, $\sum k_n a_n$, where $k_n \in \{0,1\}$ and of its signed sums, $\sum j_n a_n$, where $j_n \in \{-1,1\}$. As we will see, both questions are related, and moreover can be treated simultaneously studying the series $\sum m_n a_n$, with $m_n \in \{-1,0,1\}$.

For a sequence of real numbers c_n we will say that the sequence f_n, given by the quotient between the number of A's in the list $c_1, c_2, \ldots, c_n$ and n, is the *sequence of densities of* A's associated to c_n. If $\lim f_n = f \in [0,1]$ exists we will say that f is the density of A's of the sequence c_n.

We characterize the convergence of the series in terms of properties of the sequences of densities of 1's in k_n and j_n. As usual, when $\lim a_n/b_n = 1$ we will write $a_n \sim b_n$. We split our main results in Theorems A and B, the first one concerning with subsums and the second one with signed sums. As we will see in Theorem C, some points can be treated together.

A key tool in many of our proofs will be a restricted version of the celebrated Toeplitz Theorem about the summability of weighted sequences, see for instance [6]. For completeness, in Section 2 we present a simple proof in the restrictive case of non-negative weights.

Theorem A. *Let a_n be a positive monotonous sequence such that* $\lim a_n = 0$, $\sum a_n = \infty$ *and set* $U_n = \sum_{i=1}^n a_i$. *Let k_n be a sequence with* $k_n \in \{0,1\}$, $f_n = \frac{\sum_{i=1}^n k_i}{n}$ *be the associate sequence of densities of 1's and* $S_n = \sum_{i=1}^n k_i a_i$ *be its associated sequence of partial sums. Then the following assertions hold:*

(a) If S_n converges then $\lim n a_n f_n = 0$. *In particular* $\lim f_n = 0$ *when* $\liminf n a_n > 0$. *Moreover if* $\lim n a_n f_n = 0$ *then S_n converges if and only if* $\sum_{i=1}^n i(a_i - a_{i+1}) f_i$ *converges.*

(b) $\liminf f_n \leq \liminf \frac{S_n}{U_n} \leq \limsup \frac{S_n}{U_n} \leq \limsup f_n$. *Hence, if* S_n *converges then* $\liminf f_n = 0$, *but there are cases where* $\lim f_n$ *does not exist. Additionally, if* $\liminf f_n > r > 0$ *then* $S_n > rU_n$ *for* n *large enough. Moreover,* $\lim \frac{S_n}{U_n} = r$ *when* $\lim f_n = r$, *and if* $r \neq 0$, *then* $S_n \sim rU_n$.

(c) For any $l \geq 0$ *there exists a sequence* $k_n \in \{0, 1\}$ *such that* $\lim \sum_{i=1}^{n} k_i a_i = l$.

(d) Let r_n *be a monotonous sequence with* $\lim r_n = \infty$ *and* $\lim \frac{r_n}{U_n} = 0$. *Assume also that* $\lim \frac{r_n - r_{n-1}}{a_n} = 0$. *Then there exists a sequence* $k_n \in \{0, 1\}$ *such that the associate sequence of partial sums* S_n *satisfies* $S_n \sim r_n$ *and* $\liminf f_n = 0$.

Observe that items (a) and (b) give two different necessary conditions for the convergence of a subseries in terms of the corresponding density. The first one is that $\lim n a_n f_n = 0$. The second one is that $\liminf f_n = 0$. Notice that, on the one hand, when $\liminf n a_n > 0$, the first one implies $\lim f_n = 0$. This is precisely the situation for the harmonic series, see for instance [10, 13, 14, 17, 18]. However, when $\lim n a_n = 0$ this condition is automatically satisfied because $0 \leq f_n \leq 1$, and imposes no restrictions on f_n. On the other hand, when $\lim n a_n = \infty$ this first condition is stronger. It implies that $\lim f_n$ goes to zero faster than $\frac{1}{n a_n}$.

Moreover, in the harmonic case, it is also known that if $\sum \frac{k_n}{n}$ is convergent and $\lim f_n \ln n = C \in \mathbb{R}$, then $C = 0$, see [16]. This fact is also a consequence of the second part of item (a) because in this situation, and when $C \neq 0$, $n(a_n - a_{n+1}) f_n \sim \frac{C}{(n+1)\ln n}$ and the series $\sum \frac{C}{(n+1)\ln n}$ is divergent. In particular, this result can be applied to show the divergence of the sum of the inverses of the prime numbers by using the results of Hadamard and de la Vallée-Poussin. Recall that they proved independently in 1896 that the density of the prime numbers smaller than n is asymptotic to $\frac{1}{\ln n}$. In fact, using the same approach we will prove the following corollary.

Corollary 1.1. *The series* $\sum_{p>2,\ prime} \frac{1}{p(\ln p)^{\alpha}(\ln \ln p)^{\beta}}$, $\alpha \geq 0, \beta \geq 0$, *is convergent if and only if either* $\alpha > 0$ *or* $\alpha = 0$ *and* $\beta > 1$.

The above result when $\beta = 0$ is already proved in [20] by using a different approach.

If instead of taking the prime numbers we consider the sum of the inverses of the twin primes it is known that the corresponding f_n satisfies $|f_n| < \frac{C}{\ln^2 n}$, for some positive constant C, see [1, p. 313]. Hence, by using again the second part of item (a) we recover the nice result of Brun [4], who proved in 1919 the convergence of that series, because in this case, $|n(a_n - a_{n+1}) f_n| \leq \frac{C}{(n+1)\ln^2 n}$ and the series $\sum \frac{C}{(n+1)\ln^2 n}$ is convergent.

The result of item (c) is already known, see [2, 19, 21].

Notice also that by Stolz's criterion, if $\lim \frac{r_n - r_{n-1}}{a_n} = 0$, then $\lim \frac{r_n}{U_n} = 0$. Therefore, item (d) implies that, essentially, subsums of the original series with $\liminf f_n = 0$ can diverge with any speed smaller than the speed of divergence of the complete series. A celebrated concrete example for $a_n = \frac{1}{n}$ and $r_n = \ln \ln n$, is the result of 1874 of Mertens who proved that $\sum_{p \text{ prime}, p \leq n} \frac{1}{p} \sim \ln \ln n$, see [9, 11, 15].

The most famous convergent, but not absolutely convergent series, was given by Mercator in the XVII century and is $\sum \frac{(-1)^{n+1}}{n} = \ln 2$. Taking $j_n = (-1)^{n+1}$, it holds that the density of 1's in the sequence j_n is $\frac{1}{2}$. Next theorem shows that in many convergent cases this will be the situation.

Theorem B. *Let a_n be a positive monotonous sequence such that $\lim a_n = 0$, $\sum a_n = \infty$ and set $U_n = \sum_{i=1}^n a_i$. Let j_n be a sequence with $j_n \in \{-1, 1\}$, $f_n = \frac{1}{n}\sum_{i=1}^n \frac{(1+j_i)}{2}$ be its associate sequence of densities of 1's and $T_n = \sum_{i=1}^n j_i a_i$ be the associated sequence of partial sums. Then the following assertions hold:*

(i) If T_n converges then $\lim na_n(2f_n - 1) = 0$. In particular $\lim f_n = \frac{1}{2}$ when $\liminf na_n > 0$. Moreover, if $\lim na_n(2f_n - 1) = 0$, T_n converges if and only if $\sum_{i=1}^n i(a_i - a_{i+1})(2f_i - 1)$ converges.

(ii) $\liminf(2f_n - 1) \leq \liminf \frac{T_n}{U_n} \leq \limsup \frac{T_n}{U_n} \leq \limsup(2f_n - 1)$. Hence, if T_n converges then $\frac{1}{2} \in [\liminf f_n, \limsup f_n]$, but there are cases where $\lim f_n$ does not exist. Additionally, if $\liminf f_n > r > \frac{1}{2}$ (resp. $\limsup f_n < s < \frac{1}{2}$) then $T_n > (2r-1)U_n$ (resp. $T_n < (2s-1)U_n$) for n large enough. Moreover, if $\lim f_n = r$ then $\lim \frac{T_n}{U_n} = 2r - 1$ and, in particular, when $r \neq \frac{1}{2}$, $T_n \sim (2r-1)U_n$.

(iii) For any $l \in \mathbb{R}$ there exists a sequence $j_n \in \{-1, 1\}$ such that $\lim \sum_{i=1}^n j_i a_i = l$.

(iv) Let r_n be a monotonous sequence with $\lim r_n = \infty$ and $\lim \frac{r_n}{U_n} = 0$. Assume also that $\lim \frac{r_n - r_{n-1}}{a_{n+1}} = 0$. Then there exists a sequence $j_n \in \{-1, 1\}$ such that the associate sequence of partial sums T_n satisfies $T_n \sim r_n$ and $\frac{1}{2} \in [\liminf f_n, \limsup f_n]$

Notice that $0 < \frac{r_n - r_{n-1}}{a_n} < \frac{r_n - r_{n-1}}{a_{n+1}}$. Hence, as in Theorem A, by Stolz criterion the hypothesis $\lim \frac{r_n - r_{n-1}}{a_{n+1}} = 0$ implies that $\lim \frac{r_n}{U_n} = 0$.

As we will see, the proofs of the first two items of the above two theorems will be a consequence of the following more general result about series $\sum m_n a_n$, with $m_n \in \{-1, 0, 1\}$.

Theorem C. *Let a_n be a positive monotonous sequence sequence such that $\lim a_n = 0$, $\sum a_n = \infty$ and set $U_n = \sum_{i=1}^n a_i$. Let m_n be a sequence with $m_n \in \{-1, 0, 1\}$, such that f_n and g_n are the associated sequences of densities of 1's and -1's, respectively, and let $P_n = \sum_{i=1}^n m_i a_i$ be its associated sequence of partial sums. Then the following assertions hold:*

(a) If P_n converges then $\lim na_n(f_n - g_n) = 0$. In particular $\lim(f_n - g_n) = 0$ when $\liminf na_n > 0$. Moreover if $\lim na_n(f_n - g_n) = 0$ then P_n converges if and only if $\sum_{i=1}^n i(a_i - a_{i+1})(f_i - g_i)$ converges.

(b) $\liminf(f_n - g_n) \leq \liminf \frac{P_n}{U_n} \leq \limsup \frac{P_n}{U_n} \leq \limsup(f_n - g_n)$. Hence, if P_n converges then $\liminf(f_n - g_n) = 0$, but there are cases where $\lim(f_n - g_n)$ does not exist. Additionally, if $\liminf(f_n - g_n) > r > 0$ (resp. $\limsup(f_n - g_n) < s < 0$) then $P_n > rU_n$ (resp. $P_n < sU_n$) for n large enough. Moreover, $\lim \frac{P_n}{U_n} = r$ when $\lim(f_n - g_n) = r$, and in particular, if $r \neq 0$, then $P_n \sim rU_n$.

In the above theorem, two items more, similar to the last two items of Theorems A and B could be added. We have decided to omit them because the statements of these theorems are stronger. For instance they allow to obtain any finite sum without using -1's (Theorem A).

Theorems A, B and C can be extended, without major difficulties, to positive divergent series with non increasing terms going to zero, but they are no more true for general positive sequences

a_n going to zero. Under these assumptions it is possible to have a convergent subseries with density f arbitrarily close to one. For instance we can split the integers in the set of squares $\mathbb{S} := [1^2, 2^2, 3^2, 4^2, \ldots]$ and the complementary one, $\mathbb{N} \setminus \mathbb{S} = [2, 3, 5, 6, 7, 8, 10, \ldots]$. Then, for any fixed $m \in \mathbb{N}$, construct a_n taking the inverses of the first m elements of $\mathbb{S}$ and the inverse of the first element in $\mathbb{N} \setminus \mathbb{S}$, and continue similarly with the remaining elements. In this way, $\sum a_n$ is divergent and the subseries of the inverses of the squares is convergent. Hence, writing it as $\sum k_n a_n$, with $k_n = 1$ when $1/a_n$ is a square, and $k_n = 0$ otherwise, we get that for this series is convergent and the density of 1's is $f = \frac{m}{m+1}$. Similarly, a convergent series with k_n with density $f = 1$ can be constructed.

Consider a discrete random variable W, with the following distribution: $P\{W = 1\} = p$, $P\{W = -1\} = q$ and $P\{W = 0\} = 1 - p - q$, with $p + q > 0$. For short we will denote this distribution by $\mathcal{W}(p, q)$.

The convergence of the random series $\sum a_n W_n$, where W_n are independent identically distributed (i.i.d.) random variables, with distribution $\mathcal{W}(p, q)$, can be studied with the celebrated Kolmogorov's Three-Series Theorem, see Theorem 2.8 or [3, 12]. In Section 4 we will prove next theorem by using it. That section also includes a discussion of how, in some particular cases, this result can be reobtained as a consequence of Theorem C and of the Law of the Iterated Logarithm.

Theorem D. *Consider the random series $Z = \sum a_n W_n$, where W_n is a sequence of i.i.d. random variables, with distribution $\mathcal{W}(p, q)$. Assume that a_n is a positive sequence, that tends to 0 and such that $\sum a_n = \infty$. Then Z converges a.s. if and only if $p = q$ and $\sum a_n^2$ is convergent. Otherwise, Z is a.s. divergent.*

2 Preliminary results

In the first part of this section we collect several results about series. In the second one we recall some results about random variables and random series.

One of the main tools that we will use is the so called Toeplitz Theorem ([6, Sec. 3.1-3.6], that we state and prove here with more restrictive assumptions appropriate for our interests.

Theorem 2.1. *Let $c_{n,m}, (n, m) \in \mathbb{N} \times \mathbb{N}$ be a double sequence satisfying the following properties:*

(a) $c_{n,m} \geq 0$ *for all* $(n, m) \in \mathbb{N} \times \mathbb{N}$ *and* $c_{n,m} = 0$ *when* $m > n$.

(b) $\lim \sum_{i=1}^{n} c_{n,i} = 1$.

(c) For any fixed $m \in \mathbb{N}$, $\lim c_{n,m} = 0$.

Then for any sequence x_n we get that the sequence $y_n = \sum_{i=1}^{n} c_{n,i} x_i$ satisfies

$$\liminf x_n \leq \liminf y_n \leq \limsup y_n \leq \limsup x_n.$$

In particular if $\lim x_n = l \in \mathbb{R} \cup \{\pm\infty\}$ *then* $\lim y_n = l$.

Proof. Let us denote $a := \liminf x_n$ and $b := \limsup x_n$ and let z be an accumulation point of y_n. We will assume that a and b are finite. The other cases follow by obvious adaptations of the proof. To prove the theorem we need to show that $z \in [a,b]$. To do this we will show that for any $\epsilon > 0$, $z \in [a-\epsilon, b+\epsilon]$.

Note first that if $x_n \in [A,B]$ for all $n \in \mathbb{N}$, then since $c_{n,i} \geq 0$ we get

$$y_n \in \big[\big(\sum_{i=1}^{n} c_{n,i}\big)A, \big(\sum_{i=1}^{n} c_{n,i}\big)B\big]$$

for all $n \in \mathbb{N}$. Therefore, since $\lim \sum_{i=1}^{n} c_{n,i} = 1$, any accumulation point of y_n belongs to $[A,B]$.

The second observation is that if we change a finite number of terms of the sequence x_n, obtaining a new sequence x'_n, then the set of accumulation points of the corresponding sequence y'_n does not change. This is due to the fact that if, given a fixed r, we change x_r by x'_r then the corresponding sequence y'_n satisfies $y'_n = y_n + c_{n,r}(x'_r - x_r)$ and since $\lim c_{n,r} = 0$, the sequences y_n and y'_n share the same set of accumulation points.

Let z be an accumulation point of y_n and let $\epsilon > 0$. Then the sequence x_n has only a finite number of terms outside $[a-\epsilon, b+\epsilon]$. We construct another sequence x'_n by changing the terms outside $[a-\epsilon, b+\epsilon]$ by a for example. Then from the second observation z is also an accumulation point of the sequence y'_n and by the first observation it follows that $z \in [a-\epsilon, b+\epsilon]$. This ends the proof of the theorem. $\square$

A consequence of the above result is the next known extension of the celebrated Stolz's criterion that we state and prove in the next lemma.

Lemma 2.2. *(Stolz's criterion) Let $\frac{a_n}{b_n}$ be a sequence of real numbers with b_n a monotonous sequence tending to ∞. Then*

$$\liminf \frac{a_n - a_{n-1}}{b_n - b_{n-1}} \leq \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} \leq \limsup \frac{a_n - a_{n-1}}{b_n - b_{n-1}}$$

In particular, if the sequence $\frac{a_n - a_{n-1}}{b_n - b_{n-1}}$ converges to $l \in \mathbb{R} \cup \{\pm\infty\}$ the same holds for $\frac{a_n}{b_n}$.

Proof. It follows applying Theorem 2.1 to

$$x_n = \begin{cases} \frac{a_1}{b_1}, & \text{if } n = 1, \\ \frac{a_n - a_{n-1}}{b_n - b_{n-1}} & \text{otherwise,} \end{cases} \quad \text{with} \quad c_{n,i} = \begin{cases} \frac{b_1}{b_n}, & \text{if } i = 1, \\ \frac{b_i - b_{i-1}}{b_n}, & \text{if } 1 < i \leq n, \\ 0, & \text{if } i > n, \end{cases}$$

because $y_n = a_n/b_n$. $\square$

Lemma 2.3. *Let a_n be a non-vanishing sequence, let m_n be a sequence with $m_n \in \{-1,0,1\}$ and f_n (resp. g_n) its associate sequence of densities of 1's (resp. -1's) and let $P_n = \sum_{i=1}^{n} m_i a_i$ be the corresponding sequence of partial sums. Then*

$$(1)\quad n a_n (f_n - g_n) = P_n - a_n \sum_{i=1}^{n-1} \left(\frac{1}{a_{i+1}} - \frac{1}{a_i}\right) P_i \quad \textit{and} \quad P_n = n a_n (f_n - g_n) + \sum_{i=1}^{n-1} i(a_i - a_{i+1})(f_i - g_i).$$

Proof. We have $m_1 = \frac{P_1}{a_1}$ and for $i > 1$, $m_i = \frac{P_i - P_{i-1}}{a_i}$, $f_n = \frac{1}{2n}\sum_{i=1}^n m_i(m_i+1)$ and $g_n = \frac{1}{2n}\sum_{i=1}^n m_i(m_i-1)$. Hence, $f_n - g_n = \frac{1}{n}\sum_{i=1}^n m_i$ and, as a consequence,

$$f_n - g_n = \frac{1}{n}\left(\sum_{i=2}^n \frac{P_i - P_{i-1}}{a_i} + \frac{P_1}{a_1}\right) = \frac{P_n}{na_n} - \frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{1}{a_{i+1}} - \frac{1}{a_i}\right)P_i$$

and the first assertion follows. Similarly, $n(f_n - g_n) = \sum_{i=1}^n m_i$ and holds that $m_1 = f_1 - g_1$ and for $i > 1$, $m_i = i(f_i - g_i) - (i-1)(f_{i-1} - g_{i-1})$. Hence

$$P_n = \sum_{i=2}^n \big(i(f_i - g_i) - (i-1)(f_{i-1} - g_{i-1})\big)a_i + (f_1 - g_1)a_1 = na_n(f_n - g_n) + \sum_{i=1}^{n-1} i(a_i - a_{i+1})(f_i - g_i).$$

□

The following result is well-known. We include its proof for completeness.

Lemma 2.4. *Let $f : [1,\infty) :\longrightarrow (0,\infty]$ be a continuous decreasing function. Then the sequence $A_n = A_n(f) = \sum_{i=1}^n f(i) - \int_1^n f(s)ds$ is convergent.*

Proof. Clearly $\sum_{i=1}^n f(i) - \int_1^n f(s)ds \geq 0$. Moreover $A_{n+1} - A_n = f(n+1) - \int_n^{n+1} f(s)ds \leq 0$. Thus A_n is a positive decreasing sequence. Therefore it is convergent. □

Recall that Euler, in 1731, proved that $\lim\big(\sum_{i=1}^n \frac{1}{i} - \ln n\big) = \gamma \in [0,1)$, where $\gamma \approx 0.577218$ is precisely the nowadays called Euler's constant, see [7, 9]. Notice that if $f_0(x) = 1/x$ then $A_n(f_0)$ precisely converges to the gamma constant, γ.

Next lemma facilitates the application of the second part of item (a) of Theorem A.

Lemma 2.5. *Let a_n be a sequence of positive numbers given by a smooth function $h : \mathbb{R}^+ \to \mathbb{R}^+$, that is $a_n = h(n)$, such that $\lim h'(n+1)/h'(n) = 1$. If for x big enough, $h'(x) < 0$ and $h''(x) > 0$, then $a_{n+1} - a_n \sim h'(n)$.*

Proof. By the Intermediate Value Theorem, $a_{n+1} - a_n = h(n+1) - h(n) = h'(z_n)$, where $z_n \in (n, n+1)$. Since h' is increasing $h'(n) < h(n+1) - h(n) < h'(n+1)$. Hence,

$$\frac{h'(n+1)}{h'(n)} < \frac{h(n+1) - h(n)}{h'(n)} < 1.$$

Taking limit in both sides, by the hypotheses, the lemma follows. □

2.1 Random series

Some of the results of this section are extracted from [3, 8]. We start stating the Law of the Iterated Logarithm, that quantifies the speed of convergence towards the expected value given by the Strong Law of Large Numbers.

Theorem 2.6. *Let X_n be a sequence of i.i.d. random variables with $E(X_1^2) < \infty$. Then,*

$$(2) \qquad \liminf \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{2n\ln\ln n}} = -1 \quad (a.s.) \quad \text{and} \quad \limsup \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{2n\ln\ln n}} = +1 \quad (a.s.),$$

where $E(X_1) = \mu$ and $\mathrm{Var}(X_1) = \sigma^2$.

Corollary 2.7. *Let W_n be a sequence of i.i.d. with distribution $\mathcal{W}(p,q)$, and let F_n (resp. G_n) be the random variables that give the density of 1's (resp. -1's) in the sequence. Then*

$$\liminf \frac{\sqrt{n}\,\big(F_n - G_n - (p-q)\big)}{\sqrt{\ln\ln n}} = -L \ (a.s.) \quad \text{and} \quad \limsup \frac{\sqrt{n}\,\big(F_n - G_n - (p-q)\big)}{\sqrt{\ln\ln n}} = L \ (a.s.),$$

where $L = \sqrt{2(p+q-(p-q)^2)}$. Moreover, for almost all ω, there exists $K(\omega) > L$ such that

$$\big|F_n(\omega) - G_n(\omega) - (p-q)\big| \le K(\omega)\sqrt{\frac{\ln\ln n}{n}}.$$

Proof. Recall that the discrete random variable $\mathcal{W}(p,q)$, takes the values $1, -1$ and 0 with respective probabilities p, q and $1-p-q$ and that $p+q>0$. Notice that $\sum_{i=1}^{n} W_i = n(F_n - G_n)$, $E(W_i) = \mu = p - q$ and $\mathrm{Var}(W_i) = \sigma^2 = p + q - (p-q)^2$. Hence

$$\sum_{i=1}^{n} W_i - n\mu = n\big(F_n - G_n - (p-q)\big).$$

Since $E(W_i^2) = p + q > 0$ we can apply Theorem 2.6. Replacing the above equality in (2) we get the results. $\square$

The following result, known as Kolmogorov's Three-Series Theorem, is the general tool for studying sums of independent random variables. Notice that it allows to know the behavior of a random series studying three deterministic series. As usual, 1_B denotes the indicator function of the set B, that is $1_B(x) = 1$ if $x \in B$ and $1_B(x) = 0$ otherwise.

Theorem 2.8. *Let X_n be a sequence of independent random variables. The series $\sum X_n$ is a.s. convergent if and only if, for some $A > 0$, next three conditions are satisfied:*

(a) $\sum P\big(|X_n| \ge A\big)$ is convergent,

(b) If $Y_n = X_n \cdot 1_{\{|X_n| \le A\}}$, $\sum E(Y_n)$, the series of the expected values of Y_n, is convergent,

(c) $\sum \mathrm{Var}(Y_n)$ is convergent.

Finally we state a corollary of Kolmogorov's $0-1$ Law.

Corollary 2.9. *Let X_n be a sequence of independent random variables. Then, the series $\sum X_n$ is either a.s. convergent or a.s. divergent.*

3 Proofs of the main results

We start this section proving Theorem C. Afterwards, using it we will prove the first two items of Theorems A and B.

Proof of Theorem C. (a) Left-hand equality (1) in Lemma 2.3 is

$$na_n(f_n - g_n) = P_n - a_n \sum_{i=1}^{n-1} \left(\frac{1}{a_{i+1}} - \frac{1}{a_i}\right) P_i.$$

Putting

$$c_{n,i} = \begin{cases} \frac{a_n}{a_{i+1}} - \frac{a_n}{a_i}, & \text{if } i < n, \\ 0, & \text{otherwise,} \end{cases}$$

we get that $\lim \sum_{i=1}^{n} c_{n,i} = \lim \left(1 - \frac{a_n}{a_1}\right) = 1$ and for any fixed i, $\lim c_{n,i} = 0$. Thus if $\lim P_n = l$, applying Theorem 2.1 to the sequence P_n and the coefficients $c_{n,i}$ we get $\lim a_n \sum_{i=1}^{n-1} \left(\frac{1}{a_{i+1}} - \frac{1}{a_i}\right) P_i = l$ and hence $\lim n a_n (f_n - g_n) = 0$. Clearly this last equality implies $\lim(f_n - g_n) = 0$ when $\liminf n a_n > 0$. The other statement follows directly from the right-hand equality (1) of Lemma 2.3. Moreover, dividing both sides of this equality by U_n we get

$$\frac{P_n}{U_n} = \frac{n a_n}{U_n}(f_n - g_n) + \sum_{i=1}^{n-1} \frac{i(a_i - a_{i+1})}{U_n}(f_i - g_i).$$

To prove item (b), take

$$c_{n,i} = \begin{cases} \frac{i(a_i - a_{i+1})}{U_n}, & \text{if } i < n, \\ \frac{n a_n}{U_n}, & \text{if } i = n, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $\sum_{i=1}^{n} c_{n,i} = 1$ and for all $i \in \mathbb{N}$, $\lim c_{n,i} = 0$. Thus the claimed chain of inequalities follows from Theorem 2.1. From these inequalities all the other results but one follow. Only remains to show that in general the condition about the convergence to zero of $f_n - g_n$ is not a necessary condition for the convergence of a subseries. More specifically, we want to prove the existence of sequences a_n and m_n under the hypotheses of the theorem and such that $\sum m_n a_n$ converges (then $\liminf(f_n - g_n) = 0$) but $\lim(f_n - g_n)$ does not exist. Note that since $\lim n a_n (f_n - g_n)$ must be zero it follows that we need to choose a_n such that $\liminf n a_n = 0$. To do this we will specify both sequences, a_n and m_n, based on a nice example constructed by Šalat [18], with a different objective. In fact, to simplify the problem we will construct a series $\sum k_n a_n$, with $k_n \in \{0, 1\}$. In this way $g_n \equiv 0$ and $f_n - g_n \equiv f_n$.

We will prove that if $b_n = \frac{1}{m^{m+2}}$, where m is the unique natural number such that $n \in [m^m, (m+1)^{m+1})$, $a_n = b_n + \frac{1}{n^2}$ and we set

$$k_n = \begin{cases} 1, & \text{if } n \in [m^m, 2m^m) \text{ for some } m \in \mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \tag{3}$$

then the following assertions hold:

(I) a_n is a monotonous positive sequence, $\lim n a_n = 0$ and the series $\sum a_n$ is divergent.

(II) $\limsup f_n \geq \frac{1}{2}$ and $\sum k_n a_n$ is convergent.

To prove (I), note first that the sequence a_n is clearly positive and monotone because the sequence b_n is non-increasing and $\frac{1}{n^2}$ is monotonous decreasing. To see that $\lim n a_n = 0$ it suffices to show that $\lim n b_n = 0$. To prove this, observe that for $n \in [m^m, (m+1)^{m+1})$ we have

$$n b_n = \frac{n}{m^{m+2}} < \frac{(m+1)^{m+1}}{m^{m+2}} = \frac{1}{m}\left(1 + \frac{1}{m}\right)^{m+1}$$

that clearly tends to zero. Lastly note that

$$\sum_{i=m^m}^{(m+1)^{m+1}-1} b_i = \big((m+1)^{m+1} - m^m\big)\frac{1}{m^{m+2}} > \frac{m^{m+1}}{m^{m+2}} = \frac{1}{m}.$$

Thus $\sum b_n$ is divergent and hence the same occurs for $\sum a_n$.

To prove (II), note that $f_{2m^m} \geq \frac{1}{2}$, because from (3) at least half of the k_n until $n = 2m^m$ are 1's. Then $\limsup f_n \geq \frac{1}{2}$. Finally, to see that $\sum k_n a_n$ converges it suffices to show that $\sum k_n b_n$ is convergent. We have

$$\sum_{i=m^m}^{(m+1)^{m+1}-1} k_i b_i = \sum_{i=m^m}^{2m^m-1} b_i = \frac{m^m}{m^{m+2}} = \frac{1}{m^2}.$$

Therefore $\sum k_n b_n$ is convergent and the same happens for $\sum k_n a_n$.

□

Proof of Theorem A. Items (a) and (b) are a direct consequence of Theorem C and its proof.

(c) We consider first the case $l > a_1$. The opposite situation follows similarly and we skip the details. Let n_0 be the first natural number such that $\sum_{i=1}^{n_0+1} a_i \geq l$. We choose $k_i = 1$ for $i = 1, \ldots, n_0$ and $k_{n_0+1} = 0$. Now let $n_1 > n_0$ be the first natural number greater than n_0 such that $S_{n_0} + a_{n_1} < l$. We choose $k_i = 0$ for all $i \in \{n_0+1, \ldots, n_1-1\}$. Note that $l - S_i = l - S_{n_0+1} < a_{n_0+1}$ for all i in this set. We also choose $k_{n_1} = 1$ and let n_2 be the first natural number greater than n_1 such that

$$S_{n_1} + a_{n_1+1} + \ldots + a_{n_2+1} \geq l.$$

Now we choose $k_i = 1$ for all $i \in \{n_1, \ldots, n_2\}$ and $k_{n_2+1} = 0$. In this way we obtain a sequence k_n such that $l - S_n$ is a positive non increasing sequence verifying that $l - S_{n_j} < a_{n_j+1}$ for some partial n_j, with j even. This shows that $\lim S_n = l$.

(d) Now we will construct a sequence $k_n \in \{0,1\}$ such that for n large enough

$$r_n < S_n = \sum_{i=1}^{n} k_i a_i \leq r_n + a_{\varphi(n)}$$

where $\varphi(n)$ is a non-decreasing sequence going to infinity.

Notice that since $\lim \frac{r_n - r_{n-1}}{a_n} = 0$, it is not restrictive to assume that $r_n - r_{n-1} < a_n$. Assume for instance that $a_1 < r_1$ and let n_1 be the first integer such that $\sum_{i=1}^{n_1} a_i > r_{n_1}$. Observe that such a n_1 exists because $\lim \frac{r_n}{U_n} = 0$. We choose $k_i = 1$ for $i \in \{1, \ldots, n_1\}$. Now let n_2 be the first natural number greater than n_1 such that $r_{n_2} > S_{n_1}$. Now we choose $k_i = 0$ for all $i \in \{n_1+1, \ldots, n_2-1\}$ and $k_{n_2} = 1$. Clearly we have $r_i \leq S_i < r_i + a_{n_1}$ for all $i \in \{n_1+1, \ldots, n_2-1\}$ and $S_{n_2} = S_{n_1} + a_{n_2} \geq r_{n_2-1} + a_{n_2} > r_{n_2}$ because $r_{n_2} - r_{n_2-1} < a_{n_2}$. Moreover $S_{n_2} - r_{n_2} < a_{n_2}$. Proceeding in this way we obtain a sequence n_i and a sequence k_n in such a way we have $r_n < S_n = \sum_{i=1}^{n} k_i a_i \leq r_n + a_{n_j}$ for all $n \in \{n_j, \ldots, n_{j+1}\}$. Thus we have

$$0 < \frac{S_n - r_n}{r_n} \leq a_{n_j},$$

for all $n \in \{n_j, \ldots, n_{j+1}\}$. Passing to the limit we get $\lim \frac{S_n}{r_n} = 1$. Observe now that $\liminf f_n = 0$. If not, $\liminf \frac{S_n}{U_n} = \liminf \frac{r_n}{U_n} > 0$ giving a contradiction. This ends the proof of the Theorem. □

Proof of Corollary 1.1. Recall that by the results of Hadamard and de la Vallée-Poussin the density of the prime numbers smaller than n is asymptotic to $\frac{1}{\ln n}$. Hence, by the second part of item (a) of Theorem A the convergence of the series involving the prime numbers depends on the convergence of the series with terms $n(a_n - a_{n+1})f_n \sim \frac{n}{\ln n}(a_n - a_{n+1})$, where $a_n = h(n)$, with $h(x) = \frac{1}{x(\ln x)^\alpha (\ln \ln x)^\beta}$. It is not difficult to see for any $\alpha \geq 0, \beta \geq 0$, we are under the hypotheses of Lemma 2.5. Hence $a_{n+1} - a_n \sim h'(n) \sim -\frac{1}{n^2(\ln n)^\alpha(\ln \ln n)^\beta}$ and, as a consequence,

$$n(a_n - a_{n+1})f_n \sim \frac{n}{\ln n}(a_n - a_{n+1}) \sim \frac{1}{n(\ln n)^{\alpha+1}(\ln \ln n)^\beta}.$$

By the Integral Convergence test, the series $\sum \frac{1}{n(\ln n)^\delta}$ is convergent if and only if $\delta > 1$. Hence, since $n(\ln n)^{\alpha+1}(\ln \ln n)^\beta \geq n(\ln n)^{\alpha+1}$, the result follows for $\alpha > 0$.

Consider now $\alpha = 0$. Again by the Integral Convergence test, the series $\sum \frac{1}{n \ln n (\ln \ln n)^\beta}$ is convergent if and only if $\beta > 1$, because for $\beta \neq 1$, $\left(\frac{1}{1-\beta}\frac{1}{(\ln \ln x)^{\beta-1}}\right)' = \frac{1}{x \ln x (\ln \ln x)^\beta}$ and $(\ln \ln \ln x)' = \frac{1}{x \ln x (\ln \ln x)}$. Hence the corollary follows. In fact, these series were used by Hardy [5, 14], with $\beta = 1, 2$, to illustrate the series that either diverge or converge very slowly. □

A direct consequence of the second part of statement (a) of Theorem A is next result that gives a kind of comparison test between the densities of 1's.

Corollary 3.1. *Let k_n, k'_n be sequences such that $k_n, k'_n \in \{0, 1\}$, let f_n, f'_n be their corresponding sequences of densities of 1's and let S_n, S'_n be their corresponding sequences of partial sums. Assume that* $\lim n a_n f_n = \lim n a_n f'_n = 0$. *Then the following statements hold.*

- *If $f_n \leq f'_n$ and S'_n converges, then S_n converges.*
- *If $f_n \geq f'_n$ and S'_n diverges, then S_n diverges.*
- *If* $\lim \frac{f_n}{f'_n} = C \neq 0$ *then both series have de same character.*

Proof of Theorem B. Similarly that in Theorem A, items (i) and (ii) are a direct consequence of Theorem C. Only one example of a convergent signed subseries such that the $\lim(f_n - g_n) = \lim(2f_n - 1)$ does not exist, must be given. Such an example can be constructed similarly to the one with $g_n \equiv 0$ given in the proof of Theorem C. We omit the details.

(iii) Assume without loss of generality that $l \geq 0$. If $l \geq a_1$ put $k_1 = 1$ and let i_1 be the first natural number such that $S_{i_1} = \sum_{i=1}^{i_1} a_i > l$. We choose $k_i = 1$ if $i \in \{1, \ldots, i_1\}$. Now let $i_2 > i_1$ be the first natural number such that $S_{i_2} = \sum_{i=1}^{i_1} a_i - \sum_{i=i_1+1}^{i_2} a_i < l$. Then we choose $k_i = -1$ if $i \in \{i_1 + 1, \ldots, , i_2\}$. Note that $|S_i - l| < a_{i_1}$ for any $i \in [i_1, i_2]$. In this way we can define an increasing sequence i_j and $k_n \in \{1, -1\}$ given by

$$k_n = \begin{cases} 1, & \text{if } n \in \{i_j + 1, \ldots, i_{j+1}\} \text{ for some even } j, \\ -1, & \text{if } n \in \{i_j + 1, \ldots, i_{j+1}\} \text{ for some odd } j, \end{cases}$$

where $i_0 := 0$. Note that we will get $|S_i - l| < a_{i_j}$ if $i > i_j$. This ends the proof when $l \geq a_1$. The other cases follow by obvious adaptations of this proof.

To prove item (iv) we consider the sequences $b_n = a_{2n}$ and $c_n = a_{2n-1}$. Set $\widetilde{r}_n = \frac{1}{2} r_{2n}$. We have

$$0 < \frac{\tilde{r}_n - \tilde{r}_{n-1}}{b_n} = \frac{1}{2}\frac{r_{2n} - r_{2n-2}}{a_{2n}} = \frac{1}{2}\left(\frac{r_{2n} - r_{2n-1}}{a_{2n}} + \frac{r_{2n-1} - r_{2n-2}}{a_{2n}}\right).$$

Thus passing to the limit we get by hypothesis that $\lim \frac{\tilde{r}_n - \tilde{r}_{n-1}}{b_n} = 0$, because $0 < \frac{r_{2n} - r_{2n-1}}{a_{2n}} < \frac{r_{2n} - r_{2n-1}}{a_{2n+1}}$. Hence the sequence $\widetilde{r}_n$ satisfies the hypotheses of item (d) of Theorem A with respect to the series $\sum_{i=1}^n b_i$.

Therefore we get that there exists a sequence $k_n \in \{0, 1\}$ such that $\lim \frac{\sum_{i=1}^n k_i a_{2i}}{\frac{1}{2} r_{2n}} = 1$. Define

$$j_i = \begin{cases} -1 + 2k_{i/2}, & \text{if } i \text{ is even,} \\ 1, & \text{otherwise.} \end{cases}$$

We have $T_{2n} = \sum_{i=1}^{2n} j_i a_i = -\sum_{i=1}^n b_i + 2\sum_{i=1}^n k_i b_i + \sum_{i=1}^n c_i$. Hence

$$\frac{T_{2n}}{r_{2n}} = \frac{\sum_{i=1}^n c_i - \sum_{i=1}^n b_i + 2\sum_{i=1}^n k_i b_i}{r_{2n}}.$$

Now we claim that

$$\lim\left(\sum_{i=1}^n c_i - \sum_{i=1}^n b_i\right) = M \in \mathbb{R}.$$

To see this let $h : [1, \infty) :\longrightarrow (0, \infty)$ be a continuous decreasing function satisfying $h(n) = a_n$ for all $n \in \mathbb{N}$ and let $H(x) = \int_1^x h(s)ds$. From Lemma 2.4 there exist $L_1, M_1 \in \mathbb{R}$ such that

$$\lim\left(\sum_{i=1}^n b_i - \int_1^n h(2s)ds\right) = L_1 \quad \text{and} \quad \lim\left(\sum_{i=1}^n c_i - \int_1^n h(2s-1)ds\right) = M_1.$$

Thus we get

$$\lim\left(\sum_{i=1}^n b_i - \sum_{i=1}^n c_i\right) = L_1 - M_1 - \frac{1}{2}\int_1^2 h(s)ds + \frac{1}{2}\lim\int_{2n-1}^{2n} h(s)ds = M$$

and the claim follows. Hence,

$$\lim \frac{T_{2n}}{r_{2n}} = \lim \frac{2\sum_{i=1}^n k_i b_i}{r_{2n}} = \lim \frac{\sum_{i=1}^n k_i a_{2i}}{\frac{1}{2} r_{2n}} = 1.$$

Similar computations show that also $\lim \frac{T_{2n+1}}{r_{2n+1}} = 1$, and so $\lim \frac{T_n}{r_n} = 1$.

Lastly we get $\frac{1}{2} \in [\liminf f_n, \limsup f_n]$. Otherwise, either $\liminf f_n > \frac{1}{2}$ or $\limsup f_n < \frac{1}{2}$. In the first case from item (ii) we obtain $\liminf \frac{T_n}{U_n} \geq \liminf(2f_n - 1) > 0$ and hence $\frac{T_n}{U_n} > r$ for some $r > 0$ and n large enough, in contradiction with the hypothesis. In the other case we arrive to contradiction arguing with $\limsup \frac{T_n}{U_n}$. □

4 Random series

In this section we prove Theorem D and compare it with a corollary of our theorems on deterministic series. We end with a test example.

Proof of Theorem D. We will apply the Kolmogorov's Three-Series Theorem to the sequence of independent random variables $X_n = a_n W_n$. Since a_n tends to zero, given any $A > 0$, there exits n_0 such that for all $n > n_0$, $P\big(|X_n| \geq A\big) = 0$. Hence the $\sum_{i=1}^n P\big(|X_i| \geq A\big)$ always converges and item (a) of the theorem is satisfied for any $A > 0$.

Similarly, for any n big enough, $Y_n = X_n \cdot 1_{\{|X_n| \leq A\}} = X_n$. Hence $E(Y_n) = a_n E(W_n) = (p-q)a_n$. Then $\sum_{i=n_0}^n E(Y_i) = (p-q)\sum_{i=n_0}^n a_i$, that converges if and only if $p = q$, because $\sum_{i=1}^n a_i$ diverges. Hence item (b) of the theorem holds if and only if $p = q$.

Finally, for $p = q$ and again for $n > n_0$,

$$\mathrm{Var}(Y_n) = \mathrm{Var}(X_n) = \mathrm{Var}\left(a_n W_n\right) = a_n^2 \,\mathrm{Var}(W_1).$$

Since $\mathrm{Var}(W_1) = 2p > 0$, $\sum_{i=1}^n \mathrm{Var}(Y_i)$ converges if and only if $\sum_{i=1}^n a_i^2$ converges, as we wanted to prove.

When either $p \neq q$ or $\sum_{i=1}^n a_i^2$ diverges, by Corollary 2.9, the random series diverges a.s. and the theorem follows. □

As we will see, the results obtained applying Theorem C together with a consequence of the Law of the Iterated Logarithm (Corollary 2.7) are weaker that the ones stated by using the Kolmogorov's Three-Series Theorem, given in Theorem D. Nevertheless, we briefly discuss them because we believe that it is nice to see how from the study of properties of related deterministic series it is also possible to get as a consequence results about a.s. convergence of random series.

Corollary 4.1. *Let a_n be a positive sequence that tends monotonically to 0 and such that $\sum a_n = \infty$. Consider the random series $Z = \sum a_n W_n$, where W_n are i.i.d. random variables with distribution $\mathcal{W}(p,q)$. The following holds:*

(i) If $p \neq q$ then Z is a.s. divergent.

(ii) If $p = q$ and $a_n \geq \frac{R}{\sqrt{n \ln\ln n}}$, for $n \geq n_0$ and some $R > 0$, then Z is a.s. divergent.

(iii) If $p = q$ and $\sum \sqrt{n \ln\ln n}\,\big(a_n - a_{n+1}\big)$ converges then Z is a.s. convergent.

Proof. Let F_n (resp. G_n) be the random variable that gives the density of 1's (resp. -1's) in the sequence W_n. We know from Corollary 2.7 that for almost all ω, the sequence $f_n = F_n(\omega)$ and $g_n = G_n(\omega)$ satisfy

$$(4) \qquad \liminf \frac{\sqrt{n}\,\big(f_n - g_n - (p-q)\big)}{\sqrt{\ln\ln n}} = -L \qquad \text{and} \qquad \limsup \frac{\sqrt{n}\,\big(f_n - g_n - (p-q)\big)}{\sqrt{\ln\ln n}} = L,$$

with $L = \sqrt{2(p+q-(p-q)^2)}$. Fixed two of these sequences f_n and g_n we also know that

$$(5) \qquad \big|f_n - g_n - (p-q)\,\big| \leq K\sqrt{\frac{\ln\ln n}{n}},$$

for some $K > L$. Therefore, to prove the corollary we can assume that the densities of 1's and -1's in the series $Z(\omega)$ are given by sequences f_n and g_n satisfying (4) and (5).

(i) As a consequence of Theorem C we get that when $\sum m_n a_n$ is convergent and $f_n - g_n$ converges, then it converges towards 0. Since we know that $f_n - g_n$ converges towards $p - q$, we get that Z is a.s. divergent when $p - q \neq 0$.

(ii) To prove that Z is a.s. divergent we will use item (a) of Theorem C and (4). Since $p = q$, notice that

$$\limsup n a_n \Big| f_n - g_n \Big| \geq \limsup \frac{R\sqrt{n}}{\sqrt{\ln\ln n}} \Big| f_n - g_n \Big| = RL > 0.$$

Hence $\lim n a_n (f_n - g_n) \neq 0$ and Z is a.s. divergent.

(iii) Again, by Theorem C, to prove that Z is a.s. convergent it suffices to show that $\sum n(a_n - a_{n+1})(f_n - g_n)$ is convergent. We will use that the sequence f_n satisfies (5) and $p = q$. This holds because

$$n\big(a_n - a_{n+1}\big)\Big| f_n - g_n \Big| \leq Kn\sqrt{\frac{\ln\ln n}{n}}\big(a_n - a_{n+1}\big) = K\sqrt{n\ln\ln n}\,\big(a_n - a_{n+1}\big),$$

which, by hypothesis (iii), is precisely the general term of a convergent series. □

We end the paper comparing the results of applying Theorem D and the above corollary to a test family of random series. We consider

$$W^{\alpha,\beta} = \sum \frac{W_n}{n^\alpha \ln^\beta n}, \qquad \alpha > 0, \beta \geq 0,$$

where W_n are i.i.d. random variables with distribution $\mathcal{W}(p, q)$. Both results imply that $W^{\alpha,\beta}$ is a.s. divergent unless $p = q$. Hence we fix $p = q$. Then, notice that by Theorem D, $W^{\alpha,\beta}$ is a.s. convergent if and only if $\sum \frac{1}{n^{2\alpha} \ln^{2\beta} n}$ is convergent, and otherwise $W^{\alpha,\beta}$ is a.s. divergent. Table 1 collects the final results obtained when $p = q$. There we have also used the trivial result that if b_n is a positive sequence such that $\sum b_n$ is convergent then the random sequence $\sum b_n W_n$ is also (always) convergent .

	$0 < \alpha < \frac{1}{2}$	$\alpha = \frac{1}{2}$	$\frac{1}{2} < \alpha < 1$	$\alpha = 1$	$\alpha > 1$
$0 \leq \beta \leq \frac{1}{2}$	a.s. div.	a.s. div. $(*)$	a.s. conv.	a.s. conv.	conv.
$\frac{1}{2} < \beta \leq 1$	a.s. div.	a.s. conv. $(*)$	a.s. conv.	a.s. conv.	conv.
$\beta > 1$	a.s. div.	a.s. conv.	a.s. conv.	conv.	conv.

Table 1: Behavior of the random series $W^{\alpha,\beta}$, with $p = q$, according to α and β. Cases with $(*)$ are covered by the Kolmogorov's Three-Series Theorem but not by our approach.

If, instead of Theorem D, we apply items (ii) and (iii) of Corollary 4.1 we recover all the results of that table except the ones corresponding to $\alpha = \frac{1}{2}$ and $0 \leq \beta \leq 1$. In this range of values our approach does not decide the behavior of the random series.

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